Department of Economics
The Ohio State University
Midterm Answers–Econ 8712

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1. (40 points)

Consider an economy with n agents and three time periods (period 0, period 1, and period 2). Each agent is a von Neumann-Morgenstern expected utility maximizer, with a Bernoulli utility function that depends on whether the agent is "patient" or "impatient." In period 0, each agent is endowed with one unit of an asset that yields 1 unit of consumption if the asset is liquidated in period 1, and $R > 1$ units of consumption if the asset is liquidated in period 2. Suppose that, at the beginning of period 1, πn of the agents will learn that they are "impatient" and the remaining $(1-\pi)n$ agents will learn that they are "patient," so the probability of being impatient is π. Denoting consumption in period 1 and period 2 by (c_1, c_2), an impatient agent only cares about consumption in period 1, so her Bernoulli utility function is

$$u^I(c_1, c_2) = \frac{(c_1)^{1-a}}{1-a}$$

for $a > 0$. A patient consumer treats consumption in each period as perfect substitutes, so her Bernoulli utility function is

$$u^P(c_1, c_2) = \frac{(c_1 + c_2)^{1-a}}{1-a}.$$

(a) (10 points) If left to her own devices, an agent will liquidate her investment in period 1 when she is impatient and will liquidate her investment in period 2 when she is patient. Taking expectations at time zero, before an agent learns whether she is impatient or patient, what is her expected utility if she is left to her own devices?

(b) (15 points) What is the certainty equivalent of the risky consumption an agent faces in part (a)? That is, at time zero, how much consumption, c, would she agree to accept in exchange for her investment, assuming that she would consume c units in period 1 if she is impatient and c units in period 2 if she is patient?

For part (c), suppose that the n agents decide to pool their resources and form a bank. The impatient agents will withdraw their deposits in period 1 and receive c_1 units of consumption, and the patient agents will withdraw their deposits in period 2 and receive c_2 units of consumption. To meet its obligations, the bank will have to liquidate $\pi n c_1$ units of its invested deposits in period 1, leaving $n - \pi n c_1$ units invested until period 2 and $(n - \pi n c_1)R$ units to allocate to the $(1-\pi)n$ patient agents. Therefore, the bank's resource constraint requires

$$c_2 = \frac{(1 - \pi c_1)R}{(1 - \pi)}.$$

(c) (15 points) Since all agents face the identical uncertainty at time 0, they will all have the same expected utility when facing a deposit contract, (c_1, c_2). *Find the deposit contract that maximizes ex ante expected utility. (Specifying the optimal* c_1 *is enough, because* c_2 *is determined by the resource constraint).*

Answer:

(a) With probability π, the agent is impatient and liquidates her investment in period 1, consuming 1 unit. With probability $1 - \pi$, she is patient and liquidates her investment in period 2, consuming R units. Therefore, her expected utility is

$$\begin{aligned} & \pi\left(\frac{1^{1-a}}{1-a}\right) + (1-\pi)\left(\frac{R^{1-a}}{1-a}\right) \\ = \quad & \frac{\pi + (1-\pi)R^{1-a}}{1-a}. \end{aligned}$$

(b) If she consumes c units of consumption, during period 1 if she is impatient and during period 2 if she is patient, then the utility of the certainty equivalent is

$$\begin{aligned} & \pi\left(\frac{c^{1-a}}{1-a}\right) + (1-\pi)\left(\frac{c^{1-a}}{1-a}\right) \\ = \quad & \frac{c^{1-a}}{1-a}. \end{aligned}$$

The certainty equivalent gives her the same utility as she would receive under the risky consumption from part (a), so c must solve

$$\frac{c^{1-a}}{1-a} = \frac{\pi + (1-\pi)R^{1-a}}{1-a}.$$

Solving for c, we have

$$c = [\pi + (1-\pi)R^{1-a}]^{1/(1-a)}.$$

(c) Given the promised period-1 consumption of the impatient consumers, c_1, the patient consumers will receive $c_2 = \frac{(1-\pi c_1)R}{(1-\pi)}$ in period 2. The expected utility is then given by

$$\pi\left(\frac{(c_1)^{1-a}}{1-a}\right) + (1-\pi)\left(\frac{[\frac{(1-\pi c_1)R}{(1-\pi)}]^{1-a}}{1-a}\right).$$

To find the optimal c_1, differentiate the above expression, yielding the first order condition,

$$\pi(c_1)^{-a} + (1-\pi)[\frac{(1-\pi c_1)R}{(1-\pi)}]^{-a}[-\frac{\pi R}{1-\pi}] = 0.$$

After simplifying and bringing the second term to the right hand side, we have

$$(c_1)^{-a} = R[\frac{(1-\pi c_1)R}{(1-\pi)}]^{-a}.$$

Taking both sides to the $-\frac{1}{a}$ power, we have

$$c_1 = R^{-1/a}[\frac{(1-\pi c_1)R}{(1-\pi)}].$$

This is a linear equation in c_1. Solving, we have

$$c_1 = \frac{R^{(a-1)/a}}{1-\pi+\pi R^{(a-1)/a}}.$$

2. (20 points)

Find a counterexample to the FFTWE when not all utility functions satisfy local nonsatiation. That is, find a competitive equilibrium with an allocation that is not strongly Pareto optimal. It is enough to present a carefully constructed and labelled Edgeworth box diagram, along with an explanation for why your diagram constitutes a counterexample.

