Department of Economics
The Ohio State University
Final Exam Questions and Answers–Econ 8712

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1. (40 points)

The following economy has two consumers, two firms, and three goods. Good 3 is leisure/labor. For $i = 1, 2$, consumer i has the initial endowment vector, $\omega_i = (1, 1, 1)$, and the utility function,

$$u(x_i^1, x_i^2, x_i^3) = \log(x_i^1) + \log(x_i^2).$$

Notice that the third good provides no utility, so the consumers will demand 0 units whatever the prices. Consumer 1 owns firm 1 and consumer 2 owns firm 2.

Firm 1 produces good 1 using labor and good 2 as inputs. Using the standard notation in which y_f^j is firm f's net output of good j, firm 1's production function (the boundary of the production set) is given by:

$$\begin{aligned} y_1^1 &= (y_1^2 y_1^3)^{1/2}, \\ \text{where } y_1^2 &\leq 0 \text{ and } y_1^3 \leq 0. \end{aligned}$$

Firm 2 produces good 2 using labor and good 1 as inputs. Firm 2's production function (the boundary of the production set) is given by:

$$\begin{aligned} y_2^2 &= (y_2^1 y_2^3)^{1/2}, \\ \text{where } y_2^1 &\leq 0 \text{ and } y_2^3 \leq 0. \end{aligned}$$

(a) (10 points) Define a competitive equilibrium for this economy.

(b) (30 points) Compute the competitive equilibrium price vector and allocation.

Answer:

(a) A CE is a price vector, (p^{1*}, p^{2*}, p^{3*}), and an allocation, $(x_1^{1*}, x_1^{2*}, x_1^{3*}, x_2^{1*}, x_2^{2*}, x_2^{3*})$ and $(y_1^{1*}, y_1^{2*}, y_1^{3*}, y_2^{1*}, y_2^{2*}, y_2^{3*})$, such that:

(i) $(x_1^{1*}, x_1^{2*}, x_1^{3*})$ solves:

$$\begin{aligned} & \max \log(x_1^1) + \log(x_1^2) \\ & \text{subject to} \\ p^{1*}x_1^1 + p^{2*}x_1^2 + p^{3*}x_1^3 &\leq p^{1*} + p^{2*} + p^{3*} + \pi_1 \\ x_1 &\geq 0, \end{aligned}$$

(ii) $(x_2^{1*}, x_2^{2*}, x_2^{3*})$ solves:

$$\begin{aligned} & \max \log(x_2^1) + \log(x_2^2) \\ & \text{subject to} \\ p^{1*}x_2^1 + p^{2*}x_2^2 + p^{3*}x_2^3 &\leq p^{1*} + p^{2*} + p^{3*} + \pi_2 \\ x_2 &\geq 0, \end{aligned}$$

(iii) $(y_1^{1*}, y_1^{2*}, y_1^{3*})$ solves

$$\begin{aligned} \max \pi_1 &= p^{1*}y_1^1 + p^{2*}y_1^2 + p^{3*}y_1^3 \\ & \text{subject to} \\ y_1^1 &\leq (y_1^2 y_1^3)^{1/2}, \\ y_1^2 &\leq 0 \text{ and } y_1^3 \leq 0, \end{aligned}$$

(iv) $(y_2^{1*}, y_2^{2*}, y_2^{3*})$ solves

$$\begin{aligned} \max \pi_2 &= p^{1*}y_2^1 + p^{2*}y_2^2 + p^{3*}y_2^3 \\ & \text{subject to} \\ y_2^2 &\leq (y_2^1 y_2^3)^{1/2}, \\ y_2^1 &\leq 0 \text{ and } y_2^3 \leq 0, \end{aligned}$$

(v) market clearing:

$$\begin{aligned} x_1^{1*} + x_2^{1*} &\leq 2 + y_1^{1*} + y_2^{1*} \\ x_1^{2*} + x_2^{2*} &\leq 2 + y_1^{2*} + y_2^{2*} \\ x_1^{3*} + x_2^{3*} &\leq 2 + y_1^{3*} + y_2^{3*}. \end{aligned}$$

(b) First, normalize $p^3 = 1$ and notice that, due to strict monotonicity, budget and market clearing inequalities will hold as equalities, and firms will be on their production frontiers. Substituting the technological constraint into firm 1's profit expression, firm 1 chooses y_1^2 and y_1^3 to maximize

$$p^{1*}(y_1^2 y_1^3)^{1/2} + p^{2*}y_1^2 + y_1^3.$$

Differentiating the above expression with respect to y_1^2 yields

$$-\frac{1}{2}p^{1*}(y_1^2 y_1^3)^{-1/2}y_1^3 = p^{2*} \tag{1}$$

and differentiating with respect to y_1^3 yields

$$-\frac{1}{2}p^{1*}(y_1^2 y_1^3)^{-1/2}y_1^2 = 1.$$

Dividing left sides and right sides of the above two equations yields the condition that marginal rates of technical substitution equal input price ratios:

$$\frac{y_1^3}{y_1^2} = p^{2*}. \tag{2}$$

Due to constant returns to scale, if we substitute (2) into one of the first order conditions, say (1), we do not find a supply function, but rather a restriction on prices such that firm 1 is willing to produce a positive finite quantity:

$$\begin{aligned} -\frac{1}{2}p^{1*}(y_1^2(p^{2*}y_1^2))^{-1/2}p^{2*}y_1^2 &= p^{2*}, \\ \frac{1}{2}p^{1*}(p^{2*})^{-1/2} &= 1, \\ p^{1*} &= \sqrt{4p^{2*}}. \end{aligned} \tag{3}$$

