The Ohio State University
Department of Economics
Econ 805–Extra Problems on Production and Uncertainty: Questions and
Answers

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Prof. Peck

(1) In the following economy, there are two consumers, two firms, and two goods (labor/leisure and food). For i $= 1,2$, consumer i is endowed with zero units of food and 1 unit of leisure, $\omega_i = (0, 1)$. Letting x_i denote consumer i's consumption of food and ℓ_i denote consumer i's consumption of leisure, the utility function is: $\log(x_i) + \log(\ell_i)$.

Let y_1 denote firm 1's output of food and L_1 denote firm 1's labor input (so that L_1 must be nonnegative). Then firm 1's production function, the frontier of its production set, is given by: $y_1 = AL_1$, where the parameter A is a positive real number. Firm 1 is owned by consumer 1.

Let y_2 denote firm 2's output of food and L_2 denote firm 2's labor input (so that L_2 must be nonnegative). Then firm 2's production function, the frontier of its production set, is given by: $y_2 = (L_2)^{1/2}$. Firm 2 is owned by consumer 2.

(a) Define a competitive equilibrium for this economy.

(b) Calculate the competitive equilibrium price vector and allocation, as a function of the parameter, A. Assume that we have an interior solution, where both firms produce output.

(c) For what values of the parameter, A, will we have a corner solution, where one of the firms produces zero output?

Answer:

(a) Normalizing the price of food to be 1 and denoting the price of labor as p, a Competitive Equilibrium is a price vector, $(1, p)$, and an allocation, $(x_1, \ell_1, x_2, \ell_2, y_1, L_1, y_2, L_2)$, such that:

(i) (x_1, ℓ_1) solves:

$$\max \log(x_1) + \log(\ell_1)$$
$$\text{subject to}$$
$$x_1 + p\ell_1 = p$$
$$(x_1, \ell_1) \geq 0.$$

This relies on the fact that utility is monotonic and firm 1 has CRS and receives zero profits at the CE.

(ii) (x_2, ℓ_2) solves

$$\max \log(x_2) + \log(\ell_2)$$
$$\text{subject to}$$
$$x_2 + p\ell_2 = p + \pi_2$$
$$(x_2, \ell_2) \geq 0.$$

(iii) (y_1, L_1) solves

$$\max y_1 - pL_1$$
$$\text{subject to}$$
$$y_1 \;=\; AL_1$$
$$L_1 \;\geq\; 0.$$

(iv) (y_2, L_2) solves

$$\max y_2 - pL_2$$
$$\text{subject to}$$
$$y_2 \;=\; (L_2)^{1/2}$$
$$L_2 \;\geq\; 0.$$

(v)

$$x_1 + x_2 \;=\; y_1 + y_2$$
$$\ell_1 + \ell_2 + L_1 + L_2 \;=\; 2.$$

(Equalities follow from strict monotonicity of utility.)

(b) Starting with the profit maximization conditions, for an interior solution where firm 1 produces a finite positive quantity, we must have zero profits: $AL_1 - pL_1 = 0$, or $p = A$.

Plugging the constraint into the profit expression for firm 2, we have the unconstrained problem:

$$\max_{L_2}(L_2)^{1/2} - pL_2.$$

Setting the derivative equal to zero and solving yields:

$$L_2 = \frac{1}{4p^2}, \qquad y_2 = \frac{1}{2p}, \qquad \text{and } \pi_2 = \frac{1}{4p}.$$

The first order conditions for consumer 1's utility maximization problem are:

$$x_1 \;=\; p\ell_1 \quad \text{(MRS condition)}$$
$$x_1 + p\ell_1 \;=\; p \quad \text{(budget)}$$

Solving, we get:

$$x_1 = \frac{p}{2} \quad \text{and} \quad \ell_1 = \frac{1}{2}.$$

The first order conditions for consumer 2 are:

$$x_2 \;=\; p\ell_2$$
$$x_2 + p\ell_2 \;=\; p + \frac{1}{4p}.$$

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Solving, we get

$$x_2 = \frac{p}{2} + \frac{1}{8p} \quad \text{and} \quad \ell_2 = \frac{1}{2} + \frac{1}{8p^2}.$$

To get the final allocation, substitute $p = A$ into the demand functions. Market clearing comes in, not to allow us to solve for the price, but instead to allow us to determine firm 1's input and output. We get:

$$x_1 = \frac{A}{2}, \ell_1 = \frac{1}{2}, x_2 = \frac{A}{2} + \frac{1}{8A}, \ell_2 = \frac{1}{2} + \frac{1}{8A^2},$$
$$L_2 = \frac{1}{4A^2}, \; y_2 = \frac{1}{2A}.$$

Market clearing for labor/leisure implies

$$\frac{1}{2} + (\frac{1}{2} + \frac{1}{8A^2}) + L_1 + \frac{1}{4A^2} = 2.$$

Solving for L_1, we have

$$L_1 = 1 - \frac{3}{8A^2}, \text{ and therefore, } y_1 = A - \frac{3}{8A}.$$

(c) Firm 1 will shut down if the previous expression for L_1 is negative, which occurs if A is sufficiently small. The condition is, $A < (\frac{3}{8})^{1/2}$. If you are wondering what the CE would be in this case, we could recalculate the CE, assuming firm 1 does not produce. The equilibrium wage will end up being $p = (\frac{3}{8})^{1/2}$, and indeed firm 1 does best not to produce.

