Department of Economics
The Ohio State University
Midterm Answers–Econ 8712

Prof. James Peck
Fall 2017

Directions: *Answer all questions, be neat, and show all work. If it is appropriate, you are allowed to use propositions presented in class without proving them here.*

1. (40 points)

A decision maker (DM) is a von Neumann-Morgenstern expected utility maximizer with Bernoulli utility function over final wealth x given by the strictly concave, strictly monotonic, and twice continuously differentiable function $u(x)$, and initial wealth W. Suppose that the DM will have a minor accident with probability π_1, in which case she suffers damages of D_1. The DM will have a major accident with probability π_2, in which case she suffers damages D_2. With the remaining probability, $\pi_3 = 1 - \pi_1 - \pi_2$, she does not have an accident and suffers no damages. Assume that all three probabilities are strictly positive and that we have $0 < D_1 < D_2$.

Now suppose that the DM has an opportunity to purchase as much insurance as she wants from an insurance company offering her fair odds. However, the amount of money that the DM receives from the insurance company when she has an accident is the same for a minor accident as for a major accident. That is, if she purchases α units of insurance, her payment to the insurance company is $\alpha(\pi_1 + \pi_2)$, she receives nothing when she does not have an accident, and she receives α when she has any kind of accident.

(i) (15 points) Set up the optimization problem faced by the DM and find the first order conditions for an interior solution.

(ii) (25 points) Show that there is a unique optimal insurance level, α^*, and that it satisfies $D_1 < \alpha^* < D_2$. [Do not try to solve for α^*.]

Answer:

(i)

$$\max_{\alpha} \pi_1 u(W - \alpha(\pi_1 + \pi_2) - D_1 + \alpha) + \pi_2 u(W - \alpha(\pi_1 + \pi_2) - D_2 + \alpha)$$
$$+(1 - \pi_1 - \pi_2)u(W - \alpha(\pi_1 + \pi_2))$$

Differentiating expected utility with respect to α, we have

$$\begin{aligned} &\pi_1 u'(W - \alpha(\pi_1 + \pi_2) - D_1 + \alpha)(1 - \pi_1 - \pi_2) \\ &+\pi_2 u'(W - \alpha(\pi_1 + \pi_2) - D_2 + \alpha)(1 - \pi_1 - \pi_2) \\ &-(1 - \pi_1 - \pi_2)u'(W - \alpha(\pi_1 + \pi_2))(\pi_1 + \pi_2). \end{aligned} \quad (1)$$

Setting (1) equal to zero and simplifying, we have the first order condition,

$$\begin{aligned}&\pi_1 u'(W-\alpha(\pi_1+\pi_2)-D_1+\alpha)\\&+\pi_2 u'(W-\alpha(\pi_1+\pi_2)-D_2+\alpha)\\&-u'(W-\alpha(\pi_1+\pi_2))(\pi_1+\pi_2)=0.\end{aligned} \tag{2}$$

(ii) Dividing (1) by the positive constant, $(1-\pi_1-\pi_2)$, and differentiating with respect to α, we have

$$\begin{aligned}&\pi_1 u''(W-\alpha(\pi_1+\pi_2)-D_1+\alpha)(1-\pi_1-\pi_2)\\&+\pi_2 u''(W-\alpha(\pi_1+\pi_2)-D_2+\alpha)(1-\pi_1-\pi_2)\\&+u''(W-\alpha(\pi_1+\pi_2))(\pi_1+\pi_2)^2.\end{aligned}$$

By the concavity of u, it follows that this expression is negative, so the second order conditions are satisfied. Therefore, the DM's optimization problem has a unique solution.