Alternatively, one could derive (3) by substituting (2) into the profit expression, noticing that profits are linear in y_1^2, and concluding that profits must be zero for a solution to the profit maximization problem with positive output.

Going through the same steps for firm 2 yields the mrts condition,

$$\frac{y_2^3}{y_2^1} = p^{1*}, \tag{4}$$

and the restriction on prices

$$p^{2*} = \sqrt{4p^{1*}}. \tag{5}$$

Solving (3) and (5), we find the equilibrium price vector,

$$p^{1*} = p^{2*} = 4, p^{3*} = 1.$$

Note–because labor is inelastically supplied, it is clear that at least one firm must be producing, and because of the symmetry of the problem in terms of goods 1 and 2, it is clear that both firms will be producing. Thus, both (3) and (5) must hold.

Because the two consumers have the same utility function and endowment vectors, and because profit incomes are zero, they will have the same demand functions. Simultaneously solving the budget equation and the marginal rate of substitution condition, we have

$$\begin{aligned} p^{1*}x_i^1 + p^{2*}x_i^2 &= p^{1*} + p^{2*} + 1 \\ \frac{x_i^2}{x_i^1} &= \frac{p^{1*}}{p^{2*}} \end{aligned}$$

and the demand functions

$$\begin{aligned} x_i^1 &= \frac{p^{1*} + p^{2*} + 1}{2p^{1*}} \\ x_i^2 &= \frac{p^{1*} + p^{2*} + 1}{2p^{2*}}. \end{aligned}$$

Subsituting the equilibrium price vector yields

$$\begin{aligned} x_i^1 &= \frac{9}{8} \\ x_i^2 &= \frac{9}{8}. \end{aligned}$$

Now we use market clearing for goods 1 and 2:

$$x_1^{1*} + x_2^{1*} = \frac{9}{4} = 2 + y_1^{1*} + y_2^{1*} \tag{6}$$

$$x_1^{2*} + x_2^{2*} = \frac{9}{4} = 2 + y_1^{2*} + y_2^{2*} \tag{7}$$

From firm 1's production function and (2), we have

$$y_1^{1*} = (y_1^{2*} y_1^{3*})^{1/2} = (y_1^{2*} 4 y_1^{2*})^{1/2} = -2y_1^{2*}.$$

From firm 2's production function and (4), we have

$$y_2^{2*} = (y_2^{1*} y_2^{3*})^{1/2} = (y_2^{1*} 4 y_2^{1*})^{1/2} = -2y_2^{1*}.$$

Substituting these conditions into the market clearing equations yields

$$\frac{9}{4} = 2 - 2y_1^{2*} + y_2^{1*} \tag{8}$$

$$\frac{9}{4} = 2 + y_1^{2*} - 2y_2^{1*}. \tag{9}$$

Simultaneously solving (8) and (9) yields

$$y_1^{2*} = y_2^{1*} = -\frac{1}{4},$$

which implies

$$y_1^{1*} = y_2^{2*} = \frac{1}{2}.$$

Therefore, the equilibrium price vector is $(4, 4, 1)$, and the equilibrium allocation is $x_1 = x_2 = (\frac{9}{8}, \frac{9}{8}, 0)$, $y_1 = (\frac{1}{2}, -\frac{1}{4}, -1)$, $y_2 = (-\frac{1}{4}, \frac{1}{2}, -1)$.

2. (30 points)

The following pure-exchange economy has 2 consumers, 2 states of nature, and one physical commodity per state of nature. For $s = 1, 2$, the probability of state s is denoted by π_s. For $s = 1, 2$ and $i = 1, 2$, denote the consumption of consumer i in state s by x_i^s. The initial endowment vectors are given by $(\omega_1^1, \omega_1^2) = (1, 3)$ and $(\omega_2^1, \omega_2^2) = (3, 1)$. For $i = 1, 2$, consumer i is an expected

utility maximizer, but with a "Bernoulli" utility function, given in brackets below, that depends on the state. Specifically, overall utility functions, $V_1(x_1^1, x_1^2)$ and $V_2(x_2^1, x_2^2)$, are given by

$$\begin{aligned} V_1(x_1^1, x_1^2) &= \pi_1[2u_1(x_1^1)] + \pi_2[u_1(x_1^2)], \\ V_2(x_2^1, x_2^2) &= \pi_1[u_2(x_2^1)] + \pi_2[2u_2(x_2^2)]. \end{aligned}$$

Assumption 1: The functions, $u_1(\cdot)$ and $u_2(\cdot)$, are differentiable, strictly concave, and strictly monotonic.

