Department of Economics
The Ohio State University
Final Exam Answers–Econ 8712

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Fall 2024

Directions: *Answer all questions. An answer is not correct unless you show all work and provide proper explanations. When appropriate, you can use propositions proven in class without proving them here.*

1. (30 points)

Consider the following pure exchange economy with three consumers and K goods. For $i = 1, 2, 3$, consumer i has the initial endowment vector, ω_i, and the utility function, $u_i(x_i)$. Suppose that there are three consumption vectors, x^A, x^B, and x^C, satisfying

$$x^A + x^B + x^C = \omega_1 + \omega_2 + \omega_3$$

and

$$\begin{aligned} u_1(x^A) &> u_1(x^B) > u_1(x^C) \\ u_2(x^B) &> u_2(x^C) > u_2(x^A) \\ u_3(x^C) &> u_3(x^A) > u_3(x^B). \end{aligned}$$

Prove that $(x_1, x_2, x_3) = (x^B, x^C, x^A)$ cannot be a competitive equilibrium allocation. What assumptions are you making on the utility functions? (The fewer the assumptions you need to make, the more credit you receive.)

Answer:

Here is a pretty good answer that only assumes local non-satiation: Suppose that $(x_1, x_2, x_3) = (x^B, x^C, x^A)$ is a competitive equilibrium allocation. Under local non-satiation, we can apply the FFTWE to conclude that this allocation is strongly Pareto optimal. However, the allocation (x^A, x^B, x^C) is feasible and Pareto dominates (x^B, x^C, x^A), a contradiction.

Here is an answer that does not assume anything about the utility functions: Suppose that $(x_1, x_2, x_3) = (x^B, x^C, x^A)$ is a competitive equilibrium allocation with a price vector, p^*. Since a solution to consumer 1's utility maximization problem is x^B and $u_1(x^A) > u_1(x^B)$ holds, x^A must not be affordable. That is, we have

$$p^* \cdot x^A > p^* \cdot x^B.$$

Similarly, when we consider the utility maximizing choices of consumer 2 and consumer 3, we have

$$\begin{aligned} p^* \cdot x^B &> p^* \cdot x^C \\ p^* \cdot x^C &> p^* \cdot x^A. \end{aligned}$$

By summing the left sides and the right sides, these three inequalities imply

$$p^* \cdot x^A + p^* \cdot x^B + p^* \cdot x^C > p^* \cdot x^B + p^* \cdot x^C + p^* \cdot x^A,$$

a contradiction. Intuitively, at least one of the consumers will have the highest income and be able to afford any bundle that the other consumers demand.

2. (40 points)

The following economy has two consumers, two firms, and two goods. Good 1 is food consumption and good 2 is labor/leisure. For $i = 1, 2$, consumer i has a utility function given by

$$u_i(x_i^1, x_i^2) = \log(x_i^1) + \log(x_i^2).$$

Consumer 1 has the initial endowment vector $\omega_1 = (0, 1)$ and consumer 1 owns firm 1. Consumer 2 has the initial endowment vector $\omega_2 = (0, 2)$ and consumer 2 owns firm 2.

Firm 1 has the production function,

$$y_1^1 = (L_1)^{1/2},$$

where y_1^1 is output of good 1 and L_1 is their nonnegative input of labor.

Firm 2 has the production function,

$$y_2^1 = AL_2,$$

where y_2^1 is output of good 1, L_2 is their nonnegative input of labor, and A is a positive parameter.

(a) (10 points) Define a competitive equilibrium for this economy.

(b) (30 points) Calculate the competitive equilibrium price vector and allocation for this economy, as a function of the parameter A. For what values of A does firm 2 produce positive output in the competitive equilibrium?

Answer:

(a) A competitive equilibrium is a price vector, (p^{1*}, p^{2*}), and an allocation, $(x_1^{1*}, x_1^{2*}, x_2^{1*}, x_2^{2*}, y_1^{1*}, L_1^*, y_2^{1*}, L_2^*)$, such that

(1) (x_1^{1*}, x_1^{2*}) solves (where π_1 is the profit of firm 1)

$$\begin{aligned} & \max \log(x_1^1) + \log(x_1^2) \\ & \text{subject to} \\ p^{1*}x_1^1 + p^{2*}x_1^2 \quad \leq \quad & p^{2*} + \pi_1 \\ x_1 \quad \geq \quad & 0 \end{aligned}$$

