Department of Economics
The Ohio State University
Final Exam Answers–Econ 8712

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Fall 2017

1. (35 points)

The following economy has three consumers, one firm, and four goods. Good 1 is the labor/leisure of consumer 1, good 2 is the labor/leisure of consumer 2, good 3 is the labor/leisure of consumer 3, and good 4 is a consumption good produced by the firm. For $i = 1, 2, 3$, consumer i receives utility from good 4 and his/her own leisure, according to the utility function,

$$\log(x_i^i) + \log(x_i^4).$$

It is not feasible to consume someone else's leisure, negative amounts of one's own leisure, or more than one's own endowment of leisure. Endowments are given by $\omega_1 = (1, 0, 0, 0)$, $\omega_2 = (0, 1, 0, 0)$, and $\omega_3 = (0, 0, 1, 0)$. The firm is owned by consumer 1.

The firm produces good 4 using goods 1-3 as inputs. For convenience, denote the (nonnegative) inputs as L^1, L^2, and L^3. Inputs are perfect substitutes, so the firm's production function (the boundary of the production set) is given by:

$$\begin{aligned} y^4 &= (L^1 + L^2 + L^3)^{1/2}, \\ &\quad \text{where } L^1, L^2, \text{ and } L^3 \text{ are nonnegative.} \end{aligned}$$

(a) (10 points) Define a competitive equilibrium for this economy.

(b) (25 points) Compute the competitive equilibrium price vector and allocation.

Answer:

(a) A competitive equilibrium (imposing one cannot consume someone else's leisure) is a price vector, (p^1, p^2, p^3, p^4), and an allocation, $(x_1^1, x_1^4, x_2^2, x_2^4, x_3^3, x_3^4, L^1, L^2, L^3, y^4)$, such that

(i) (x_1^1, x_1^4) solves

$$\begin{aligned} & \max \log(x_1^1) + \log(x_1^4) \\ & \text{subject to} \\ p^1 x_1^1 + p^4 x_1^4 &\leq p^1 + \pi, \\ x_1 &\geq 0, \\ x_1^1 &\leq 1, \end{aligned}$$

(ii) (x_2^2, x_2^4) solves

$$\begin{aligned}
& \max \log(x_2^2) + \log(x_2^4) \\
& \text{subject to} \\
p^2 x_2^2 + p^4 x_2^4 &\leq p^2, \\
x_2 &\geq 0, \\
x_2^2 &\leq 1,
\end{aligned}$$

(iii) (x_3^3, x_3^4) solves

$$\begin{aligned}
& \max \log(x_3^3) + \log(x_3^4) \\
& \text{subject to} \\
p^3 x_3^3 + p^4 x_3^4 &\leq p^3, \\
x_3 &\geq 0, \\
x_3^3 &\leq 1,
\end{aligned}$$

(iv) (L^1, L^2, L^3, y^4) solves

$$\begin{aligned}
\max \pi &\equiv p^4 y^4 - p^1 L^1 - p^2 L^2 - p^3 L^3 \\
& \text{subject to} \\
y^4 &\leq (L^1 + L^2 + L^3)^{1/2},
\end{aligned}$$

(v) markets clear:

$$\begin{aligned}
x_1^1 + L^1 &\leq 1, \\
x_2^2 + L^2 &\leq 1, \\
x_3^3 + L^3 &\leq 1, \\
x_1^4 + x_2^4 + x_3^4 &\leq y^4.
\end{aligned}$$

(b) Normalize $p^4 = 1$. We know that consumers 2 and 3 must be supplying labor in the CE, or else their utility is negative infinity. So either we have an equilibrium in which all interior first order conditions are satisfied, in which case due to perfect substitutes we must have $p^1 = p^2 = p^3 \equiv p$, or we have a corner solution in which consumer 1 strictly prefers not to supply labor and $p^1 > p^2 = p^3$ holds. We will assume, and then verify, that the equilibrium is of the former type.

