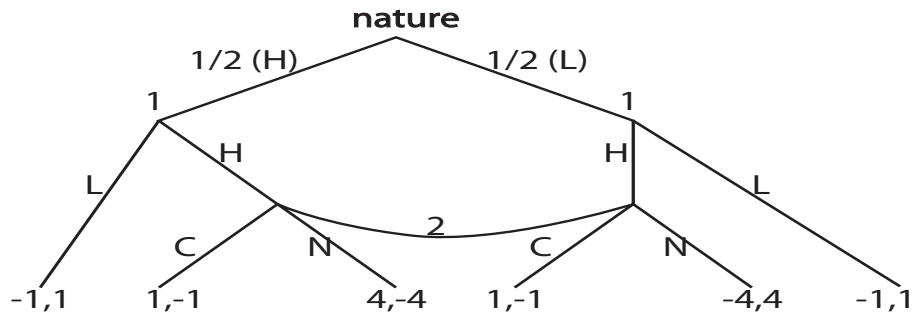


Spring 2013
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Homework #3 Answers

1. O-R, exercise 217.3. You can formulate the game as defined in O-R, or as a carefully labelled game tree. Find all mixed-strategy Nash equilibria.

Answer: The game tree is as follows.



Since we are looking for a Nash equilibrium, we can use the corresponding strategic game with the following payoff matrix

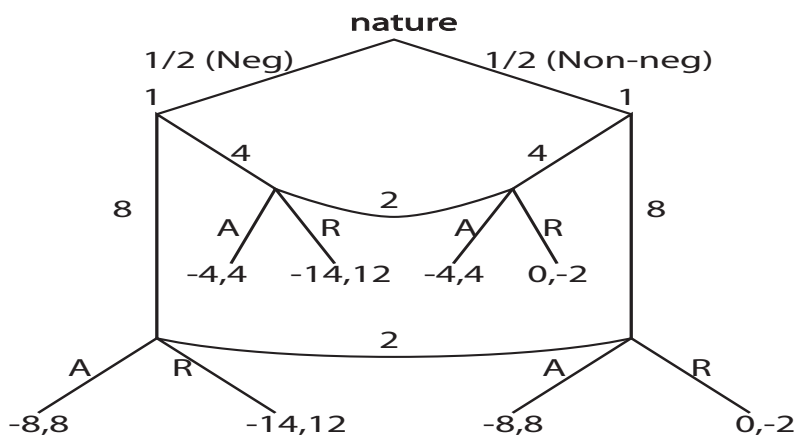
	C	N
LH	0, 0	$-\frac{5}{2}, \frac{5}{2}$
LL	-1, 1	-1, 1
HL	0, 0	$\frac{3}{2}, -\frac{3}{2}$
HH	1, -1	0, 0

where the first letter is player 1's claim when nature chooses H and the second letter is player 1's claim when nature chooses L. For player 2, C stands for concede and N stands for "not conceding." Notice that the payoffs represent lotteries over terminal histories. For example, the strategy profile (LH,N) yields a payoff of $(-1, 1)$ when nature chooses H, and a payoff of $(-4, 4)$ when nature chooses L. First, notice that player 1 will never claim to have a low card when nature gives her a high card. Thus, LH is strictly dominated by HH, and LL is strictly dominated by HL. Once we have eliminated those strategies, it is easy to see that there is only one Nash equilibrium. Player 1 chooses HL with probability $\frac{2}{5}$ and HH with probability $\frac{3}{5}$. Player 2 chooses C with probability $\frac{3}{5}$ and N with probability $\frac{2}{5}$.

2. Player 1 is involved in an accident with player 2 with damages of \$12,000. Player 1 knows whether or not she is negligent. Player 2 does not know player 1's type, but instead assigns prior probability $\frac{1}{2}$ to each of the two types (negligent or not negligent). Player 1 must send player 2 a pre-trial take-it-or-leave-it offer, which must be either \$4000 or \$8000, after which player 2 decides either to accept or reject. If he accepts the offer, $m \in \{4000, 8000\}$, then player 1's payoff is $-m$ and player 2's payoff is m . If he rejects the offer, the case goes to trial, which player 1 wins if she is not negligent, and player 2 wins if player 1 is negligent. The loser pays the court costs of \$2000. Therefore, if the case goes to trial and player 1 is negligent, payoffs are $(-14000, 12000)$. If the case goes to trial and player 1 is not negligent, payoffs are $(0, -2000)$.

