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Misbehavior in Common Value Auctions: Bidding Rings and Shills†

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We characterize the optimal misbehavior by bidding rings or an auctioneer in the ascending English auction with common values. We also show, in an extended game, that in equilibrium potential members join and truthfully reveal their signals. Under a separability assumption, behavior does not change if nonring bidders are informed about the ring's existence. In general, misbehavior in dynamic settings is more profitable than in outcome-equivalent static settings. However, under a stronger separability assumption, the ring can do no better in the dynamic English format than in the outcome-equivalent, static Sophi format. (*JEL* D44, D82)

Misbehavior in auctions by a subset of bidders, or by the selling side (auctioneer), have long been known to exist. Cassady (1967) documents misbehavior by a group of bidders, or a "ring," who cooperate (collude) to acquire the item at a lower price. Examples of auctions with bidder rings, sometimes small rings of two or three bidders, include auctions for antiques, fish, timber, and wool.[1] He also documents the auctioneer's practice of spiking the price by the use of phantom bids, or "bids from the chandelier," and by cooperating with a confederate or "shill," sometimes called "shill bidding" or "trotting."

Most of the theoretical literature on misbehavior in auctions was done within the independent private values model. In this paper, we study misbehavior in the common value environment, where bidders have affiliated private signals or estimates of the value at the time bidding takes place but where the ex post value of the object is the same for all. There is a common value component of many auctions, such as mineral rights or oil and gas leases (see Wilson 1977; Milgrom 1981; or Hendricks, Porter, and Tan 2008). The common value environment alters the optimal (mis)behavior of a bidding ring, since the ring can manipulate the beliefs of

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[1] According to Cassady (1967, 177), "The term 'ring' apparently derives from the fact that in a settlement sale following the auction, members of the collusive arrangement form a circle or ring to facilitate observation of their trading behavior by the ring leader."

the bidders as well as the auction price. In our analysis, the ring consists of a subset of bidders (assumed for simplicity to be just two) who share the payoff when they win the item. The characterization of optimal misbehavior by an existing ring is of central importance, because, as we demonstrate, it is valid when potential members can choose whether to form a ring and whether to reveal their signals to each other after observing their private signals. We are thinking about an environment in which bidders do not suspect a ring and act as the standard theory specifies. However, misbehavior does occur, and we are interested in understanding the form it takes. Furthermore, under a separability assumption, equilibrium outcomes do not change when bidders are aware of the ring.

Our findings reveal the optimal misbehavior by the bidding ring in the English auction. One of the ring members will exit at the first possible opportunity. Under a separability assumption, Assumption TS, we characterize the exit point of the other ring member, based on an indifference condition. For signal realizations below a threshold, there is an interior solution for how long to remain in the auction. For signal realizations above the threshold, the ring remains in the auction and guarantees securing the object. Surprisingly, the nonring bidders choose the same strategies that they would employ with full knowledge of the existence of the ring and its equilibrium strategy!

The equilibrium to our model of auctioneer misbehavior, in which the bidders are not aware of the presence of the shill, would not be consistent with equilibrium if the presence of the shill were common knowledge. For this reason, our results for auctioneer misbehavior are less applicable than our results for the bidding ring. We should be thinking of a situation in which the auctioneer has been maintaining a reputation for honest dealing without shills and does not appear willing to risk the reputational or legal punishments of being caught.

We find that the auctioneer delays exiting the auction (i.e., having its shill exit the auction) until only one other bidder remains. If the auctioneer's signal exceeds, or is close enough to, the inferred second-order statistic of the bidders, then the auctioneer remains in the auction to further push up the price, taking the risk of retaining the item at an expected loss.

Another contribution of this paper is that we compare the profitability of misbehavior in dynamic mechanisms and in static mechanisms that are outcome-equivalent without misbehavior. We provide a general result that misbehavior in the indirect mechanism is more profitable (often strictly so) than in the direct mechanism. To illustrate, we also present Sophi, a static mechanism, the equilibrium of which in the absence of misbehavior is outcome-equivalent to that of the English auction.[2] Since the dynamic mechanism allows the misbehaving party to tailor their misbehavior to what they observe, it provides weakly higher profits than the static mechanism, strictly higher profits in almost all cases. However, under a much stronger separability assumption than what we use for our

[2] See Levin and Reiss (2020). Ausubel (1999) introduced the generalized Vickrey auction in a model with interdependent values. For the special case of common values, the generalized Vickrey auction simplifies to the Sophi auction.

main results, it turns out that the English auction and the Sophi auction yield the same outcomes.

Our analysis assumes that the ring forms and its members agree to an equal share of the profit before they observe their signals. We augment the game to include a stage that allows the potential ring members to decide, only after observing their private signals, whether to join the ring or not. We then show that the potential members prefer to join the ring and truthfully report their signals, no matter what their signals are.

Literature Review.—There is substantial literature on bidding rings in models with independent private values, mostly considering all-inclusive rings; see Graham and Marshall (1987); Mailath and Zemsky (1991); and McAfee and McMillan (1992). Marshall and Marx (2007) study both first- and second-price auctions with heterogeneous bidders and a ring that is not all-inclusive.

