Department of Economics
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Econ 8817–Game Theory

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Homework #1 Answers

1. O-R, exercise 19.1.

Answer: There are n players, and each player i has the action set, $A_i = \{out\} \cup [0, 1]$. Each player prefers an action profile with more votes than any other player than one in which he/she ties for the most votes; prefers to tie than to be out; and prefers to be out rather than lose.

With two players, the unique equilibrium is for both players to choose the median of the distribution, $m = F^{-1}(\frac{1}{2})$. This is a NE, because both firms tie, and any deviation will cause that firm to lose or be out. To see that this NE is unique, there cannot be a NE in which one of the players is out, because that player could guarantee at least a tie by choosing the right position. There cannot be a NE in which the players choose different positions, because a player standing to lose could guarantee at least a tie, and a player standing to tie could move closer to the other player and thereby win. There cannot be a NE in which the players choose the same position other than the median, because they would stand to tie, but a player could deviate closer to the median and win.

With three players, there cannot be a NE. If all three are out, then one player could choose a position and win. If two are out, then one of those players could guarantee at least a tie by choosing the right position. If one player is out, then the other two must be choosing the median voter position; in that case, the player that is out could choose a position close to the median voter position and receive almost half the votes, thereby winning. Finally, suppose all three voters choose positions. They must tie for first, because otherwise being out is preferred by a loser. If the players choose three distinct points, then one of the outside players could move closer to the middle and win. If two of the players choose the same position, then the other player could move closer to them and win. If all three players choose the same position, then one of them could move slightly away and receive nearly half the votes (or more), thereby winning.

2. O-R, exercise 28.1.

Answer: There are two players, $N = \{1, 2\}$, the set of states is $\Omega = S \times S$, where the probability of each state (s_1, s_2) is the product of the probability of s_1 and the probability of s_2. The signal fucntion of player i is given by

$\tau_i(s_1, s_2) = s_i$. The action set for player i is $A_i = \{\text{exchange, don't exchange}\}$. The utility function is

$$\begin{aligned} u_i(X, Y, s_1, s_2) &= s_{-i} \quad \text{if } X = Y = \textit{exchange} \\ u_i(X, Y, s_1, s_2) &= s_i \quad \text{otherwise.} \end{aligned}$$

Let x be the smallest prize. First consider the cases in which one or more players choose the strategy of never exchanging. The only possible NE of this form consist of one player never exchanging and the other player either never exchanging or only exchanging prize x. To see this, if say player 1 is willing to exchange a prize greater than x and player 2 never exchanges, then player 2 is not best responding.

Next consider the cases in which both players are willing to exchange at least one prize, and let S_i be the highest type of player i that chooses to exchange. If $S_1 > S_2$ held at the Bayesian NE, then player 1 is better off keeping his/her prize with type S_1. A symmetric argument for player 2 establishes that $S_1 = S_2$ holds. If we have $S_1 = S_2 > x$, then player 1 with type S_1 is receiving the same prize with positive probability and receiving a lower prize with positive probability; no exchange results in higher expected payoff.

3. O-R, exercise 35.2.

Answer: There are two players, $N = \{1, 2\}$, and the action space for player i is $A_i = [0, 1]$. Thus, the space of mixed strategies is the space of probability distributions over the unit interval. The payoff resulting from a pair of actions is given by

$$\begin{aligned} u_i(a_1, a_2) &= -a_i \quad \text{if } a_i < a_{-i} \\ u_i(a_1, a_2) &= \frac{1}{2} - a_i \quad \text{if } a_i = a_{-i} \\ u_i(a_1, a_2) &= 1 - a_i \quad \text{if } a_i > a_{-i}. \end{aligned}$$

First we have to rule out mass points and gaps in the mixed strategy distribution over the action space, $[0, 1]$. The supports of the distributions must be the same for both players, because if for example player 1 puts positive probability over an interval that is outside the support of player 2's distribution, then player 1 should instead transfer probability to smaller values within the interval. The common support cannot have gaps, because then a player should not put weight on an action to the "right" of a gap. The lower support must be at zero, because otherwise a player is willing to choose an action that yields negative profits (unless both players have a mass point at the lower support, which cannot be, because a slightly higher investment is more profitable). Thus, equilibrium profits must be zero. There cannot be a mass point in the interior of the support, because then the other player should not be putting positive weight on actions just below the mass point. Player i cannot have a mass point at zero, because then player $-i$ can guarantee positive profits. The upper support must be 1, because otherwise positive profits are available.