(2) In the following economy, there are 2 consumers, one good, x, and 2 states of nature, a and b. The probabilities of the states are A and B, where A + B = 1. The utility functions and endowments of the consumers are given by

$$\begin{aligned}
V_1 &= A\log(x_1^a) + B\log(x_1^b) \\
(\omega_1^a, \omega_1^b) &= (2,0) \\
V_2 &= Ax_2^a + Bx_2^b \\
(\omega_2^a, \omega_2^b) &= (1,1)
\end{aligned}$$

(a) Draw an Edgeworth box diagram, indicating the endowment point and the contract curve.

(b) Calculate the C.E. of this economy, as a function of the parameters, A and B. Be careful to remember the constraint that consumption must be nonnegative.

(c) Interpret the C.E. as an insurance market. In other words, who is offering the insurance, what is the amount of the premium, what is the amount of the claim in the event of an "accident," and how does the premium compare with the expected value of the claim?

Answer:

(a) The contract curve follows the 45-degree line from (0,0) to (1,1), then follows the top of the box from (1,1) to (3,1). On the flat segment, marginal rates of substitution are not equal, but these points are Pareto optimal.
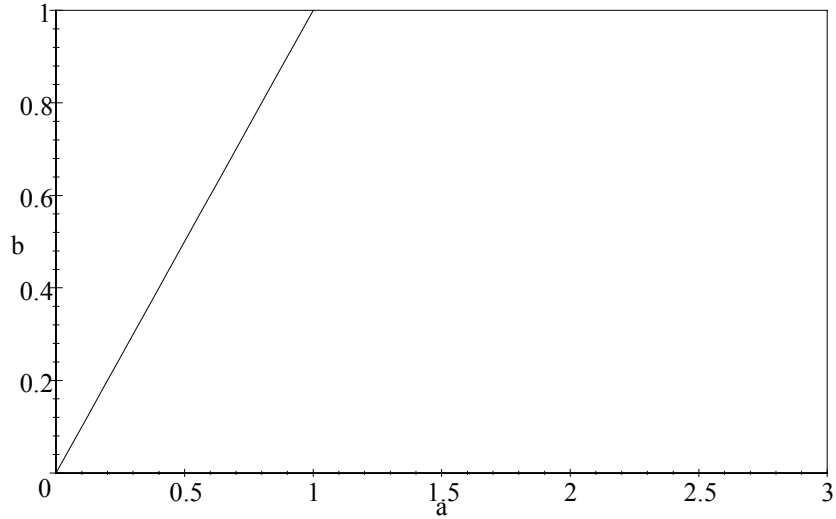
(b) To calculate the CE, we first notice that the probabilities will determine whether we have an interior solution, with all consumptions positive, or a corner solution, with $x_2^b = 0$. Let us calculate the demand function for consumer 1. Normalizing $p^b = 1$, the first order conditions are

$$\frac{Ax_1^b}{Bx_1^a} = p^a$$

$$p^a x_1^a + x_1^b = 2p^a.$$

Solving, we have

$$\frac{Ax_1^b}{B} + x_1^b = 2p^a, \quad \text{or}$$

$$x_1^b = \frac{2Bp^a}{A+B} = 2Bp^a.$$

$$x_1^a = \frac{Ax_1^b}{Bp^a} = 2A.$$

If we have an interior solution, consumer 2's marginal rate of substitution must equal the price ratio, so

$$\frac{A}{B} = p^a.$$

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The allocation is found by plugging the price to determine consumer 1's consumption, and using market clearing to determine consumer 2's consumption. We have

$$
\begin{aligned}
x_1^a &= 2A, \quad x_1^b = 2A, \\
x_2^a &= 3 - 2A, \quad x_2^b = 1 - 2A.
\end{aligned}
$$

This interior equilibrium applies if consumption is nonnegative, $A \leq \frac{1}{2}$.

