Partial Equilibrium

As a prelude to general equilibrium, we will put together demand and supply in the context of a single market.

To contain the analysis to a single market, we envision this market as a small part of the overall economy with no close substitutes or complements:

1. Because the good is a small part of a consumer's budget, income effects will be negligible.

2. Changes in this market will not affect or be affected by prices in other markets. Therefore, we can consider the bundle of all other goods to be a single "composite" commodity, which we call the numeraire.

The price of the good will be in terms of the numeraire. (The price of the numeraire is defined to be one.)

Consumers

Based on the preceding motivation, we consider a two-good model. For $i = 1, ..., n$, x_i is consumer i's consumption of the good, and m_i is her consumption of the numeraire.

Utility is quasi-linear, which we will see reflects no income effects:

$$u_i(m_i, x_i) = m_i + \phi_i(x_i)$$

We will assume that consumers will not be at a corner solution where they spend all of their income on the good. No nonnegativity constraints on m_i.

Assume $\phi_i(0) = 0$, $\phi_i'(x_i) > 0$, and $\phi_i''(x_i) < 0$.

Firms

For $f = 1, ..., F$, firm f uses the numeraire as an input to produce the good as an output. The minimum input required to produce y_f units of output is $c_f(y_f)$. Since the price of the numeraire is one, $c_f(y_f)$ is the cost function.

Note: alternatively, we can write the firm's technology, using the convention that inputs are negative outputs, as follows:

$$Y_f = \{(-z_f, y_f) : y_f \geq 0 \text{ and } z_f \geq c_f(y_f)\}.$$

Assume that the cost function satisfies $c'_f(y_f) > 0$ and $c''_f(y_f) \geq 0$.

Competitive Equilibrium

In a competitive equilibrium, incomes are determined by the ownership of resources, and cannot be taken as exogenous.

We assume that consumer i has a zero endowment of the good, an endowment of money given by ω_i^m, and a claim on firm f's profits equal to T_{if}.

Def. A competitive equilibrium for the partial equilibrium economy is a price, p^*, and an allocation, $(m_i^*, x_i^*)_{i=1,\ldots,n}$, $(y_f^*)_{f=1,\ldots,F}$, such that:

(i) utility maximization: for each i, (m_i^*, x_i^*) solves

$$\max_{m_i, x_i} m_i + \phi_i(x_i)$$

$$\text{subject to}$$

$$p^* x_i + m_i \leq \omega_i^m + \sum_{f=1}^{F} T_{if}(p^* y_f^* - c_f(y_f^*)),$$

(ii) profit maximization: for each f, y_f^* solves

$$\max_{y_f \geq 0} p^* y_f - c_f(y_f),$$

(iii) markets clear:

$$\begin{aligned} \sum_{i=1}^{n} x_i^* &= \sum_{f=1}^{F} y_f^* \\ \sum_{i=1}^{n} m_i^* + \sum_{f=1}^{F} c_f(y_f^*) &= \sum_{i=1}^{n} \omega_i^m. \end{aligned}$$

Suppose we have an interior solution in which all firms are producing and all consumers are purchasing the good.

The first order conditions (necessary and sufficient) are that for all i and f:

$$\phi_i'(x_i^*) = p^* = c_f'(y_f^*).$$

Firms choose output to equate the marginal cost to the price.

Conumers equate marginal utility of the good to the price. The quasi-linear utility specification ensures that demand depends only on the price of the good (relative to money) and not on income.

The downward sloping function $\phi_i'(x_i)$ is the inverse demand function for consumer i.

The Walrasian demand function for consumer i is given by $x_i(p) = [\phi_i']^{-1}(p)$.

The aggregate demand function is given by $x(p) = \sum_{i=1}^{n} x_i(p)$.

Similarly, the inverse supply function for firm f is the upward sloping function $c_f'(y_f)$, the supply function for firm f is given by $y_f(p) = [c_f']^{-1}(p)$, and the aggregate supply function is given by $y(p) = \sum_{f=1}^{F} y_f(p)$.

We can graphically find the equilibrium price by the unique intersection of supply and demand. (Price is the dependent variable, so add horizontally.)

If production exhibits constant returns to scale, $c''_f(y_f) = 0$, and the supply curve is flat at the price equal to the constant marginal cost.

Under constant returns to scale, the supply curve determines the equilibrium price, and demand determines the aggregate quantity. Individual firm output is indeterminate.

A First Pass at Welfare Economics

Def. An allocation for the partial equilibrium economy, $(m_i, x_i)_{i=1,\ldots,n}$, $(y_f)_{f=1,\ldots,F}$, is feasible if we have

$$\begin{aligned} \sum_{i=1}^{n} x_i &\leq \sum_{f=1}^{F} y_f \\ \sum_{i=1}^{n} m_i + \sum_{f=1}^{F} c_f(y_f) &\leq \sum_{i=1}^{n} \omega_i^m. \end{aligned}$$

Def. An allocation for the partial equilibrium economy, $(m_i, x_i)_{i=1,\ldots,n}$, $(y_f)_{f=1,\ldots,F}$, is Pareto optimal if there is no other feasible allocation, $(m_i', x_i')_{i=1,\ldots,n}$, $(y_f')_{f=1,\ldots,F}$, such that $u_i(m_i', x_i') \geq u_i(m_i, x_i)$ for all i, and $u_h(m_h', x_h') > u_h(m_h, x_h)$ for some h.