Let us solve the firm's problem, denoting $L \equiv L^1 + L^2 + L^3$. We can express profit as

$$\pi = L^{1/2} - pL.$$

The first order condition is

$$\begin{aligned}
\frac{1}{2} L^{-1/2} &= p, \text{ or} \\
L &= \frac{1}{4(p)^2}.
\end{aligned}$$

This gives

$$\begin{aligned} y^4 &= \frac{1}{2p} \\ \pi &= \frac{1}{4p}. \end{aligned}$$

The solution to consumer 1's problem is given by the first order conditions,

$$\begin{aligned} \frac{x_1^4}{x_1^1} &= p \\ px_1^1 + x_1^4 &= p + \frac{1}{4p} \end{aligned}$$

yielding the demand functions,

$$\begin{aligned} x_1^1 &= \frac{1}{2} + \frac{1}{8(p)^2} \\ x_1^4 &= \frac{p}{2} + \frac{1}{8p}. \end{aligned}$$

Similar calculations (without the profit terms) yields the demand functions of consumers 2 and 3,

$$\begin{aligned} x_2^2 &= x_3^3 = \frac{1}{2} \\ x_2^4 &= x_3^4 = \frac{p}{2}. \end{aligned}$$

To find the value of p, use market clearing for good 4.

$$\begin{aligned} \frac{p}{2} + \frac{1}{8p} + \frac{p}{2} + \frac{p}{2} &= \frac{1}{2p}, \\ \frac{3p}{2} &= \frac{3}{8p}, \\ p &= \frac{1}{2}. \end{aligned}$$

Therefore, the C.E. price vector is $(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, 1)$. The consumptions are given by

$$\begin{aligned} x_1^1 &= 1, x_1^4 = \frac{1}{2}, \\ x_2^2 &= x_3^3 = \frac{1}{2}, x_2^4 = x_3^4 = \frac{1}{4}. \end{aligned}$$

From firm's supply and labor demand, we have

$$y^4 = 1, L = 1.$$

From the market clearing conditions and the consumers' leisure allocation, we have $L^1 = 0$, and $L^2 = L^3 = \frac{1}{2}$. Although consumer 1 consumes his/her entire leisure endowment, this is the unique equilibrium. If the price of consumer 1's leisure/labor is higher, there would be excess supply of good 1.

2. (35 points)

The following Arrow Securities economy has 2 consumers, 2 states of nature, and one physical commodity per state. For $s = 1, 2$, denote the probability of state s by π_s. For $i = 1, 2$, consumer i is an expected utility maximizer, with a strictly monotonic, strictly concave, and differentiable "Bernoulli" utility function given by $u_i(x_i)$. Consumer 1's initial endowment vector is $(\omega_1(1), \omega_1(2)) = (3, 2)$ and consumer 2's intitial endowment vector is $(\omega_2(1), \omega_2(2)) = (1, 2)$. Normalize prices so that $p(1) = p(2) = q^2 = 1$ holds.

Important Note: This economy has no aggregate uncertainty.

(a) (10 points) Define a competitive equilibrium for this economy.

(b) (25 points) Compute the competitive equilibrium price vector and allocation, and explain your reasoning. You can refer to results presented in class without proving them here.

Answer:

(a) A C.E. is a set of prices, $(q^1 q^2, p(1), p(2))$, and an allocation, $(b_1^1, b_1^2, b_2^1, b_2^2)$, and consumption $(x_1(1), x_1(2), x_2(1), x_2(2))$ such that

(i) $(x_1(1), x_1(2), b_1^1, b_1^2)$ solves the utility maximization problem

$$\max_{x_1(s), b_1^s} \sum_{s=1}^{2} \pi_s u_1(x_1(s))$$

subject to

$$\begin{aligned} \sum_{s=1}^{2} q^s b_1^s &\leq 0 \\ p(1)x_1(1) &\leq 3p(1) + b_1^1, \\ p(2)x_1(2) &\leq 2p(2) + b_1^2, \\ x_1(s) &\geq 0, \end{aligned}$$

(ii) $(x_2(1), x_2(2), b_2^1, b_2^2)$ solves the utility maximization problem

$$\max_{x_2(s), b_2^s} \sum_{s=1}^{2} \pi_s u_2(x_2(s))$$

subject to

$$\begin{aligned}
\sum_{s=1}^{2} q^s b_2^s &\leq 0 \\
p(1)x_2(1) &\leq p(1) + b_2^1, \\
p(2)x_2(2) &\leq 2p(2) + b_2^2, \\
x_2(s) &\geq 0,
\end{aligned}$$