Evaluating the left side of (2) at $\alpha = D_1$we have

$$\begin{aligned}&\pi_1 u'(W-D_1(\pi_1+\pi_2))\\&+\pi_2 u'(W-D_1(\pi_1+\pi_2)-D_2+D_1)\\&-u'(W-D_1(\pi_1+\pi_2))(\pi_1+\pi_2).\end{aligned} \tag{3}$$

Because u is strictly concave and $D_1 < D_2$ holds, we have

$$u'(W-D_1(\pi_1+\pi_2)-D_2+D_1) > u'(W-D_1(\pi_1+\pi_2)),$$

so (3) is strictly greater than

$$\begin{aligned}&\pi_1 u'(W-D_1(\pi_1+\pi_2))\\&+\pi_2 u'(W-D_1(\pi_1+\pi_2))\\&-u'(W-D_1(\pi_1+\pi_2))(\pi_1+\pi_2)=0.\end{aligned}$$

Evaluating the left side of (2) at $\alpha = D_2$we have

$$\begin{aligned}&\pi_1 u'(W-D_2(\pi_1+\pi_2)-D_1+D_2)\\&+\pi_2 u'(W-D_2(\pi_1+\pi_2))\\&-u'(W-D_2(\pi_1+\pi_2))(\pi_1+\pi_2).\end{aligned} \tag{4}$$

Because u is strictly concave and $D_1 < D_2$ holds, we have

$$u'(W-D_2(\pi_1+\pi_2)-D_1+D_2) < u'(W-D_2(\pi_1+\pi_2)),$$

so (4) is strictly less than

$$\begin{aligned}&\pi_1 u'(W-D_2(\pi_1+\pi_2))\\&+\pi_2 u'(W-D_2(\pi_1+\pi_2))\\&-u'(W-D_2(\pi_1+\pi_2))(\pi_1+\pi_2)=0.\end{aligned}$$

Since the objective is strictly concave, the slope is positive at $\alpha = D_1$, and the slope is negative at $\alpha = D_2$, it follows that there is a unique solution between D_1 and D_2.

2. (20 points)

The aggregate excess demand functions for a pure exchange economy with two goods and n consumers are given below. The n consumers have utility functions that are continuous, strictly monotonic, and strictly quasi-concave. In these expressions, a and b are real numbers.

$$\begin{aligned} Z^1(p^1, p^2) &= \frac{\frac{p^1}{p^2} + 2}{a\frac{p^1}{p^2}} + \frac{\frac{p^1}{p^2} + 1}{\frac{p^1}{p^2}} - 3, \\ Z^2(p^1, p^2) &= \frac{\frac{p^1}{p^2} + 2}{2} + \frac{\frac{p^1}{p^2} + 1}{b} - 4. \end{aligned}$$

(a) (10 points) Solve for the values of a and b. [Hint: Use Walras' Law.]

(b) (10 points) Calculate the competitive equilibrium price of good 1 relative to good 2, $\frac{p^1}{p^2}$.

Answer:

Since Walras' Law applies to all price vectors, it applies to normalized price vectors where the price of good 1 is denoted by p and the price of good 2 is normalized to 1. Using Walras' Law, we have

$$\begin{aligned} pZ^1(p) + Z^2(p) &= 0, \text{ or} \\ p[\frac{p+2}{ap} + \frac{p+1}{p} - 3] + [\frac{p+2}{2} + \frac{p+1}{b} - 4] &= 0. \end{aligned}$$

This simplifies to

$$p[\frac{1}{a} - \frac{3}{2} + \frac{1}{b}] + [\frac{2}{a} + \frac{1}{b} - 2] = 0. \tag{5}$$

Since (5) holds for all p, it follows that both expressions in brackets must be zero, which can be easily solved for $a = 2$ and $b = 1$.

Since we know the values of a and b, we can write

$$Z^1(p) = \frac{p+2}{2p} + \frac{p+1}{p} - 3.$$

Because all utility functions are strictly monotonic, we have a competitive equilibrium if and only if the relative price, p, satisfies

$$\frac{p+2}{2p} + \frac{p+1}{p} - 3 = 0,$$

which we can solve for $p = \frac{4}{3}$.

