Department of Economics
The Ohio State University
Final Exam Questions and Answers–Econ 8712

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1. (35 points)

The following economy has one consumer, two firms, and four goods. Goods 1 and 2 are consumption goods, good 3 is capital, and good 4 is labor. The consumer is endowed with one unit of capital and one unit of labor, $\omega = (0, 0, 1, 1)$, and only receives utility from consumption of goods 1 and 2. Thus, capital and labor will be inelastically supplied to the market. The consumer's utility function is given by

$$u(x^1, x^2) = 4\log(x^1) + \log(x^2).$$

The consumer owns both firms.

Firm 1 produces good 1 using capital and labor as inputs. For convenience, denote the (nonnegative) capital input used by firm 1 as K_1 and the (nonnegative) labor input used by firm 1 as L_1. Firm 1's production function (the boundary of the production set) is given by:

$$\begin{aligned} y_1^1 &= (K_1 L_1)^{1/2}, \\ \text{where } L_1 &\geq 0 \text{ and } K_1 \geq 0. \end{aligned}$$

Firm 2 produces good 2 using capital and labor as inputs. For convenience, denote the (nonnegative) capital input used by firm 2 as K_2 and the (nonnegative) labor input used by firm 2 as L_2. Firm 2's production function (the boundary of the production set) is given by:

$$\begin{aligned} y_2^2 &= (K_2 L_2)^{1/2}, \\ \text{where } L_2 &\geq 0 \text{ and } K_2 \geq 0. \end{aligned}$$

(a) (10 points) Define a competitive equilibrium for this economy.

(b) (25 points) Compute the competitive equilibrium price vector and allocation.

Answer:

(a) A CE is a price vector, (p^1, p^2, p^3, p^4) and an allocation, (x^1, x^2, x^3, x^4) and $(y_1^1, K_1, L_1, y_2^2, K_2, L_2)$, such that

(i) (x^1, x^2, x^3, x^4) solves

$$\begin{aligned} & \max 4\log(x^1) + \log(x^2) \\ & \text{subject to} \\ p^1 x^1 + p^2 x^2 + p^3 x^3 + p^4 x^4 &\leq p^3 + p^4 + \pi_1 + \pi_2 \\ x &\geq 0, \end{aligned}$$

(ii) y_1^1, K_1, L_1 solves

$$
\begin{aligned}
& \max p^1 y_1^1 - p^3 K_1 - p^4 L_1 \\
& \text{subject to} \\
y_1^1 &\leq (K_1 L_1)^{1/2} \\
L_1 &\geq 0 \text{ and } K_1 \geq 0
\end{aligned}
$$

(iii) y_2^2, K_2, L_2 solves

$$
\begin{aligned}
& \max p^2 y_2^2 - p^3 K_2 - p^4 L_2 \\
& \text{subject to} \\
y_2^2 &\leq (K_2 L_2)^{1/2} \\
L_2 &\geq 0 \text{ and } K_2 \geq 0
\end{aligned}
$$

(iv) markets clear:

$$
\begin{aligned}
x^1 &\leq y_1^1 \\
x^2 &\leq y_2^2 \\
x^3 + K_1 + K_2 &\leq 1 \\
x^4 + L_1 + L_2 &\leq 1.
\end{aligned}
$$

(b) First, we know that any CE will have budget, technology, and market clearing inequalities hold as equalities, and we will have $x^3 = x^4 = 0$. Also, we know that both firms must be producing in equilibrium, in order to avoid utility of negative infinity. We will normalize $p^4 = 1$.

Firm 1 profit maximization implies the marginal rate of technical substitution condition,

$$L_1 = p^3 K_1. \tag{1}$$

Substituting the production function into the profit expression yields

$$p^1 (K_1 L_1)^{1/2} - p^3 K_1 - L_1$$

From (1), the profit expression becomes

$$
\begin{aligned}
& p^1 (K_1 p^3 K_1)^{1/2} - p^3 K_1 - p^3 K_1, \text{ or} \\
& [p^1 (p^3)^{1/2} - 2p^3] K_1,
\end{aligned}
$$

which requires (this is due to constant returns to scale) the term in brackets to be zero, or

$$p^1 = 2\sqrt{p^3}. \tag{2}$$

Similarly, firm 2 profit maximization implies the marginal rate of technical substitution condition,

$$L_2 = p^3 K_2. \tag{3}$$

Substituting the production function into the profit expression yields

$$p^2(K_2L_2)^{1/2} - p^3K_2 - L_2$$

From (3), the profit expression becomes

$$\begin{aligned}&p^2(K_2p^3K_2)^{1/2} - p^3K_2 - p^3K_2, \text{ or}\\ &[p^2(p^3)^{1/2} - 2p^3]K_2,\end{aligned}$$

which requires (this is due to constant returns to scale) the term in brackets to be zero, or

$$p^2 = 2\sqrt{p^3}. \tag{4}$$

Solving the utility maximization problem yields the marginal rate of substitution condition,

$$\frac{4x^2}{x^1} = \frac{p^1}{p^2},$$

which, along with the budget equation (imposing zero profit income), allows us to solve for the demand functions,

$$\begin{aligned}x^1 &= \frac{4(p^3+1)}{5p^1}\\ x^2 &= \frac{(p^3+1)}{5p^2}.\end{aligned}$$

