Department of Economics
The Ohio State University
Final Exam Answers–Econ 8712

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1. (30 points)

The following economy has one consumer, two firms, and three goods. Good 1 is food, good 2 is clothing, and good 3 is leisure/labor. The consumer has the initial endowment vector, $\omega = (0, 0, 1)$, and the utility function,

$$u(x^1, x^2, x^3) = \log(x^1) + \log(x^2).$$

Notice that the third good provides no utility, so the consumer will demand 0 units whatever the prices.

Firm 1 produces food using labor as an input. Denoting firm 1's output of food by y_1^1 and its non-negative input of labor by L_1, the firm's production function (the boundary of the production set) is given by:

$$y_1^1 = (L_1)^{1/2}.$$

Firm 2 produces clothing using labor as an input. Denoting firm 2's output of clothing by y_2^2 and its non-negative input of labor by L_2, the firm's production function (the boundary of the production set) is given by:

$$y_2^2 = (2L_2)^{1/2}.$$

(a) (10 points) Define a competitive equilibrium for this economy.

(b) (20 points) Compute the competitive equilibrium price vector and allocation.

Answer:

(a) A C.E. is a price vector, (p^1, p^2, p^3), and an allocation, $(x^1, x^2, x^3, y_1^1, L_1, y_2^2, L_2)$, such that:

(i) (x^1, x^2, x^3) solves

$$\begin{array}{rcl} & & \max \log(x^1) + \log(x^2) \\ & & \text{subject to} \\ p^1x^1 + p^2x^2 + p^3x^3 & \leq & p^3 + \pi_1 + \pi_2 \\ (x^1, x^2, x^3) & \geq & 0 \end{array}$$

(where π_1 and π_2 are the profits of firms 1 and 2),

(ii) (y_1^1, L_1) solves

$$\begin{aligned} & \max p^1 y_1^1 - p^3 L_1 \\ & \text{subject to} \\ y_1^1 \quad \leq \quad & (L_1)^{1/2} \end{aligned}$$

(iii) (y_2^2, L_2) solves

$$\begin{aligned} & \max p^2 y_2^2 - p^3 L_2 \\ & \text{subject to} \\ y_2^2 \quad \leq \quad & (2L_2)^{1/2} \end{aligned}$$

(iv) markets clear:

$$\begin{aligned} x^1 & \leq y_1^1 \\ x^2 & \leq y_2^2 \\ x^3 + L_1 + L_2 & \leq 1. \end{aligned}$$

(b) Normalize $p^3 = 1$ and note that all inequalities will hold as equalities, due to strict monotonicity of utility. Substituting the constraint into firm 1's profit expression and differentiating with respect to L_1, we have the first order condition

$$\frac{1}{2}(L_1)^{-1/2} p^1 = 1,$$

which we can solve for

$$\begin{aligned} L_1 &= \frac{(p^1)^2}{4}, \\ y_1^1 &= \frac{p^1}{2}, \\ \pi_1 &= \frac{(p^1)^2}{4}. \end{aligned}$$

Substituting the constraint into firm 2's profit expression and differentiating with respect to L_2, we have the first order condition

$$2 \cdot \frac{1}{2}(2L_2)^{-1/2} p^2 = 1,$$

which we can solve for

$$\begin{aligned} L_2 &= \frac{(p^2)^2}{2}, \\ y_2^2 &= p^2, \\ \pi_2 &= \frac{(p^2)^2}{2}. \end{aligned}$$

The solution to the consumer's problem is found by imposing $x^3 = 0$ and solving the budget equation and the marginal rate of substitution condition for the remaining demands,

$$\begin{aligned} p^1x^1 + p^2x^2 &= 1 + \pi_1 + \pi_2 \\ \frac{x^2}{x^1} &= \frac{p^1}{p^2}, \end{aligned}$$

yielding

$$\begin{aligned} x^1 &= \frac{1+\pi_1+\pi_2}{2p^1} = \frac{1 + \frac{(p^1)^2}{4} + \frac{(p^2)^2}{2}}{2p^1} \\ x^2 &= \frac{1+\pi_1+\pi_2}{2p^2} = \frac{1 + \frac{(p^1)^2}{4} + \frac{(p^2)^2}{2}}{2p^2}. \end{aligned}$$

Now we solve for the prices using market clearing for goods 1 and 2. Good 1 market clearing gives us

$$\begin{aligned} \frac{1 + \frac{(p^1)^2}{4} + \frac{(p^2)^2}{2}}{2p^1} &= \frac{p^1}{2}, \text{ or} \\ 1 + \frac{(p^1)^2}{4} + \frac{(p^2)^2}{2} &= (p^1)^2. \qquad (1) \end{aligned}$$