Let (p^*, x^*) be a competitive equilibrium with complete contingent commodity markets. Then **exactly one** of the following statements is true:

1. We have $x_1^{1*} > x_1^{2*}$, for all functions, $u_1(\cdot)$ and $u_2(\cdot)$, satisfying Assumption 1.

2. We have $x_1^{1*} = x_1^{2*}$, for all functions, $u_1(\cdot)$ and $u_2(\cdot)$, satisfying Assumption 1.

3. We have $x_1^{1*} < x_1^{2*}$, for all functions, $u_1(\cdot)$ and $u_2(\cdot)$, satisfying Assumption 1.

(30 points) Indicate whether statement 1, statement 2, or statement 3 is true, and give a proof of the correct statement. (Half credit for a correctly worked out example illustrating which statement is true, since this is not a proof.)

Answer:

Statement 1 is true, and here is the proof.

Because Assumption 1 holds, the C.E. allocation is characterized by equal marginal rates of substitution. Therefore, denoting derivatives by "primes", we have

$$\begin{aligned} \frac{\pi_1[2u_1'(x_1^{1*})]}{\pi_2[u_1'(x_1^{2*})]} &= \frac{\pi_1[u_2'(x_2^{1*})]}{\pi_2[2u_2'(x_2^{2*})]}, \text{ or} \\ \frac{2u_1'(x_1^{1*})}{u_1'(x_1^{2*})} &= \frac{u_2'(x_2^{1*})}{2u_2'(x_2^{2*})}. \end{aligned} \tag{10}$$

Suppose by way of contradiction that statement 1 is not true. Then for some functions satisfying Assumption 1, we have $x_1^{1*} \leq x_1^{2*}$. By concavity, $x_1^{1*} \leq x_1^{2*}$ implies $u_1'(x_1^{1*}) \geq u_1'(x_1^{2*})$, so the left side of (10) is greater than or equal to 2. Because there is no aggregate uncertainty (endowments add up to 4 in each state), $x_1^{1*} \leq x_1^{2*}$ implies $x_2^{1*} \geq x_2^{2*}$. Therefore, it follows from concavity that we have $u_2'(x_2^{1*}) \leq u_2'(x_2^{2*})$, so the right side of (10) is less than or equal to one half. Since the two sides are equal, the left side cannot be greater than 2 and

the right side less than one half, contradicting the supposition that statement 1 is not true.

3. (30 points)

Suppose that we add to the Rothschild-Stiglitz insurance model a government that only cares about the welfare of the low-risk drivers. A fraction, $\frac{9}{10}$, of the drivers are of the low-risk type, with an accident probability, $p^L = 0.2$. A fraction, $\frac{1}{10}$, of the drivers are of the high-risk type, with an accident probability, $p^H = 0.7$. Drivers differ only in their accident probabilities, with the Bernoulli utility function $u(x) = \log(x)$, an initial wealth of 1, and damages of 1 when they have an accident. (That is, the endowment in (W_1, W_2) space is $(1, 0)$.)

Suppose that the government can exclude all private firms from offering insurance contracts, and that the goverment chooses a single contract to offer to all drivers. (The goverment cannot observe a driver's type, and therefore must offer the same contract to everyone.) Also assume that the government is required to balance it's budget, so the contract it offers must yield itself zero expected profits.

Compute the goverment's optimal contract, to maximize the utility of the low-risk drivers subject to the break-even constraint.

Answer:

Because both high-risk types and low-risk types will choose the contract offered by the government, the break-even constraint requires the government to offer a contract that is on or below the pooled fair odds line. The accident probability for a driver randomly selected from the population is given by

$$\overline{p} = (.9)(.2) + (.1)(.7) = \frac{1}{4}.$$

Then the equation of the pooled fair odds line equates expected consumption with expected endowment wealth, using the accident probability $\overline{p} = \frac{1}{4}$:

$$\frac{3}{4}W_1 + \frac{1}{4}W_2 = \frac{3}{4}.$$

The government's problem, then, is to choose (W_1, W_2) to solve

$$\begin{aligned} & \max \frac{4}{5}\log(W_1) + \frac{1}{5}\log(W_2) \\ & \text{subject to} \\ \frac{3}{4}W_1 + \frac{1}{4}W_2 \quad \leq \quad & \frac{3}{4}. \end{aligned}$$

The solution is characterized by the constraint holding with equality and the marginal rate of substitution condition,

$$\frac{\frac{4}{5}W_2}{\frac{1}{5}W_1} = 3.$$

Substituting $W_1 = \frac{4}{3}W_2$ (from the mrs condition) into the constraint, and solving, we have $W_1 = \frac{4}{5}$ and $W_2 = \frac{3}{5}$.