(iii) markets clear,

$$\begin{aligned}
b_1^1 + b_2^1 &\leq 0 \\
b_1^2 + b_2^2 &\leq 0 \\
x_1(1) + x_2(1) &\leq 4 \\
x_1(2) + x_2(2) &\leq 4
\end{aligned}$$

(b) Because there is no aggregate uncertainty, we know what the C.E. allocation would be in the contingent commodities model. Consumption in state 1 equals consumption in state 2, and equals the expected endowment consumption (since prices are proportional to probabilities). Because the consumption in the Arrow securities model is the same as in the contingent commodities model by Arrow's theorem, we have

$$\begin{aligned}
x_1(1) &= x_1(2) = 3\pi_1 + 2\pi_2 = \pi_1 + 2 \\
x_2(1) &= x_2(2) = \pi_1 + 2\pi_2 = 2 - \pi_1.
\end{aligned}$$

Normalizing $p(1) = p(2) = q^2 = 1$, and substituting the consumptions above into the the spot market budget constraints, we have

$$\begin{aligned}
b_1^1 &= -\pi_2, b_1^2 = \pi_1, \\
b_2^1 &= \pi_2, b_2^2 = -\pi_1.
\end{aligned}$$

Substituting the security holdings of consumer 1 into his/her securities market constraint, we have

$$\begin{aligned}
q^1(-\pi_2) + \pi_1 &= 0, \\
q^1 &= \frac{\pi_1}{\pi_2}.
\end{aligned}$$

3. (30 points)

In the following Rothschild-Stiglitz economy, initial wealth and damages are both equal to 1, so the initial endowment in consumption space is given by $E = (1, 0)$. The consumers are expected utility maximizers with Bernoulli utility function $u(w) = \log(w)$. The high risk type has accident probability $p^H = \frac{3}{4}$, and the low risk type has accident probability $p^L = \frac{1}{4}$. There is an equal number of each risk type, $\lambda = \frac{1}{2}$.

(a) (10 points) Find the contract, $\alpha = (W_1, W_2)$, that is on the pooled fair odds line and also on the 45 degree line.

(b) (20 points) If the contract α from part (a) is the only other contract offered, find a new contract, β, that could be profitably offered, where the low risk types purchase β and the high risk types purchase α.

Answer:

(a) The accident probability for a person taken at random from the population is given by

$$\overline{p} = \frac{1}{2}p^H + \frac{1}{2}p^L = \frac{1}{2}\frac{3}{4} + \frac{1}{2}\frac{1}{4} = \frac{1}{2}.$$

Therefore, the equation of the pooled fair odds line is

$$\begin{aligned} \frac{1}{2}W_1 + \frac{1}{2}W_2 &= \frac{1}{2}1 + \frac{1}{2}0, \text{ or} \\ W_1 + W_2 &= 1. \end{aligned}$$

The point on the pooled fair odds line that is also on the 45 degree line, where $W_1 = W_2$ holds, is therefore $\alpha = (\frac{1}{2}, \frac{1}{2})$.

(b) On the 45 degree line, the slope (in absolute value) of the high risk indifference curves is $\frac{1-p^H}{p^H}$, which equals $\frac{1}{3}$, and the slope of the low risk indifference curves is $\frac{1-p^L}{p^L}$, which equals 3. Therefore, at the contract α, the slope of the pooled fair odds line (in absolute value) is 1, the high risk indifference curve through α is flatter, and the low risk indifference curve through α is steeper. Thus, a contract on the pooled fair odds line that is very close to α, but slightly closer to the endowment, satisfies the conditions of part (b). That is, it is on a lower indifference curve for the high risks, it is on a higher indifference curve for the low risks, and it is highly profitable because only the low risks will accept the contract. For example, $(0.501, 0.499)$ will work.

Of course, there are many answers, and I wrote a simple program to determine if your answer was correct. Also, a diagram will help illustrate your answer, but if you did not give an actual contract, you are relying on your diagram being drawn accurately enough to have all of the right features and also be readable.