Department of Economics
The Ohio State University
Final Exam Questions and Answers–Econ 8712

Prof. Peck
Fall 2019

1. (30 points)

Consider a pure exchange economy with $n \geq 3$ consumers and $K \geq 2$ commodities. For $i = 1, ..., n$, denote the non-negative consumption vector of consumer i as x_i and the utility function (defined over all non-negative consumption vectors) as $u_i(x_i)$. Prove the following statement:

For any strongly Pareto optimal allocation, x^*, there exists a consumer, h, who prefers her consumption to any other consumer's consumption. That is, we have

$$u_h(x_h^*) \geq u_h(x_i^*) \quad \textit{for all } i = 1, ..., n.$$

(Full credit for proving this without any additional assumptions on utility functions or the number of consumers. Partial credit if you need to make additional assumptions.)

Answer:

Suppose that the conclusion is false, so that for any consumer, h, there exists another consumer whose bundle consumer h strictly prefers. Start with an arbitrary consumer, who we label as $h1$. Consumer $h1$ strictly prefers the bundle of another consumer, who we will label as $h2$. Consumer $h2$ strictly prefers the bundle of another consumer. If this consumer is $h1$, we have a contradiction to strong Pareto optimality, since the allocation where $h1$ and $h2$ exchange bundles dominates x^*. Otherwise, consumer $h2$ strictly prefers the bundle of another consumer, who we label as $h3$.

We proceed in this fashion, looking at the consumer whose bundle consumer $h3$ strictly prefers, and so on, until eventually we have a consumer, hk, who strictly prefers the bundle of one of the consumers we considered earlier, hj, for $j < k$. This "cycle" must eventually occur. Then consider the allocation in which consumer hj receives the bundle of $hj + 1$, and so on until consumer hk receives the bundle of hj. Those consumers i not in the cycle receive x_i^*. This allocation Pareto dominates x^*, a contradiction.

2. (35 points)

The following economy has two consumers, two firms, and two goods. Good 2 is leisure/labor. For $i = 1, 2$, consumer i has the utility function,

$$u(x_i^1, x_i^2) = \log(x_i^1) + \log(x_i^2).$$

Consumer 1 has no endowments of goods, $\omega_1 = (0, 0)$, but owns both firms. Consumer 2 has the initial endowment vector, $\omega_2 = (0, 1)$.

Firm 1 produces good 1 using good 2 as an input. For convenience, denote the (nonnegative) labor input used by firm 1 as L_1. Firm 1's production function (the boundary of the production set) is given by:

$$\begin{aligned} y_1^1 &= (\frac{1}{3}L_1)^{1/2}, \\ \text{where } L_1 &\geq 0. \end{aligned}$$

Firm 2 also produces good 1 using good 2 as an input. For convenience, denote the (non-negative) labor input used by firm 2 as L_2. Firm 2's production function (the boundary of the production set) is given by:

$$\begin{aligned} y_2^1 &= \frac{1}{4}L_2, \\ \text{where } L_2 &\geq 0. \end{aligned}$$

(a) (10 points) Define a competitive equilibrium for this economy.
(b) (25 points) Calculate the competitive equilibrium for this economy.

Answer:

(a) A CE is a price vector (p^1, p^2) and an allocation $(x_1^1, x_1^2, x_2^1, x_2^2, y_1^1, L_1, y_2^1, L_2)$ satisfying

(i) (x_1^1, x_1^2) solves

$$\begin{aligned} & \max \log(x_1^1) + \log(x_1^2) \\ & \text{subject to} \\ p^1 x_1^1 + p^2 x_1^2 &\leq \pi_1 + \pi_2 \\ x_1 &\geq 0 \end{aligned}$$

(ii) (x_2^1, x_2^2) solves

$$\begin{aligned} & \max \log(x_2^1) + \log(x_2^2) \\ & \text{subject to} \\ p^1 x_2^1 + p^2 x_2^2 &\leq p^2 \\ x_2 &\geq 0 \end{aligned}$$

(iii) (y_1^1, L_1) solves

$$\begin{aligned} \max \pi_1 &= p^1 y_1^1 - p^2 L_1 \\ & \text{subject to} \\ y_1^1 &\leq \sqrt{\frac{1}{3}L_1} \\ L_1 &\geq 0 \end{aligned}$$

(iv) (y_2^1, L_2) solves

$$\begin{aligned} \max \pi_2 &= p^1 y_2^1 - p^2 L_2 \\ & \text{subject to} \\ y_2^1 &\leq \frac{L_2}{4} \\ L_2 &\geq 0 \end{aligned}$$

(v) markets clear:

$$\begin{aligned} x_1^1 + x_2^1 &\leq y_1^1 + y_2^1 \\ x_1^2 + x_2^2 + L_1 + L_2 &\leq 1. \end{aligned}$$

(b) Normalize the price of good 2 to be 1 and denote the price of good 1 as p. Let us start with the profit maximization problems.