Good 2 market clearing gives us

$$\begin{aligned} \frac{1 + \frac{(p^1)^2}{4} + \frac{(p^2)^2}{2}}{2p^2} &= p^2, \text{ or} \\ 1 + \frac{(p^1)^2}{4} + \frac{(p^2)^2}{2} &= 2(p^2)^2. \qquad (2) \end{aligned}$$

Since the left side of (1) and (2) are the same, we can equate the right sides, yielding $(p^1)^2 = 2(p^2)^2$. Substituting this relationship into (1), we have

$$\begin{aligned} 1 + \frac{2(p^2)^2}{4} + \frac{(p^2)^2}{2} &= 2(p^2)^2, \text{ or} \\ p^2 &= 1, \text{ and therefore} \\ p^1 &= \sqrt{2}. \end{aligned}$$

Thus, the equilibrium price vector is $(\sqrt{2}, 1, 1)$, and the allocation is given by

$$\begin{aligned} x^1 &= \frac{\sqrt{2}}{2}, x^2 = 1, x^3 = 0, \\ y_1^1 &= \frac{\sqrt{2}}{2}, L_1 = \frac{1}{2}, y_2^2 = 1, L_2 = \frac{1}{2}. \end{aligned}$$

2. (30 points)

The following pure-exchange economy has 2 consumers, S states of nature, and one physical commodity per state of nature. For $s = 1, ..., S$, denote the consumption of consumer i in state s by x_i^s. For $i = 1, 2$, consumer i is a von Neumann-Morgenstern expected utility maximizer with the Bernoulli utility function $u_i(x_i^s)$, which is strictly monotonic, continuously differentiable, and strictly concave. For $s = 1, ..., S$, the endowment of consumer i in state s is denoted by ω_i^s. Before the state of nature is observed, consumers trade state-contingent commodities.

We allow the two consumers to have different probability beliefs over states. Denote the probability that consumer i assigns to state s by π_i^s, and assume that $\pi_i^s > 0$ for all i and s. (Note: Any differences in probability assessments do not reflect informational differences–the two consumers may understand that their beliefs are different and "agree to disagree.")

Let (p^*, x^*) be a competitive equilibrium for this economy with contingent commodity markets. For the following statements, either prove the statement (if the statement is true) or find a counterexample (if the statement is false).

(a) (15 points) If $\omega_1^s + \omega_2^s > \omega_1^{s'} + \omega_2^{s'}$ holds for states s and s', then at the competitive equilibrium we have $(x_1^s)^* > (x_1^{s'})^*$.

(b) (15 points) If there is no aggregate uncertainty, so $\omega_1^s + \omega_2^s = \omega_1^{s'} + \omega_2^{s'}$ holds for all states s and s', and if we have

$$\frac{\pi_1^1}{\pi_1^S} > \frac{\pi_2^1}{\pi_2^S},$$

then at the competitive equilibrium we have $(x_1^1)^* > (x_1^S)^*$.

Answer:

(a) This statement is false. Intuitively, the condition will not hold if consumer 1 assigns a high enough probability and consumer 2 assigns a low enough probability to state s'. Here is a counterexample. Both consumers have "log" utility, and there are two states with s being state 1 and s' being state 2. The initial endowments are given by $\omega_1 = (2, 0)$ and $\omega_2 = (0, 1)$. Consumer 1 has the beliefs $(\pi_1^1, \pi_1^2) = (\pi, 1 - \pi)$ and consumer 2 has the beliefs $(\pi_1^1, \pi_1^2) = (\frac{1}{2}, \frac{1}{2})$. We will compute the CE and show that the condition fails for some values of the parameter π. Normalize the price of state-2 consumption to be 1 and denote the price of state-1 contingent consumption as p.

Consumer 1 demand is found by solving the MRS equation and the budget equation:

$$\begin{aligned}
px_1^1 + x_1^2 &= 2p \\
\frac{\pi x_1^2}{(1-\pi)x_1^1} &= p, \text{ yielding (skipping some algebra you should show)} \\
x_1^1 &= 2\pi, x_1^2 = 2p(1-\pi).
\end{aligned}$$

Consumer 2 demand is found by solving the MRS equation and the budget equation:

$$\begin{aligned} px_2^1 + x_2^2 &= 1 \\ \frac{x_2^2}{x_2^1} &= p, \text{ yielding} \\ x_2^1 &= \frac{1}{2p}, x_2^2 = \frac{1}{2}. \end{aligned}$$

Market clearing for good 2 yields

$$\begin{aligned} 2p(1-\pi) + \frac{1}{2} &= 1, \\ p &= \frac{1}{4-4\pi}. \end{aligned}$$

Thus, consumer 1's consumption at the CE is

$$\begin{aligned} x_1^1 &= 2\pi \\ x_1^2 &= 2(\frac{1}{4-4\pi})(1-\pi) = \frac{1}{2}. \end{aligned}$$

If π is less than one quarter, then $x_1^2 > x_1^1$, which contradicts the condition $(x_1^s)^* > (x_1^{s'})^*$.

