Department of Economics
The Ohio State University
Midterm Exam Answers–Econ 8712

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1. (30 points)
A decision maker (DM) is a von Neumann-Morgenstern expected utility maximizer with Bernoulli utility function over final wealth x given by

$$u(x) = x^{1/2}.$$

Suppose that the DM has an opportunity to bet on the big game. He must decide on whether to bet on the "home" team winning the game or the "visiting" team winning the game, and he must also decide on the nonnegative bet size, b. The DM has initial wealth, W. The DM's final wealth is $W - b$ if the team he bets on loses the game, and his final wealth is $W + \frac{10}{11}b$ if the team he bets on wins the game. The DM believes that the home team will win the game with probability p and that the visiting team will win the game with probability $1 - p$.

(a) Calculate the values of p for which the expected utility maximizing bet size is zero.
(b) Calculate the expected utility maximizing bet size when $p = \frac{11}{16}$.

Answer:
(a) When the DM bets on the home time, the expected utility is

$$p(W + \frac{10}{11}b)^{1/2} + (1 - p)(W - b)^{1/2}.$$

For an interior solution, the first order condition (differentiating with respect to b) is

$$\frac{1}{2}p(W + \frac{10}{11}b)^{-1/2}\frac{10}{11} - \frac{1}{2}(1 - p)(W - b)^{-1/2} = 0, \text{ or}$$

$$\frac{(W + \frac{10}{11}b)^{-1/2}}{(W - b)^{-1/2}} = \frac{11(1 - p)}{10p}. \quad (1)$$

Solving (1) for b gives the solution, and the optimal bet size is an increasing function of p, but we reach a corner solution where the optimal bet size on the home team is zero when p falls below a threshold. To find the threshold, find the value of p solving (1) when $b = 0$. Since the left side will equal 1, the threshold is given by $p = \frac{11}{21}$, which is exactly the probability at which betting on the home team carries fair odds. Thus, bet a positive amount on the home team whenever the odds are fair or better, $p \geq \frac{11}{21}$. We can do an identical computation for

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when the DM would want to bet on the visiting team, by replacing p with $1-p$. The DM would bet a positive amount on the visiting team whenever $1-p \geq \frac{11}{21}$, or in other words, $p \leq \frac{10}{21}$. Thus, the optimal bet size is zero whenever the odds for both bets are unfavorable: $\frac{10}{21} < p < \frac{11}{21}$.

(b) When $p = \frac{11}{16}$, we have an interior solution where the DM bets a positive amount on the home team. Substituting p into (1) and squaring both sides, we have
$$\frac{W-b}{W + \frac{10}{11}b} = (\frac{11 \cdot 5}{10 \cdot 11})^2 = \frac{1}{4}.$$

Cross-multiplying, we have

$$4W - 4b = W + \frac{10}{11}b,$$
$$b = \frac{11}{18}W.$$

2. (30 points)

The following problem concerns a pure exchange economy with k goods and 2 consumers. Assume that initial endowments are strictly interior and that each consumer's utility function is strictly quasi-concave, strictly monotonic, and continuous. Suppose that (p^*, x^*) is a competitive equilibrium. Also suppose that x^{**} is a feasible allocation such that $x_1^* \neq x_1^{**}$ holds and $u_1(x_1^*) = u_1(x_1^{**})$ holds.

Prove that x^{**} cannot be strongly Pareto optimal.

Answer:
Suppose by way of contradiction that the conclusion is false, so that x^{**} is strongly Pareto optimal. Since $u_1(x_1^*) = u_1(x_1^{**})$ holds, x^* would Pareto dominate x^{**} if consumer 2 strictly preferred x_2^* to x_2^{**}, so we must have

$$u_2(x_2^{**}) \geq u_2(x_2^*).$$

Since (p^*, x^*) is a competitive equilibrium, by the FFTWE, x^* is strongly Pareto optimal. But x^{**} would Pareto dominate x^* if consumer 2 strictly preferred x_2^{**} to x_2^*, so we must have

$$u_2(x_2^*) \geq u_2(x_2^{**}).$$

Combining the two displayed inequalities, we conclude that $u_2(x_2^{**}) = u_2(x_2^*)$ must hold.