(2) (x_2^{1*}, x_2^{2*}) solves (where π_2 is the profit of firm 2)

$$\begin{aligned} & \max \log(x_2^1) + \log(x_2^2) \\ & \text{subject to} \\ p^{1*}x_2^1 + p^{2*}x_2^2 &\leq 2p^{2*} + \pi_2 \\ x_2 &\geq 0 \end{aligned}$$

(3) (y_1^{1*}, L_1^*) solves

$$\begin{aligned} \max \pi_1 &\equiv p^{1*}y_1^1 - p^{2*}L_1 \\ & \text{subject to} \\ y_1^1 &\leq (L_1)^{1/2} \\ L_1 &\geq 0 \end{aligned}$$

(4) (y_2^{1*}, L_2^*) solves

$$\begin{aligned} \max \pi_2 &\equiv p^{1*}y_2^1 - p^{2*}L_2 \\ & \text{subject to} \\ y_2^1 &\leq AL_2 \\ L_2 &\geq 0 \end{aligned}$$

(5) market clearing

$$\begin{aligned} x_1^{1*} + x_2^{1*} &\leq y_1^{1*} + y_2^{1*} \\ x_1^{2*} + x_2^{2*} + L_1^* + L_2^* &\leq 3. \end{aligned}$$

(b) Normalize the price vector to be $(p, 1)$ and drop the asterisks for convenience. Budget constraints will hold with equality and firms will be on their production frontiers. Because firm 2's technology exhibits constant returns to scale, we have $\pi_2 = 0$. The first order condition for firm 1's profit maximization problem is

$$\frac{1}{2}p(L_1)^{-1/2} - 1 = 0.$$

Solving, we have

$$L_1 = \frac{p^2}{4}, y_1^1 = \frac{p}{2}, \pi_1 = \frac{p^2}{4}.$$

Consumer 1's demand functions are characterized by the marginal rate of substitution equaling the price ratio and the budget equation,

$$\begin{aligned} \frac{x_1^2}{x_1^1} &= p \\ px_1^1 + x_1^2 &= 1 + \frac{p^2}{4}. \end{aligned}$$

Solving, we have

$$\begin{aligned} x_1^1 &= \frac{1}{2p} + \frac{p}{8} \\ x_1^2 &= \frac{1}{2} + \frac{p^2}{8}. \end{aligned}$$

Consumer 2's first order conditions are given by

$$\begin{aligned} \frac{x_2^2}{x_2^1} &= p \\ px_2^1 + x_2^2 &= 2. \end{aligned}$$

Solving, we have

$$\begin{aligned} x_2^1 &= \frac{1}{p} \\ x_1^2 &= 1. \end{aligned}$$

To solve for the equilibrium price, we use market clearing. There are two cases. If firm 2 produces positive output, the zero profit condition requires $p = \frac{1}{A}$. At this price, we have

$$\begin{aligned} x_1^1 &= \frac{A}{2} + \frac{1}{8A}, x_1^2 = \frac{1}{2} + \frac{1}{8A^2} \\ x_2^1 &= A, x_1^2 = 1 \\ L_1 &= \frac{1}{4A^2}, y_1^1 = \frac{1}{2A}. \end{aligned}$$

Firm 2's input demand and output are determined by market clearing. Market clearing for good 1 is

$$\begin{aligned} \frac{A}{2} + \frac{1}{8A} + A &= \frac{1}{2A} + y_2^1, \text{ or} \\ y_2^1 &= \frac{3A}{2} - \frac{3}{8A}. \end{aligned}$$

Firm 2's production function then gives $L_2 = \frac{3}{2} - \frac{3}{8A^2}$. We will have an equilibrium of this form whenever $L_2 > 0$ holds, or

$$\begin{aligned} \frac{3}{2} - \frac{3}{8A^2} &> 0, \text{ or} \\ A &> \frac{1}{2}. \end{aligned}$$

When we have $A \leq \frac{1}{2}$, firm 2 does not produce. In this case, we could solve for the competitive equilibrium as if firm 2 did not exist. Now the market clearing condition for good 1 is given by

$$\begin{aligned} \frac{1}{2p} + \frac{p}{8} + \frac{1}{p} &= \frac{p}{2}, \text{ or} \\ \frac{3}{2p} &= \frac{3p}{8}, \\ p^2 &= 4, \\ p &= 2. \end{aligned}$$