(b) This statement is true. A CE allocation must be Pareto optimal, so marginal rates of substitution are equated for any pair of states, implying (I am omitting the asterisks)

$$\frac{\pi_1^1 u_1'(x_1^1)}{\pi_1^S u_1'(x_1^S)} = \frac{\pi_2^1 u_2'(x_2^1)}{\pi_2^S u_2'(x_2^S)}. \tag{3}$$

From (3) and the fact that $\frac{\pi_1^1}{\pi_1^S} > \frac{\pi_2^1}{\pi_2^S}$ holds, we know

$$\frac{u_1'(x_1^1)}{u_1'(x_1^S)} < \frac{u_2'(x_2^1)}{u_2'(x_2^S)}. \tag{4}$$

Suppose, by way of contradiction, that the claim is false, so that $x_1^1 \leq x_1^S$ holds. From $x_1^1 \leq x_1^S$, concavity implies $u_1'(x_1^1) \geq u_1'(x_1^S)$, so the left side of (4) is greater than or equal to 1. Since there is no aggregate uncertainty and the PO allocation is nonwasteful, we also have $x_2^1 \geq x_2^S$. Concavity implies $u_2'(x_2^1) \leq u_2'(x_2^S)$, so the right side of (4) is less than or equal to 1. This contradicts (4).

3. (40 points)

Consider the following standard Arrow securities market, with 3 consumers, 2 states of nature, and 1 physical commodity per state. For $i = 1, 2, 3$, consumer

i is a von Neumann-Morgenstern expected utility maximizer with Bernoulli utility function $u_i(x_i(s)) = \log(x_i(s))$. Consumers 1 and 2 have the state-contingent endowment vector equal to $(2, 1)$, and consumer 3 has the state contingent endowment vector equal to $(1, 2)$. The two states are equally likely, so $\pi_1 = \pi_2 = \frac{1}{2}$.

Before the state of nature is revealed, consumers trade a complete set of Arrow securities, after which the state is revealed, securities are redeemed, and we have a spot market. Prices are written as $(q^1, q^2, p(1), p(2))$, where (q^1, q^2) are the securities prices and $p(1)$ is the price on the state-1 spot market, and $p(2))$ is the price on the state-2 spot market.

(a) (10 points) Define a competitive equilibrium for this economy with Arrow securities markets.

(b) (15 points) Normalize the price on each spot market to be one and the price of security 1 to be one: $p(1) = p(2) = q^1 = 1$. Compute the competitive equilibrium price of security 2, the allocation of consumption, and the security holdings.

(c) (15 points) Is there a competitive equilibrium for this economy with $p(1) = q^1 = q^2 = 1$? If yes, find the competitive equilibrium and justify your answer; if no, explain why not.

Answer:

A competitive equilibrium is a vector of prices, $(q^1, q^2, p(1), p(2))$, and an allocation, $(x_1(1), x_1(2), b_1^1, b_1^2, x_2(1), x_2(2), b_2^1, b_2^2, x_3(1), x_3(2), b_3^1, b_3^2,)$, such that

(i) $x_1(1), x_1(2), b_1^1, b_1^2$ solves

$$\begin{aligned}
& \max \frac{1}{2}\log(x_1(1)) + \frac{1}{2}\log(x_1(2)) \\
& \text{subject to} \\
q^1 b_1^1 + q^2 b_1^2 &= 0 \\
p(1)x_1(1) &= 2p(1) + b_1^1 \\
p(2)x_1(2) &= p(2) + b_1^2 \\
x_1(1) &\geq 0, x_1(2) \geq 0.
\end{aligned}$$

(ii) $x_2(1), x_2(2), b_2^1, b_2^2$ solves

$$\begin{aligned}
& \max \frac{1}{2}\log(x_2(1)) + \frac{1}{2}\log(x_2(2)) \\
& \text{subject to} \\
q^1 b_2^1 + q^2 b_2^2 &= 0 \\
p(1)x_2(1) &= 2p(1) + b_2^1 \\
p(2)x_2(2) &= p(2) + b_2^2 \\
x_2(1) &\geq 0, x_2(2) \geq 0.
\end{aligned}$$