For firm 1, we can substitute the constraint into the objective, and derive the first order condition,

$$p \cdot \frac{1}{2}(\frac{1}{3}L_1)^{-1/2} \cdot \frac{1}{3} - 1 = 0,$$

from which we derive the supply function and profit (skipping some work you should show),

$$\begin{aligned} L_1 &= \frac{p^2}{12}, \\ y_1^1 &= \frac{p}{6}, \\ \pi_1 &= \frac{p^2}{12}. \end{aligned}$$

For firm 2, whose production function exhibits constant returns to scale, substituting the constraint into the objective we have profit as a function of L_2 given by $p\frac{L_2}{4} - L_2$, so this firm produces (finite) positive output if and only if $p = 4$. However, at that price, firm 1 would demand more than all of the economy's endowment of good 2, which is inconsistent with equilibrium. Therefore, firm 2 does does not produce in equilibrium.

The utility maximizing demands for consumer 1 satisfy the first order conditions,

$$\begin{aligned} \frac{x_1^2}{x_1^1} &= p, \\ p x_1^1 + x_1^2 &= \frac{p^2}{12}, \end{aligned}$$

which (skipping some work you should show) yield the demand functions,

$$\begin{aligned} x_1^1 &= \frac{p}{24}, \\ x_1^2 &= \frac{p^2}{24}. \end{aligned}$$

Similarly solving consumer 2's utility maximization problem yields the demand functions

$$\begin{aligned} x_2^1 &= \frac{1}{2p}, \\ x_2^2 &= \frac{1}{2}. \end{aligned}$$

Market clearing for good 2 requires

$$\begin{aligned} \frac{p^2}{24} + \frac{1}{2} + \frac{p^2}{12} &= 1, \\ p &= 2. \end{aligned}$$

Therefore, the allocation is

$$\begin{aligned} x_1^1 &= \frac{1}{12}, x_1^2 = \frac{1}{6}, x_2^1 = \frac{1}{4}, x_2^2 = \frac{1}{2}, \\ y_1^1 &= \frac{1}{3}, L_1 = \frac{1}{3}, y_2^1 = 0, L_2 = 0. \end{aligned}$$

3. (35 points)

The following economy has 2 consumers, 2 states of nature, and one physical commodity per state. The probability of state 1 is one-third, $\pi_1 = \frac{1}{3}$ and the probability of state 2 is two-thirds, $\pi_2 = \frac{2}{3}$. For $i = 1, 2$, consumer i is an expected utility maximizer, with a Bernoulli utility function given by $u_i(x_i) = \log(x_i)$. Consumer 1's initial endowment vector is $(\omega_1(1), \omega_1(2)) = (1, 0)$ and consumer 2's initial endowment vector is $(\omega_2(1), \omega_2(2)) = (0, 1)$.

The market structure consists of a securities market on which three securities are traded before the state is realized, and then a spot market on which physical commodities are traded after the state is realized and securities are redeemed. Security 1 is a standard Arrow security that pays 1 unit of account on the state-1 spot market, security 2 is a standard Arrow security that pays 1 unit of account on the state-2 spot market, and security 3 pays 1 unit of account on the state-1 spot market **and** one unit of account on the state-2 spot market. Normalize prices so that $p(1) = p(2) = q^3 = 1$ holds.

(a) (10 points) Define a competitive equilibrium for this economy.

(b) (15 points) Calculate a competitive equilibrium in which consumer 1's equilibrium holding of security 3 is zero, $b_1^3 = 0$.

(c) (10 points) Calculate a competitive equilibrium in which consumer 1's equilibrium holding of security 3 is one, $b_1^3 = 1$.