Substituting (1) and (3) into the market clearing condition for good 4 yields

$$1 = L_1 + L_2 = p^3K_1 + p^3K_2 = p^3(K_1 + K_2) \tag{5}$$

Market clearing for good 3 requires $K_1 + K_2 = 1$, which when substituted into (5) implies $p^3 = 1$. By (2) and (4), it follows that $p^1 = p^2 = 2$ holds, so we have all the prices.

From the demand functions, we have $x = (\frac{4}{5}, \frac{1}{5}, 0, 0)$. Market clearing for goods 1 and 2 requires $y_1^1 = \frac{4}{5}$ and $y_2^2 = \frac{1}{5}$. Using the production function and the mrts conditions, we have $K_1 = L_1 = \frac{4}{5}$ and $K_2 = L_2 = \frac{1}{5}$.

2. (30 points)

The following pure-exchange Arrow Securities economy has $n \geq 1$ consumers, $S \geq 1$ states of nature, and $K \geq 1$ physical commodities per state of nature. Assume that endowments are strictly interior, and that each consumer is a von Neumann-Morgenstern expected utility maximizer with Bernoulli utility function that is strictly concave, continuous, and strictly monotonic. Consider arbitrary prices, $\{q^s, p(s)\}|_{s=1}^S$. Note: these prices might not be equilibrium prices. For $i = 1, ..., n$, let the security holdings $\{b_i^{*s}\}|_{s=1}^S$, and consumption $\{x_i^{*j}(s)\}|_{j=1}^K|_{s=1}^S$ solve the utility maximization problem

$$\begin{aligned} & \max_{x_i(s), b_i^s} \sum_{s=1}^{S} \pi_s u_i(x_i(s)) \\ & \text{subject to} \end{aligned}$$

$$\begin{aligned} \sum_{s=1}^{S} q^s b_i^s &\leq 0 \\ \sum_{j=1}^{K} p^j(s) x_i^j(s) &\leq \sum_{j=1}^{K} p^j(s) \omega_i^j(s) + b_i^s \text{ for all } s, \\ x_i^j(s) &\geq 0. \end{aligned}$$

For the following two statements, either prove the statement (if the statement is true) or find a counterexample (if the statement is false).

Statement 1 (15 points): For all $s = 1, ..., S$, we have

$$\sum_{i=1}^{n} \sum_{j=1}^{K} p^j(s) x_i^{*j}(s) = \sum_{i=1}^{n} \sum_{j=1}^{K} p^j(s) \omega_i^j(s).$$

Statement 2 (15 points): If we have

$$\sum_{i=1}^{n} \sum_{j=1}^{K} p^j(s) x_i^{*j}(s) = \sum_{i=1}^{n} \sum_{j=1}^{K} p^j(s) \omega_i^j(s)$$

for all $s = 1, ..., S$, then $\{q^s, p(s)\}|_{s=1}^S$ are competitive equilibrium prices.

Answer:

Statement 1 is false. Intuitively, the equation cannot hold if prices are such that all consumers want to transfer income into the same state. The easiest counterexample is for $n = 1$, $K = 1$, and $S = 2$. Let the Bernoulli utility function be $u(x) = \log(x)$, and suppose $\omega(s) = 1$ and $\pi_s = \frac{1}{2}$ for $s = 1, 2$. Set

$p(1) = p(2) = q^2 = 1$. If $q^1 = 1$ holds, then the equation in the statement is satisfied, but for any other q^1, it will not be satisfied. Suppose $q^1 = 2$ holds. It is easy to compute (I leave the details to you) that $x^*(1) < 1$ and $x^*(2) > 1$ hold, so the equation does not hold for either state.