Hendricks, Porter, and Tan (2008) model an all-inclusive ring in a first-price auction model with common values. It is assumed that the bidders can commit to abide by the mechanism, including any required side payments. In general, a mechanism that achieves ex post efficiency and budget balance may not be possible, because a bidder with a very high signal may be better off going alone rather than making the required payments to the other bidders. With independent signals, however, efficiency and budget balance can be achieved. Of course, the issue is whether bidders can be incentivized to join the ring, since the object is always acquired at the reserve price in the actual auction.

It may be costly for a mechanism designer employed by the ring to approach all of the other bidders with a proposal to form a ring when there are potential criminal penalties. Due to the complication of properly modeling the costs of being discovered and the likelihood of being discovered (which might depend on the mechanism), we take a different approach that focuses on the auction itself. We assume that the set of potential ring members is exogenous and not all-inclusive (for simplicity, containing two bidders). However, this assumption is just for convenience, since we show that the ring is stable and that the analysis does not change significantly with larger rings.

There is minimal literature on shill bidding in common value auctions. Vincent (1995) provides an explanation of why an auctioneer in a second-price auction with interdependent values might want to keep the reserve price secret, where the bidders know that there is a secret reserve price. Although keeping the reserve price secret seems to violate the linkage principle of Milgrom and Weber (1982), it does not, because keeping the reserve price secret induces more participation in the auction. Lamy (2009) studies first- and second-price auctions, with interdependent values, in which the auctioneer does not receive a signal and cannot commit not to use a shill. Equilibrium in the first-price auction is immune to shill bidding, but equilibrium in the second-price auction is very complicated (even with no auctioneer signal) due to the auctioneer employing a mixed strategy. The auctioneer would be strictly better off with the ability to commit not to use a shill bid. Chakraborty and Kosmopoulou (2004) consider a model in which it is common knowledge that the auctioneer can employ a shill with a certain probability. The reason that they are able to solve an otherwise intractable model is that bidders receive one of only two possible signals.

Akbarpour and Li (2020) develop the notion of a credible mechanism, in which the mechanism designer or auctioneer has no beneficial "safe" deviations from following the mechanism's rules. We consider a different sort of auctioneer misbehavior. In our setting, the auctioneer commits to follow the auction rules, in the sense that the mapping from bidder actions to outcomes is carried out honestly. The misbehavior consists of conspiring with one of the bidders.

We briefly set up the model in Section I. Section II considers misbehavior in an English auction by a bidding ring with two members, characterizing the equilibrium and showing that the equilibrium does not change if the existence of the ring is common knowledge. Section III shows that the ring is stable and argues that the result extends to the case of larger rings. Section IV studies the use of a shill bidder in an English auction. Section V compares misbehavior in dynamic versus static settings, using the comparison between the English auction and the Sophi auction as an example. Section VI contains some concluding remarks. Many of the proofs are given in the Appendix.

I. The Common Value Model

We consider a common value auction framework with n bidders. The common value, V, and the signals, $X = (X_1, \ldots, X_n)$, are random variables drawn from a joint distribution function, $F(V,X)$, with support $[\underline{v},\bar{v}] \times [\underline{x},\bar{x}]^n$. We assume that bidders are risk neutral, F is symmetric across bidders, and the components of F are positively affiliated.[3] The interpretation is that each bidder i privately observes a signal or type (estimate), $X_i = x_i$. Denote the random variable for the j-th order statistic of the signals by Y^j and the realization as y^j. From $F(V,X)$, we can derive various marginal and conditional distributions, which we indicate by a subscript. For example, we denote the probability that the order statistics, $(Y^1, Y^2, \ldots, Y^n)$, are less than or equal to $y = (y^1, y^2, y^3, \ldots, y^n)$ by $F_{Y^1,Y^2,\ldots,Y^n}(y^1, y^2, y^3, \ldots, y^n)$. All joint signal vectors are assumed to occur with positive density. We assume that there exists a finite expected value of V, conditional on $x = (x_1, \ldots, x_n)$, denoted by the continuously differentiable function, $\mu(x) \equiv E[V|x]$. The expected value of V, conditional on y, is the continuously differentiable function denoted by $h(y) \equiv E[V|y]$, which is assumed to satisfy $\partial h(y)/\partial y^1 > 0$.[4]

The following assumption, x-separability (separability in the x_i signals), yields sharp and surprising results when it holds.

ASSUMPTION X-SEPARABILITY: *$\mu(x)$ is x-separable if for all $i \neq j$, for all $x^a_{-(i,j)}$ and $x^b_{-(i,j)}$, and for all (x_i,x_j) and $(x_i,x_j)'$, we have*

$$\mu\big((x_i,x_j),x^a_{-(i,j)}\big) \gtreqless \mu\big((x_i,x_j)',x^a_{-(i,j)}\big)$$

[3] Affiliation roughly means that large values for one variable make larger values for the other variables more likely. See Milgrom and Weber (1982) for a formal definition.