If we have $A > \frac{1}{2}$, then we have a corner solution. Consumer 2's demand is determined by the two equations,

$$
\begin{aligned}
x_2^b &= 0, \quad \text{and} \\
p^a x_2^a + x_2^b &= p^a + 1.
\end{aligned}
$$

Solving, we have

$$
x_2^a = \frac{p^a + 1}{p^a}.
$$

Using market clearing for state b, we have:

$$
\begin{aligned}
2Bp^a + 0 &= 1, \quad \text{so} \\
p^a &= \frac{1}{2B}.
\end{aligned}
$$

The allocation is:

$$
\begin{aligned}
x_1^a &= 2A, \quad x_1^b = 1, \\
x_2^a &= 1 + 2B = 3 - 2A, \quad \text{and} \quad x_2^b = 0.
\end{aligned}
$$

(c) For the interior equilibrium, consumer 2 (who is risk neutral) winds up on the same indifference curve as his initial endowment. He provides insurance to consumer 1, at fair odds. To see this, the insurance premium paid by consumer 1 is her endowment in the good state (no accident), less her consumption:

$$
\text{premium} = 2 - 2A = 2B.
$$

In the bad state, she submits a claim, where her final consumption equals her endowment plus her claim, minus her premium:

$$
\begin{aligned}
2A &= 0 + \text{ claim } - 2B, \text{ so} \\
\text{claim} &= 2.
\end{aligned}
$$

The premium is $2B$, and the expected claim is 2 multiplied by the probability of submitting the claim, B, or $2B$.

(3) Consider the following exchange economy with two von Neumann-Morgenstern expected utility maximizers, two states of nature, $s = \alpha$ and $s = \beta$, and one commodity per state. We have

$$
V_i = \sum_{s=\alpha,\beta} \pi_s u_i(x_i^s),
$$

where V_i is consumer i's utility function, π_s is the probability of state s, and x_i^s is the consumption of consumer i in state s. Assume that each u_i is twice continuously differentiable, and that we have $u_i' > 0$ and $u_i'' < 0$. Also assume that the initial endowments, ω_i^s, are strictly positive for each consumer in each state.

(a) If the following statement is true, then carefully argue why, and if it is false, then present a counterexample: If the initial endowments of state-contingent commodities are Pareto optimal for $\pi_\alpha = \pi_\beta = \frac{1}{2}$, then the endowments are Pareto optimal for any other specification of the probabilities.

(b) Define a competitive equilibrium for this economy.

(c) Does this economy satisfy the assumptions required to apply the Second Welfare Theorem?

Answer:

(a) This statement is true. Because of our assumptions of differentiability, etc., Pareto optimality at the original probabilities implies

$$\frac{\frac{1}{2}\frac{\partial u_1(\omega_1^\alpha)}{\partial x_1^\alpha}}{\frac{1}{2}\frac{\partial u_1(\omega_1^\beta)}{\partial x_1^\beta}} = \frac{\frac{1}{2}\frac{\partial u_2(\omega_2^\alpha)}{\partial x_2^\alpha}}{\frac{1}{2}\frac{\partial u_2(\omega_2^\beta)}{\partial x_2^\beta}}.$$

It follows that

$$\frac{\pi_\alpha}{\pi_\beta}\frac{\frac{\partial u_1(\omega_1^\alpha)}{\partial x_1^\alpha}}{\frac{\partial u_1(\omega_1^\beta)}{\partial x_1^\beta}} = \frac{\pi_\alpha}{\pi_\beta}\frac{\frac{\partial u_2(\omega_2^\alpha)}{\partial x_2^\alpha}}{\frac{\partial u_2(\omega_2^\beta)}{\partial x_2^\beta}} \quad \text{holds,}$$

which implies that the allocation is Pareto optimal for the new endowments.

(b) A CE is a price vector, (p^α, p^β), and an allocation, $(x_1^\alpha, x_1^\beta, x_2^\alpha, x_2^\beta)$, such that

(i) (x_1^α, x_1^β) solves

$$\max \pi_\alpha u_1(x_1^\alpha) + \pi_\beta u_1(x_1^\beta)$$
$$\text{subject to}$$
$$p^\alpha x_1^\alpha + p^\beta x_1^\beta = p^\alpha \omega_1^\alpha + p^\beta \omega_1^\beta$$
$$x_1 \geq 0,$$

(ii) (x_2^α, x_2^β) solves

$$\max \pi_\alpha u_2(x_2^\alpha) + \pi_\beta u_2(x_2^\beta)$$
$$\text{subject to}$$
$$p^\alpha x_2^\alpha + p^\beta x_2^\beta = p^\alpha \omega_2^\alpha + p^\beta \omega_2^\beta$$
$$x_2 \geq 0,$$

(iii)

$$x_1^\alpha + x_2^\alpha = \omega_1^\alpha + \omega_2^\alpha$$
$$x_1^\beta + x_2^\beta = \omega_1^\beta + \omega_2^\beta.$$

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(c) Because we have $u_i' > 0$ for each consumer, strict monotonicity is satisfied. Because we also have $u_i'' < 0$ for each consumer, the Bernoulli utility functions, and the overall utility functions V_i, are strictly concave. Strict concavity implies strict quasiconcavity. Also, the initial endowments, ω_i^s, are strictly positive for each consumer in each state. Thus, all of the assumptions required to apply the SFTWE are satisfied.