At this price, the allocation is

$$\begin{aligned} x_1 &= (\frac{1}{2}, 1) \\ x_2 &= (\frac{1}{2}, 1) \\ y_1^1 &= 1, L_1 = 1. \end{aligned}$$

3. (30 points)

The following Arrow Securities economy has 2 consumers, 2 states of nature, and one physical consumption good per state. The probability of state 1 is $\frac{1}{3}$ and the probability of state 2 is $\frac{2}{3}$. For $i = 1, 2$, consumer i is an expected utility maximizer, with a Bernoulli utility function given by $u_i(x_i) = \log(x_i)$. Consumer 1 has the initial endowment vector, $\omega_1 = (4, 2)$, and consumer 2 has the initial endowment vector, $\omega_2 = (0, 1)$. First the consumers trade Arrow securities, then the state is observed and securities are redeemed, and then the consumers trade on a spot market.

(a) (10 points) Define a competitive equilibrium for this economy.

(b) (15 points) Compute the competitive equilibrium prices, security holdings, and consumptions.

Answer:

(a) Normalize the spot prices and the price of security 2 to be one. Denote the remaining security price as q. A competitive equilibrium is a (normalized) price vector, $(q, 1, 1, 1)$, and an allocation, $(x_1(1), x_1(2), x_2(1), x_2(2), b_1^1, b_1^2, b_2^1, b_2^2)$, such that

(i) $x_1(1), x_1(2), b_1^1, b_1^2$ solves

$$\begin{aligned} & \max \frac{1}{3}\log(x_1(1)) + \frac{2}{3}\log(x_1(2)) \\ & \text{subject to} \\ qb_1^1 + b_1^2 &= 0 \\ x_1(1) &= 4 + b_1^1 \\ x_1(2) &= 2 + b_1^2 \\ x_1 &\geq 0. \end{aligned}$$

(ii) $x_2(1), x_2(2), b_2^1, b_2^2$ solves

$$\begin{aligned}
& \max \frac{1}{3}\log(x_2(1)) + \frac{2}{3}\log(x_2(2)) \\
& \text{subject to} \\
qb_2^1 + b_2^2 &= 0 \\
x_2(1) &= b_2^1 \\
x_2(2) &= 1 + b_2^2 \\
x_2 &\geq 0.
\end{aligned}$$

(iii) markets clear:

$$\begin{aligned}
x_1(1) + x_2(1) &= 4 \\
x_1(2) + x_2(2) &= 3 \\
b_1^1 + b_2^1 &= 0 \\
b_1^2 + b_2^2 &= 0.
\end{aligned}$$

(b) The easiest way to solve the consumer optimization problems is to use the spot market budget constraints to eliminate the securities variables. Then consumer 1's problem is characterized by the marginal rate of substitution condition,

$$\frac{\frac{1}{3}x_1(2)}{\frac{2}{3}x_1(1)} = q,$$

and the securities market budget equation,

$$q(x_1(1) - 4) + (x_1(2) - 2) = 0.$$

Simplifying and solving yields

$$\begin{aligned}
x_1(1) &= \frac{2 + 4q}{3q} \\
x_1(2) &= \frac{4 + 8q}{3}.
\end{aligned}$$

Consumer 2's problem is characterized by the marginal rate of substitution condition,

$$\frac{\frac{1}{3}x_2(2)}{\frac{2}{3}x_2(1)} = q,$$

and the securities market budget equation,

$$q(x_2(1) - 0) + (x_2(2) - 1) = 0.$$

Simplifying and solving yields

$$\begin{aligned}
x_2(1) &= \frac{1}{3q} \\
x_2(2) &= \frac{2}{3}.
\end{aligned}$$

To find the remaining price, use market clearing in state 2

$$\begin{aligned}\frac{4+8q}{3}+\frac{2}{3} &= 3,\\ q &= \frac{3}{8}.\end{aligned}$$

Substituting the price into the demand functions yields

$$\begin{aligned}x_1(1) &= \frac{28}{9}, x_1(2)=\frac{7}{3}\\ x_2(1) &= \frac{8}{9}, x_1(2)=\frac{2}{3}.\end{aligned}$$

From the spot market budget constraints we find the equilibrium security holdings

$$\begin{aligned}b_1^1 &= -\frac{8}{9}, b_1^2=\frac{1}{3}\\ b_2^1 &= \frac{8}{9}, b_2^2=-\frac{1}{3}.\end{aligned}$$