(iii) $x_3(1), x_3(2), b_3^1, b_3^2$ solves

$$\begin{aligned} & \max \frac{1}{2}\log(x_3(1)) + \frac{1}{2}\log(x_3(2)) \\ & \text{subject to} \\ q^1 b_3^1 + q^2 b_3^2 &= 0 \\ p(1)x_3(1) &= p(1) + b_3^1 \\ p(2)x_3(2) &= 2p(2) + b_3^2 \\ x_3(1) &\geq 0, x_3(2) \geq 0. \end{aligned}$$

(iv) markets clear:

$$\begin{aligned} b_1^1 + b_2^1 + b_3^1 &= 0 \\ b_1^2 + b_2^2 + b_3^2 &= 0 \\ x_1(1) + x_2(1) + x_3(1) &= 5 \\ x_1(2) + x_2(2) + x_3(2) &= 4. \end{aligned}$$

Note: budget and market clearing conditions are written as equalities due to strict monotonicity.

(b) Consumers 1 and 2 are identical and will have the same demand function. For consumer 1, substitute the normalized spot market constraints into the securities constraint, yielding the maximization problem

$$\begin{aligned} & \max \frac{1}{2}\log(x_1(1)) + \frac{1}{2}\log(x_1(2)) \\ & \text{subject to} \\ (x_1(1) - 2) + q^2(x_1(2) - 1) &= 0 \\ x_1(1) &\geq 0, x_1(2) \geq 0. \end{aligned}$$

The solution to this problem is found by solving the budget equation and the marginal rate of substitution condition,

$$\frac{x_1(2)}{x_1(1)} = \frac{1}{q^2},$$

yielding the demand functions (skipping the algebra which you should show)

$$\begin{aligned} x_1(1) &= x_2(1) = \frac{q^2 + 2}{2} \\ x_1(2) &= x_2(2) = \frac{q^2 + 2}{2q^2}. \end{aligned}$$

For consumer 3, substitute the normalized spot market constraints into the securities constraint, yielding the maximization problem

$$\begin{aligned}
& \max \frac{1}{2}\log(x_3(1)) + \frac{1}{2}\log(x_3(2)) \\
& \text{subject to} \\
(x_3(1)-1) + q^2(x_3(2)-2) &= 0 \\
x_3(1) &\geq 0, x_3(2) \geq 0.
\end{aligned}$$

The solution to this problem is found by solving the budget equation and the marginal rate of substitution condition,

$$\frac{x_3(2)}{x_3(1)} = \frac{1}{q^2},$$

yielding the demands (skipping the algebra which you should show)

$$\begin{aligned}
x_3(1) &= \frac{2q^2+1}{2} \\
x_3(2) &= \frac{2q^2+1}{2q^2}.
\end{aligned}$$

Now we will use market clearing for the state-1 spot market to determime the remaining price. We have

$$\begin{aligned}
2\left(\frac{q^2+2}{2}\right) + \frac{2q^2+1}{2} &= 5, \text{ or} \\
q^2 &= \frac{5}{4}.
\end{aligned}$$

Substituting the price into the demand functions and then the spot market budget constraints, we find the CE allocation:

$$\begin{aligned}
x_1(1) &= x_2(1) = \frac{13}{8}, x_3(1) = \frac{14}{8} \\
x_1(2) &= x_2(2) = \frac{13}{10}, x_3(2) = \frac{14}{10} \\
b_1^1 &= b_2^1 = -\frac{3}{8}, b_3^1 = \frac{6}{8} \\
b_1^2 &= b_2^2 = \frac{3}{10}, b_3^2 = -\frac{6}{10}.
\end{aligned}$$

The CE price vector is $(1, \frac{5}{4}, 1, 1)$.

(c) One way to answer this question is to impose the normalization $(1, 1, 1, p(2))$, then solve for the demand functions and determine if there is a value of $p(2)$ that yields market clearing on all markets. An easier way is to use the CE from part (b) and the homogeneity properties we went over in class. Consider $(1, \frac{5}{4}, 1, 1)$ to be the "un-normalized" price vector. Consumption opportunities are unchanged (and therefore we have the same utility maximizing consumptions, which we know clear markets) if we multiply q^2 by a constant and divide $p(2)$ and each b_i^2 by the same constant. Letting the constant be $\frac{4}{5}$, we have another CE given by

$$\begin{aligned}
(q^1, q^2, p(1), p(2)) &= (1, 1, 1, \frac{5}{4}) \\
x_1(1) &= x_2(1) = \frac{13}{8}, x_3(1) = \frac{14}{8} \\
x_1(2) &= x_2(2) = \frac{13}{10}, x_3(2) = \frac{14}{10} \\
b_1^1 &= b_2^1 = -\frac{3}{8}, b_3^1 = \frac{6}{8} \\
b_1^2 &= b_2^2 = \frac{3}{8}, b_3^2 = -\frac{6}{8}.
\end{aligned}$$