First Fundamental Theorem of Welfare Economics: If the price, p^*, and the allocation, $(m_i^*, x_i^*)_{i=1,\ldots,n}$, $(y_f^*)_{f=1,\ldots,F}$, are a CE, then the allocation, $(m_i^*, x_i^*)_{i=1,\ldots,n}$, $(y_f^*)_{f=1,\ldots,F}$, is Pareto optimal.

Proof sketch. Any allocation that maximizes the sum of utilities across feasible allocations is Pareto optimal. (Why?)

$$\max \sum_{i=1}^{n} (m_i + \phi_i(x_i))$$

subject to

$$\begin{aligned} \sum_{i=1}^{n} x_i &= \sum_{f=1}^{F} y_f \\ \sum_{i=1}^{n} m_i + \sum_{f=1}^{F} c_f(y_f) &= \sum_{i=1}^{n} \omega_i^m \end{aligned}$$

By substituting the constraint into the objective, we have the equivalent problem

$$\max_{x_i, y_f} \sum_{i=1}^{n} \omega_i^m - \sum_{f=1}^{F} c_f(y_f) + \sum_{i=1}^{n} \phi_i(x_i)$$

subject to

$$\sum_{i=1}^{n} x_i = \sum_{f=1}^{F} y_f$$

What matters is production and consumption of the good–the distribution of numeraire left over does not affect the sum of utilities.

The necessary and **sufficient** first order conditions are

$$\begin{aligned} \phi_i'(x_i) &= \lambda \\ -c_f'(y_f) &= -\lambda \end{aligned}$$

with solution $x_i = x_i^*$, $y_f = y_f^*$, and $\lambda = p^*$. ■

Second Fundamental Theorem of Welfare Economics: For any Pareto optimal allocation, $(m_i, x_i)_{i=1,\ldots,n}$, and $(y_f)_{f=1,\ldots,F}$, we can (re)distribute the aggregate numeraire endowment $\sum_{i=1}^n \omega_i^m$ as individual endowments such that the Pareto optimal allocation is a competitive equilibrium allocation (for some price p^*).

Proof sketch. If the allocation is Pareto optimal, it must maximize the sum of utilities subject to feasibility. [If some other feasible allocation, (m', x', y'), gave a higher sum of utilities, then we can Pareto dominate the allocation with (m'', x', y') for some m'' satisfying $\sum_{i=1}^n m_i'' = \sum_{i=1}^n m_i'$.]

Thus, the P.O. allocation satisfies the necessary conditions: for some λ, we have $\phi_i'(x_i) = \lambda = c_f'(y_f)$ for all i, f. Then set $p^* = \lambda$ and choose the numeraire endowments so that each consumer's budget constraint is satisfied. The sufficient conditions for a CE are satisfied.

Define the *Marshallian surplus* to be the total increase in utility due to production minus the cost of production,

$$S = \sum_{i=1}^{n} \phi_i(x_i) - \sum_{f=1}^{F} c_f(y_f).$$

Imagine society or a planner having a utility function over the profile of utilities received by the consumers, $W(u_1, ..., u_n)$, which is strictly increasing in each component utility. Then no matter what this "social welfare function" is, the allocation that maximizes social welfare also chooses production and allocates output to maximize the Marshallian surplus.

Why? For a given $(x_i)_{i=1,...,n}$, $(y_f)_{f=1,...,F}$, the set of feasible utility profiles is given by

$$\{(u_1, ..., u_n) : \sum_{i=1}^{n} u_i \leq \sum_{i=1}^{n} \omega_i^m + \sum_{i=1}^{n} \phi_i(x_i) - \sum_{f=1}^{F} c_f(y_f)\},$$

because the money to be allocated to consumers is the aggregate money endowment minus the money used as input to production.

Therefore, any social welfare function will choose $(x_i)_{i=1,...,n}$, $(y_f)_{f=1,...,F}$ to maximize the right side of the above inequality.

Different social welfare functions will allocate the numeraire differently, but all will choose $x_i = x_i^*$ and $y_f = y_f^*$ for all i, f.

How does the surplus depend on the total output, x, when output is produced optimally and optimally distributed to consumers?

Start with an arbitrary quantity that is efficiently produced and distributed: $(x_i)_{i=1,\dots,n}$, $(y_f)_{f=1,\dots,F}$. Now consider an infinitesimal increase $dx = \sum_{i=1}^{n} dx_i = \sum_{f=1}^{F} dy_f$.

Then we have

$$dS = \sum_{i=1}^{n} \phi_i'(x_i) dx_i - \sum_{f=1}^{F} c_f'(y_f) dy_f. \qquad (1)$$

Because x is produced and distributed efficiently, we have $\phi_i'(x_i) = P(x)$ and $c_f'(y_f) = C'(x)$, where $P(x)$ is the market inverse demand function and $C'(y)$ is the market inverse supply function.

Thus, (1) becomes $dS = [P(x) - C'(x)]dx$.

When we integrate $dS = [P(x) - C'(x)]dx$, we get the following characterization of the Marshallian surplus as a function of x:

$$S(x) = S_0 + \int_0^x [P(s) - C'(s)]ds.$$

The constant term, S_0, is the surplus associated with zero output, which equals zero if there are no fixed costs and $c_f(0) = 0$.

Therefore, the Marshallian surplus is exactly the area between the demand curve and the supply curve. Welfare is maximized when the Marshallian surplus is maximized, which occurs at the competitive equilibrium quantity, x^*.

Thus, the area between the demand curve and the supply curve is an exact measure of the welfare associated with the output x.