Thus, we have continuous, increasing distribution functions, F_1 and F_2. Player 1 must be indifferent between all choices, so we have for all a

$$F_2(a)[1-a] + (1 - F_2(a))[-a] = 0$$

which can be simplified to $F_2(a) = a$ for all a. Thus, each player chooses an investment uniformly distributed over the unit interval.

4. Consider a first-price auction with two players and certain (non-random) values. The player with the highest bid wins the object and pays his/her bid. Letting v_i denote the object's value to player i and letting b_i denote player i's bid, the payoffs are

$$\begin{aligned} u_i &= v_i - b_i \text{ if } b_i > b_{-i} \\ u_i &= \frac{v_i - b_i}{2} \text{ if } b_i = b_{-i} \\ u_i &= 0 \text{ if } b_i < b_{-i}. \end{aligned}$$

(a) If we have $v_1 = 1$ and $v_2 = 2$, show that there are no pure strategy Nash equilibria.
(b) If we have $v_1 = 1$ and $v_2 = 2$, construct a Nash equilibrium in which at least one player uses a mixed strategy.

Answer: (a) Suppose (b_1, b_2) is a NE. We cannot have $b_1 \geq b_2$, because if b_1 is greater than one, player 1 can increase his payoff by bidding zero, and if b_1 is less than or equal to one, then player 2 can increase her payoff by slightly outbidding player 1. We cannot have $b_1 < b_2$, because if b_1 is less than two, player 2 can increase her payoff by reducing her bid while maintaining $b_1 < b_2$; if b_1 is greater than or equal to two, then player 2 can increase her payoff by bidding zero.

(b) The idea is to have player 2 win the auction by bidding $b_2 = 1$, but eliminate her incentive to reduce her bid. Player 1 chooses a mixed strategy whose upper support is 1, so any advantage player 2 receives, by reducing her payment by bidding less than one, is outweighed by the probability of losing. Consider a uniform distribution, $\alpha_1 = U[b, 1]$. Clearly, player 1 is choosing a best response to $b_2 = 1$, for any value of b. If player 2 deviates to a bid, $\widehat{b} \in [b, 1]$, her payoff is

$$\left(\frac{\widehat{b} - b}{1 - b}\right)(2 - \widehat{b}).$$

Evaluated at $\widehat{b} = 1$, player 2's profit is equal to one. The deviation is unprofitable if the derivative of the above expression with respect to $\widehat{b}$ is nonnegative, which occurs if b is nonnegative. As it turns out, the following constitutes a

NE: $\alpha_1 = U[0,1]$ and $\alpha_2(1) = 1$ [That is, player 2 puts probability 1 on the action $b_2 = 1$.]

5. For the "chicken" game of Figure 47.1, find the correlated equilibrium for which the sum of the payoffs of the two players is the highest.

Answer: Without loss of generality, a correlated equilibrium can be specified in which a state of nature corresponds to the action profile that occurs in that state. Since we are looking for the highest sum of payoffs, we consider three states, $\Omega = \{a, b, c\}$, where in state a we have $(7, 2)$, in state b we have $(6, 6)$, and in state c we have $(2, 7)$. Therefore, the information partitions are $\wp_1 = \{\{a\}, \{b, c\}\}$, and $\wp_2 = \{\{a, b\}, \{c\}\}$. Let the probabilities be denoted as π_a, π_b, and π_c. Holding π_b fixed, higher π_a increases the incentive for player 1 to deviate from "T" to "B" when observing $\{b, c\}$, and higher π_c increases the incentive for player 2 to deviate from "L" to "R" when observing $\{a, b\}$. Due to the symmetry of the problem, to best satisfy both incentive constraints while putting the most possible weight on state b, we set $\pi_a = \pi_c = \pi$, and $\pi_b = 1 - 2\pi$. Now the incentive constraints become

$$\begin{aligned} \left(\frac{1-2\pi}{1-\pi}\right)6 + \left(\frac{\pi}{1-\pi}\right)2 &\geq \left(\frac{1-2\pi}{1-\pi}\right)7, \quad \text{or} \\ \pi &\geq \frac{1}{4}. \end{aligned}$$

The sum of the payoffs is decreasing in π, so the correlated equilibrium for which the sum of the payoffs of the two players is the highest is characterized by $\pi = \frac{1}{4}$, with the corresponding payoff profile $(5\frac{1}{4}, 5\frac{1}{4})$.