3. (40 points)

Consider the following pure-exchange economy with two consumers, two goods, and a government. For $i = 1, 2$, consumer i has the utility function,

$$u_i(x_i^1, x_i^2) = \log(x_i^1) + \log(x_i^2).$$

Consumer 1 has the initial endowment vector, $\omega_1 = (3, 1)$, and consumer 2 has the initial endowment vector, $\omega_2 = (1, 1)$. The government does not have any endowments. Consider the following two scenarios:

Scenario 1 (tax/transfer, then trade): The government takes one unit of good 1 away from consumer 1's endowment and adds one unit of good 1 to consumer 2's endowment. Then, with the new endowments, the consumers trade according to the competitive equilibrium.

Scenario 2 (trade, then tax/transfer): The consumers trade according to the competitive equilibrium. Then the government alters the C.E. allocation by taking one unit of good 1 away from consumer 1 and giving one unit of good 1 to consumer 2. Then the consumers consume according to the new allocation. Assume that when the consumers trade according to the C.E., they do not anticipate the tax/transfer policy that will come later.

(a) (15 points) Calculate the final allocation under Scenario 1.
(b) (15 points) Calculate the final allocation under Scenario 2.
(c) (10 points) Which scenario does consumer 1 prefer, and which scenario does consumer 2 prefer?

Answer:

It will be convenient to solve for the demand functions when a consumer has the endowment vector, $(a, 1)$. Normalize the price vector to be of the form, $(p, 1)$. The MRS and budget equations are

$$\begin{aligned} \frac{x_i^2}{x_i^1} &= p \\ px_i^1 + x_i^2 &= ap + 1. \end{aligned}$$

Solving for the demand function, we have

$$\begin{aligned} x_i^1 &= \frac{ap+1}{2p} \\ x_i^2 &= \frac{ap+1}{2}. \end{aligned}$$

(a) Under scenario 1, both consumers have the endowment vector $(2, 1)$ after tax/transfer. Substituting $a = 2$ into the demand functions, market clearing for good 2 is

$$\begin{aligned} \frac{2p+1}{2} + \frac{2p+1}{2} &= 2, \\ p &= \frac{1}{2}. \end{aligned}$$

The allocation is, for $i = 1, 2$,

$$x_i^1 = 2, \quad x_i^2 = 1.$$

(b) Before the tax/transfer, substituting $a = 3$ for consumer 1 and $a = 1$ for consumer 2, market clearing for good 2 yields

$$\begin{aligned} \frac{3p+1}{2} + \frac{p+1}{2} &= 2, \\ p &= \frac{1}{2}. \end{aligned}$$

The allocation before tax/transfer is

$$\begin{aligned} x_1^1 &= \frac{5}{2}, \quad x_1^2 = \frac{5}{4}, \\ x_2^1 &= \frac{3}{2}, \quad x_2^2 = \frac{3}{4}. \end{aligned}$$

The final allocation after the tax/transer is

$$\begin{aligned} x_1^1 &= \frac{3}{2}, \quad x_1^2 = \frac{5}{4}, \\ x_2^1 &= \frac{5}{2}, \quad x_2^2 = \frac{3}{4}. \end{aligned}$$

(c) Consumer 1's utility under scenario 1 is

$$\log(\frac{3}{2}) + \log(\frac{5}{4}) = \log(\frac{15}{8})$$

and under scenario 2 is $\log(2)$, so consumer 1 prefers scenario 1.

Consumer 2's utility under scenario 1 is

$$\log(\frac{5}{2}) + \log(\frac{3}{4}) = \log(\frac{15}{8})$$

and under scenario 2 is $\log(2)$, so consumer 2 prefers scenario 1. Intuitively, scenario 1 allows consumers to equate MRS through markets, while scenario 2 interferes with the workings of markets and leaves gains from trade unsatisfied.