Answer:

Counterexamples involve either a bliss point, a commodity bundle where a consumer has a maximum utility, or thick indifference curves, where a consumer is indifferent over all bundles within a neighborhood.

For example, suppose that there are two consumers, each with an initial endowment, $\omega_i = (1,1)$. Consumer 1 is literally indifferent between all nonnegative consumption bundles, so $u_1(x_1^1, x_1^2)$ is a constant function. Consumer 2 has the utility function, $u_2(x_2^1, x_2^2) = \log(x_2^1) + \log(x_2^2)$. The competitive equilibrium price vector is $p^* = (1,1)$ and the allocation is the initial endowment allocation, $x^* = (1,1,1,1)$. To see that this is a CE, consumer 2's bundle is the unique solution to her utility maximization problem, and any bundle solves consumer 1's utility maximization problem. Furthermore, market clearing is satisfied. To see that this allocation is not strongly PO, notice that we can take consumption away from consumer 1 and give it to consumer 2, strictly increasing the utility of consumer 2 without lowering the utility of consumer 1.

3. (40 points)

Consider the following pure-exchange economy with 300 consumers and two goods. For $i = 1, ..., 200$, consumer i has the utility function

$$u_i(x_i^1, x_i^2) = \log(x_i^1) + \log(x_i^2)$$

and the initial endowment vector, $(3, 1)$. For $i = 201, ..., 300$, consumer i has the utility function

$$u_i(x_i^1, x_i^2) = \log(x_i^1) + \log(x_i^2)$$

and the initial endowment vector, $(1, 3)$.

(a) (10 points) Define a competitive equilibrium for this economy.

(b) (20 points) Compute the competitive equilibrium price and allocation.

(c) (10 points) Is the allocation you found in part (b) strongly Pareto optimal? Explain your reasoning.

Answer:

(a) A competitive equilibrium is a price vector, (p^{*1}, p^{*2}), and an allocation, (x_i^{*1}, x_i^{*2}) for $i = 1, ..., 300$, such that

(i) For $i = 1, ..., 200$, (x_i^{*1}, x_i^{*2}) solves

$$\begin{aligned} & \max \log(x_i^1) + \log(x_i^2) \\ & \text{subject to} \\ p^{*1}x_i^1 + p^{*2}x_i^2 & \leq 3p^{*1} + p^{*2} \\ x_i & \geq 0, \end{aligned}$$

(ii) For $i = 201, ..., 300$, (x_i^{*1}, x_i^{*2}) solves

$$\begin{aligned} & \max \log(x_i^1) + \log(x_i^2) \\ & \text{subject to} \\ p^{*1}x_i^1 + p^{*2}x_i^2 & \leq p^{*1} + 3p^{*2} \\ x_i & \geq 0, \end{aligned}$$

(iii) Markets clear:

$$\begin{aligned} \sum_{i=1}^{300} x_i^1 &= 700 \\ \sum_{i=1}^{300} x_i^2 &= 500. \end{aligned}$$

(b) At the CE, all budget equations and market clearing equations hold with equality, due to the monotonicity of the utility functions. Also, the first 200 consumers are identical, facing the same UMP and the same demand function solving the UMP. We will use the subscript 1 for these consumers. Similarly, the last 100 consumers are identical, so we will use the subscript 2 for these consumers. Also, we will normalize the price vector, $p^* = (p, 1)$.

The f.o.c. for type 1 consumers are

$$\begin{aligned} \frac{x_1^2}{x_1^1} &= p \\ px_1^1 + x_1^2 &= 3p + 1. \end{aligned}$$

Solving for the demand functions, we have

$$\begin{aligned} x_1^1 &= \frac{3p+1}{2p} \\ x_1^2 &= \frac{3p+1}{2}. \end{aligned}$$

The f.o.c. for type 2 consumers are

$$\begin{aligned} \frac{x_2^2}{x_2^1} &= p \\ px_2^1 + x_2^2 &= p + 3. \end{aligned}$$

Solving for the demand functions, we have

$$\begin{aligned} x_2^1 &= \frac{p+3}{2p} \\ x_2^2 &= \frac{p+3}{2}. \end{aligned}$$

Substituting the demand functions for good 2 into the corresponding market clearing equation, we have

$$200(\frac{3p+1}{2}) + 100(\frac{p+3}{2}) = 500.$$

Solving, we have $p = \frac{5}{7}$. Substituting into the demand functions, we have $x_1 = (\frac{11}{5}, \frac{11}{7})$ and $x_2 = (\frac{13}{5}, \frac{13}{7})$. The CE is

$$\begin{aligned} p^* &= (\frac{5}{7}, 1) \\ \text{For } i &= 1, ..., 200, \; x_i = (\frac{11}{5}, \frac{11}{7}) \\ \text{For } i &= 201, ..., 300, \; x_i = (\frac{13}{5}, \frac{13}{7}). \end{aligned}$$

(c) The allocation in part (b) is strongly Pareto optimal. The utility functions satisfy strict monotonicity and therefore local nonsatiation, so we can apply the FFTWE to conclude that the CE allocation is strongly PO.