Statement 2 is false. Intuitively, the equation implies that the security market clears, but there is no reason that the spot market must clear when we have $K > 1$. The simplest counterexample is for $n = 1$, $K = 2$, and $S = 1$. Let the utility function be $u(x^1, x^2) = \log(x^1) + \log(x^2)$, and suppose $\omega^j = 1$ for $j = 1, 2$. Since there is only one state, we must have $b^* = 0$, so the spot market constraint, which must hold with equality at a maximum due to strict monotonicity, is

$$\sum_{j=1}^{2} p^j x^{*j} = \sum_{j=1}^{2} p^j \omega^j.$$

This is exactly the equation in the statement, so we know it must hold for *any* prices. We will now find prices that are *not* equilibrium prices, establishing the counterexample. It is easy to see that when we have $p^1 = p^2 = 1$, the consumer will demand exactly his endowment, so we have a CE at those prices (the security price does not matter, since there is only one state). Any other relative price is inconsistent with equilibrium. For example, set $p^1 = 2$ and $q = p^2 = 1$. It is easy to compute (I leave the details to you) that $x^{*2} > 1$ holds, so there is excess demand for good 2.

3. (35 points)

In the following Rothschild-Stiglitz model, there is one consumer, who has initial wealth of 2 and a potential accident with damages of 2, so the state-contingent initial endowment is $(2, 0)$. The consumer has a von Neumann-Morgenstern utility function with a "Bernoulli" utility of certain consumption given by $u(W) = \log(W)$. The consumer knows her risk type, but the competing insurance firms believe that she is high risk with probability $\frac{1}{2}$ and low risk with probability $\frac{1}{2}$. A high risk type has an accident probability, $p^H = \frac{1}{2}$, and a low risk type has an accident probability, $p^L = \frac{1}{3}$.

(a) (20 points) Compute the "candidate" separating equilibrium.

Now suppose instead that the consumer (call her consumer 1) can trade on a contingent commodities market before she learns her risk type or whether she will have an accident. In place of the competing insurance companies, suppose there is a single risk-neutral consumer (call him consumer 2) with an endowment of 2 in all states.

(b) (15 points) Compute the competitive equilibrium price vector and allocation.

Answer:

(a) The R-S model applies to the case of one consumer with unknown type just as well as it does to the case of a population of consumers. The contract offered to the high risk type is at the intersection of the fair odds line and the 45 degree line. The fair odds line for the high risk type has slope -1, so the contract is $\alpha^H = (W_1^H, W_2^H) = (1, 1)$.

The contract offered to the low risk type is at the intersection of the type-H indifference curve through $(1, 1)$ and the fair odds line for the low risk type. The equation of the indifference curve is

$$\begin{aligned} \frac{1}{2}\log(W_1) + \frac{1}{2}\log(W_2) &= \frac{1}{2}\log(1) + \frac{1}{2}\log(1), \text{ or} \\ W_1 W_2 &= 1. \end{aligned}$$

The equation of the fair odds line is

$$\begin{aligned} \frac{2}{3}W_1 + \frac{1}{3}W_2 &= \frac{2}{3} \times 2, \text{ or} \\ 2W_1 + W_2 &= 4. \end{aligned}$$

Substitution yields the equation

$$2(W_1)^2 - 4W_1 + 1 = 0$$

with solutions,

$$W_1 = 1 \pm \frac{\sqrt{2}}{2}.$$

Since the correct root is the one below the 45 degree line, we have the contract $\alpha^L = (1 + \frac{\sqrt{2}}{2}, 2 - \sqrt{2})$.

(b) It is an interesting philosophical question, whether there are 2 states (no accident or accident) or 4 states (no accident-H, no accident-L, accident-H, or accident-L). Does it matter that consumer 1 will learn something after the trading occurs but before delivery? In either case, consumer 2 is risk neutral with enough of an endowment to fully insure consumer 1, so the CE prices will be proportional to the probabilities and consumer 1 will consume the same in each state. Basically, the slope of consumer 2's linear indifference curves determines prices, and consumer 1 faces fair odds before she learns anything about the state.

In the two state case, the probability of an accident is

$$\frac{1}{2}\frac{1}{2} + \frac{1}{2}\frac{1}{3} = \frac{5}{12}.$$

Letting state 1 be the no-accident state, the CE price vector is $(\frac{7}{12}, \frac{5}{12})$. Consumer 1's constant consumption equals her expected endowment income, which is $\frac{7}{12} \times 2 = \frac{7}{6}$. Therefore, the CE allocation is

$$x_1 = (\frac{7}{6}, \frac{7}{6}) \text{ and } x_2 = (\frac{17}{6}, \frac{5}{6}).$$

If you set it up with 4 states, the probabilities are $(\frac{1}{4}, \frac{1}{3}, \frac{1}{4}, \frac{1}{6})$, so this is the CE price vector. The CE allocation is

$$x_1 = (\frac{7}{6}, \frac{7}{6}, \frac{7}{6}, \frac{7}{6}) \text{ and } x_2 = (\frac{17}{6}, \frac{17}{6}, \frac{5}{6}, \frac{5}{6}).$$