Formulate this as a signalling game and find all perfect Bayesian equilibria.

Answer: The game tree for this game is as follows (with payoffs and messages in thousands). Player 1's type is negligent or non-negligent, each with prior probability $\frac{1}{2}$. The message is either 4000 or 8000.



To find all pure strategy PBE, suppose player 1 chooses (8000, 4000), where the first number is her offer when she is negligent, and the second number is her offer when she is not negligent. Then player 2 can infer player 1's type from her offer, so sequential rationality requires that he reject the offer of 8000, and accept the offer of 4000. But now player 1 has a profitable deviation to switch her offers.

Suppose player 1 chooses (4000, 8000). Then player 2 must believe that player 1 is negligent when she offers 4000 and not negligent when she offers 8000, so the former is rejected and the latter is accepted. Again, player 1 has a profitable deviation to switch her offers.

Suppose player 1 chooses (8000, 8000). Then Bayes' rule requires player 2 to believe that, after an offer of 8000, both types are equally likely, so that offer is accepted. An offer of 4000 must be rejected, or else player 1 should deviate to the lower offer. However, since the lower offer is rejected, player 1 should make the lower offer when she is not negligent.

Finally, suppose player 1 chooses (4000, 4000). Then Bayes' rule requires player 2 to believe that, after an offer of 4000, both types are equally likely, so

his expected payoff from rejecting is 5000. Thus, sequential rationality requires player 2 to reject an offer of 4000. An offer of 8000 must also be rejected, or else player 1 would make the higher offer when she is negligent. For this to be consistent with a PBE, player 2 must believe that an offer of 8000 is likely to be made by a negligent player 1.

All PBE are given by the pure strategy profile, $((4000, 4000), (R, R))$, and beliefs $\mu(4000)(neg) = \mu(4000)(non-neg) = \frac{1}{2}$, $\mu(8000)(neg) = q$, and $\mu(8000)(non-neg) = 1 - q$, for any q that satisfies

$$12000q - 2000(1 - q) \geq 8000, \text{ or } q \geq \frac{5}{7}.$$

3. Consider the following cooperative game involving 5 political parties. The number of votes controlled by party i is denoted by x_i , where we have $x_1 = 45$, $x_2 = 45$, $x_3 = 3$, $x_4 = 3$, and $x_5 = 4$. Any coalition receiving a majority of the votes will be able to form a government and divide the surplus. Thus, the characteristic function is given by

$$\begin{aligned} v(S) &= 1 \quad \text{if } \sum_{i \in S} x_i \geq 51 \\ &= 0 \quad \text{otherwise.} \end{aligned}$$

- (a) Find the core of this game.
- (b) Find the Shapley value of this game.
- (c) What would be the Shapley value of the game in which the three smaller parties (3, 4, and 5) merged into a single party? Is it in their interest to merge?

Answer: (a) (Sorry for using up the variable x for the votes, so that we will have to use a different variable to denote allocations.) The core of this game is empty. For example, if $(y_1, y_2, y_3, y_4, y_5)$ is a core allocation, at least one component, say y_1 , must be positive. But then the remaining players can form a coalition to increase each of their payoffs by $y_1/4$, by dividing what was allocated to player 1 among themselves.

(b) There are $5! = 120$ possible orderings, and a party's marginal contribution is 1 if it converts a losing coalition into a winning one, 0 otherwise. Let us start with one of the smaller parties, say party 3. Party 3 has a marginal contribution of 1 in all orderings in which it follows parties (1,4), (1,5), (4,1), (5,1), (2,4), (2,5), (4,2), and (5,2). For each of these possibilities, there are 2 orderings, making a total of 16. For example party 3 has a marginal contribution of 1 for the orderings (1,4,3,2,5) and (1,4,3,5,2). Thus, the Shapley value for party 3 is $16/120$ or $2/15$. The other small parties also have a Shapley value of $2/15$, and the two large parties have a Shapley value of $3/10$.

(c) With a merged small party with 10 votes (call it party 3), its marginal contribution is 1 in the following orderings: (1,3,2) and (2,3,1). Thus, its Shapley value is $2/6=1/3$. Merging gives more power to the large parties, so the small parties will not want to merge. With 5 parties, the total Shapley value allocated to the small parties is $2/5$, which is more than $1/3$.