If we take a convex combination of the two allocations, $x^{***} = \alpha x^* + (1 - \alpha)x^{**}$ for $0 < \alpha < 1$, then each consumer i must strictly prefer x_i^{***} to x_i^* or x_i^{**}. The reason is that x_i^{***} lies on the line segment connecting two points

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on the same indifference curve, which by strict quasi-concavity must be on a higher indifference curve. But then x^{***} is feasible and Pareto dominates x^{**}, contradicting the supposition that x^{**} is strongly Pareto optimal.

3. (40 points)

Consider the following pure-exchange economy with two types of consumers and two goods. There are n_1 type-1 consumers and n_2 type-2 consumers. If consumer i is of type 1, she has the utility function

$$u_i(x_i^1, x_i^2) = 2\log(x_i^1) + \log(x_i^2)$$

and the initial endowment vector, $(1, 3)$. If consumer i is of type 2, she has the utility function

$$u_i(x_i^1, x_i^2) = \log(x_i^1) + 2\log(x_i^2)$$

and the initial endowment vector, $(3, 1)$.

(a) (10 points) Define a competitive equilibrium for this economy.
(b) (20 points) Compute the competitive equilibrium price vector.
(c) (10 points) Find the values of n_1 and n_2 for which type-1 consumers consume exactly 2 units of good 1 at the CE.

Answer:
(a) A CE is a price, (p^{1*}, p^{2*}) and an allocation, $(x_i^{1*}, x_i^{2*})_{i=1}^{n_1+n_2}$, such that
(i) for type-1 consumers, (x_i^{1*}, x_i^{2*}) solves

$$\max 2\log(x_i^1) + \log(x_i^2)$$
$$\text{subject to}$$
$$p^{1*}x_i^1 + p^{2*}x_i^2 \leq p^{1*} + 3p^{2*}$$
$$x_i \geq 0,$$

(ii) for type-2 consumers, (x_i^{1*}, x_i^{2*}) solves

$$\max \log(x_i^1) + 2\log(x_i^2)$$
$$\text{subject to}$$
$$p^{1*}x_i^1 + p^{2*}x_i^2 \leq 3p^{1*} + p^{2*}$$
$$x_i \geq 0,$$

(iii) markets clear:

$$\sum_{i=1}^{n_1+n_2} x_i^{1*} \leq n_1 + 3n_2$$

$$\sum_{i=1}^{n_1+n_2} x_i^{2*} \leq 3n_1 + n_2.$$

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(b) Because the utility functions are strictly monotonic, budget inequalities and resource inequalities will hold as equalities. By strict quasi-concavity, utility maximization problems have unique solutions, so I will denote the consumption of type-1 consumers as x_1 and the consumption of type-2 consumers as x_2. Also, I will normalize the price of good 2 to be 1 and denote the price of good 1 as p.

For type-1 consumers, the demand function is found by solving the budget equation and the MRS condition, $2x_1^2/x_1^1 = p$, yielding

$$x_1^1 = \frac{2(p+3)}{3p} \text{ and } x_1^2 = \frac{p+3}{3}.$$

For type-2 consumers, the demand function is found by solving the budget equation and the MRS condition, $x_2^2/2x_2^1 = p$, yielding

$$x_2^1 = \frac{3p+1}{3p} \text{ and } x_2^2 = \frac{2(3p+1)}{3}.$$

The market clearing condition for good 2 is then given by

$$n_1\left(\frac{p+3}{3}\right) + n_2\left(\frac{2(3p+1)}{3}\right) = 3n_1 + n_2.$$

Solving for p, we have

$$p = \frac{6n_1 + n_2}{n_1 + 6n_2}.$$

(c) Substituting the equilibrium price into the demand function, we have

$$x_1^1 = \frac{2\left(\left[\frac{6n_1+n_2}{n_1+6n_2}\right] + 3\right)}{3\left[\frac{6n_1+n_2}{n_1+6n_2}\right]},$$

which can be simplified to

$$x_1^1 = \frac{18n_1 + 38n_2}{18n_1 + 3n_2}.$$

Thus, $x_1^1 = 2$ whenever we have

$$\frac{18n_1 + 38n_2}{18n_1 + 3n_2} = 2, \text{ or}$$

$$n_2 = \frac{9}{16}n_1.$$