Answer:

(a) A competitive equilibrium is a (normalized) price vector, $(q^1, q^2, 1, 1, 1)$, and an allocation, $(x_1(1), x_1(2), x_2(1), x_2(2), b_1^1, b_1^2, b_1^3, b_2^1, b_2^2, b_2^3)$, such that

(i) $x_1(1), x_1(2), b_1^1, b_1^2, b_1^3$ solves

$$\max \frac{1}{3}\log(x_1(1)) + \frac{2}{3}\log(x_1(2))$$

subject to

$$\begin{aligned} q^1 b_1^1 + q^2 b_1^2 + b_1^3 &= 0 \\ x_1(1) &= 1 + b_1^1 + b_1^3 \\ x_1(2) &= b_1^2 + b_1^3 \\ x_1 &\geq 0. \end{aligned}$$

(ii) $x_2(1), x_2(2), b_2^1, b_2^2, b_2^3$ solves

$$\max \frac{1}{3}\log(x_2(1)) + \frac{2}{3}\log(x_2(2))$$

subject to

$$\begin{aligned} q^1 b_2^1 + q^2 b_2^2 + b_2^3 &= 0 \\ x_2(1) &= b_2^1 + b_2^3 \\ x_2(2) &= 1 + b_2^2 + b_2^3 \\ x_2 &\geq 0. \end{aligned}$$

(iii) markets clear:

$$\begin{aligned} x_1(1) + x_2(1) &= 1 \\ x_1(2) + x_2(2) &= 1 \\ b_1^1 + b_2^1 &= 0 \\ b_1^2 + b_2^2 &= 0 \\ b_1^3 + b_2^3 &= 0. \end{aligned}$$

Note: equalities are due to the monotonicity of utility functions.

(b) For consumer 1, impose $b_1^3 = 0$ and substitute the spot market budget constraints into the securities market budget constraint, yielding

$$q^1(x_1(1) - 1) + q^2 x_1(2) = 0.$$

Solving the Lagrangean gives this constraint and the marginal rate of substitution condition,

$$x_1(2) = \frac{2q^1 x_1(1)}{q^2}.$$

This yields the demand functions,

$$\begin{aligned} x_1(1) &= \frac{1}{3} \\ x_1(2) &= \frac{2q^1}{3q^2}. \end{aligned}$$

Going through the same steps for consumer 2, we substitute the spot market constraints into the securities market constraint, yielding

$$q^1 x_2(1) + q^2(x_2(2) - 1) = 0.$$

The marginal rate of substitution condition is

$$x_2(2) = \frac{2q^1 x_2(1)}{q^2}.$$

Solving, we get the demand functions,

$$\begin{aligned} x_2(1) &= \frac{q^2}{3q^1} \\ x_2(2) &= \frac{2}{3}. \end{aligned}$$

Using market clearing, we get the relative securities price, $\frac{q^2}{q^1} = 2$, and the allocation,

$$\begin{aligned} x_1 &= (\frac{1}{3}, \frac{1}{3}), b_1 = (-\frac{2}{3}, \frac{1}{3}, 0) \\ x_2 &= (\frac{2}{3}, \frac{2}{3}), b_1 = (\frac{2}{3}, -\frac{1}{3}, 0). \end{aligned}$$

Note that we have already used all of our normalizations. To find q^1 and q^2, we use the no-arbitrage condition, $q^1 + q^2 = 1$. Otherwise, consumers could either buy securities 1 and 2 and sell security 3, or sell securities 1 and 2 and buy security 3. Therefore, $q^1 = \frac{1}{3}$ and $q^2 = \frac{2}{3}$.

(c) Because the securities return matix has rank 2, this market structure is equivalent to complete markets, and all consumption allocations are given by

$$\begin{aligned} x_1 &= (\frac{1}{3}, \frac{1}{3}) \\ x_2 &= (\frac{2}{3}, \frac{2}{3}). \end{aligned}$$

To get the securities holdings, substitute the consumptions and $b_1^3 = 1$ into consumer 1's spot market budget constraints, yielding $b_1 = (-\frac{5}{3}, -\frac{2}{3}, 1)$. Market clearing then requires $b_2 = (\frac{5}{3}, \frac{2}{3}, -1)$. The equilibrium securities prices are the same as before, $q^1 = \frac{1}{3}$ and $q^2 = \frac{2}{3}$.