[4] The difference between h and μ is that h is strictly increasing in the first component and weakly increasing in the other components. Also, h is only well defined when the components of y are in descending order, but μ requires no such restriction.

implies

$$\mu\Big(\big(x_i,x_j\big),x^b_{-(i,j)}\Big) \gtreqless \mu\Big(\big(x_i,x_j\big)',x^b_{-(i,j)}\Big).$$

In words, x-separability means that if two vectors of signals differ only in the i and j components and the vector with $\left(x_i,x_j\right)$ yields a higher (respectively, lower) expectation of V than the vector with $\left(x_i,x_j\right)'$ does, then this relationship will continue to hold if we change components of the vectors other than $\left(i,j\right)$. Although x-separability is restrictive, it holds for a wide class of environments, including

$$V = \sum_{j=1}^{n}\phi\left(x_j\right) + \epsilon,$$

where ϕ is an arbitrary increasing function and ϵ is white noise. We also use a special case of x-separability, which we call Assumption AVG (for average signal), mainly for illustrative purposes.

ASSUMPTION AVG: $\mu\left(x\right) = \frac{1}{n}\left(\sum_{j=1}^{n}x_j\right)$.

Our main results require a significantly weaker separability assumption that only requires separability at "the top."

ASSUMPTION TS (Top Separability): $h\left(y\right)$ *is top separable if, for all* $j = 1,\ldots,n-1, y^1 = \ldots = y^j = \tilde{y}$ *and*

$$h\Big(\tilde{y},\ldots,\tilde{y},y_a^{j+1},\ldots,y_a^n\Big) = h\Big(\tilde{y},\ldots,\tilde{y},y_b^{j+1},\ldots,y_b^n\Big)$$

imply, for all $\hat{y} > \tilde{y}$,

$$h\Big(\hat{y},\ldots,\hat{y},y_a^{j+1},\ldots,y_a^n\Big) = h\Big(\hat{y},\ldots,\hat{y},y_b^{j+1},\ldots,y_b^n\Big).$$

Assumption TS holds, for example, if the value is the sum of functions of the order statistics,

$$h\Big(y^1,\ldots,y^n\Big) = \sum_{i=1}^{n}\phi^i\Big(y^i\Big),$$

where ϕ^i is an increasing function that varies across i. Assumption TS is much weaker than x-separability, because x-separability is likely to fail when the value weighs signals based on their order. For example, when there are three bidders and the value is of the form $V = a_1y^1 + a_2y^2 + a_3y^3$, Assumption TS is satisfied for all $\left(a_1,a_2,a_3\right) \in \mathbb{R}^3_+$, but x-separability is only satisfied if $a_1 = a_2 = a_3$ holds, as we show in the Appendix. We also show in the Appendix that x-separability implies TS.

The English Auction: Following Milgrom and Weber (1982), the English auction is modeled as an indirect mechanism, in which there is a clock that rises continuously, starting at the expected value associated with the worst possible realization of signals, $h\left(\underline{x},\ldots,\underline{x}\right)$. Bidders decide whether to remain in the auction or exit. As soon as only one bidder remains, the clock stops and that bidder wins the auction, and the payment is the clock price at that point. All bidders observe the history of exits and associated clock prices.

II. Misbehavior by a Ring of Bidders

Here we consider a model with $n - 1$ bidders as well as a "ring" of two members, each of whom observes a signal but is secretly in league with the other. We will refer to the ring members as r_1 and r_2 (where r stands for ring) and reserve the term "bidders" for those auction participants other than the ring members. We do not model the legal or reputational risks of forming a ring, so this section assumes that the ring members have committed to share the surplus equally. Extension to a larger ring is simple and does not add anything to the analysis. We assume that the bidders treat the auctions as having $n + 1$ bidders. However, we show in Proposition 2 that under Assumption TS, the bidders are choosing the same strategies that they would choose in the game in which they know that there is a ring. Let us use the previous notations for the bidders' signals, and let the signals of the ring be given by the random variables $S_1 \equiv X_1$ and $S_2 \equiv X_2$ with realizations s_1 and s_2. For $j = 1, \ldots, n-1$, we denote the j-th order statistic of the bidders as Y^j, with realization y^j. Similar to the notation for the model without the ring, define

$$h^R\left(s_1, s_2, y^1, y^2, \ldots, y^{n-1}\right)$$
$$= E\left[V \middle| \left(S_1 = s_1, S_2 = s_2, Y^1 = y^1, Y^2 = y^2, \ldots, Y^{n-1} = y^{n-1}\right)\right].$$

A. *Optimal Misbehavior by the Ring*

The analysis begins with the following observations.

Observation 1: In order to keep the payment to the auctioneer for winning as low as possible, the ring will have one of its members, say r_2, exit immediately, so the bidders incorrectly infer $S_2 = \underline{x}$.

Observation 2: Any bidder who holds signal y^j and remains active infers the vector $\left(y^{j+1}, \ldots, y^{n-1}\right)$ from the exit prices.

Observation 3: If the remaining member of the ring, r_1, is still active when the holder of the second-highest signal, y^2, exits, then both r_1 and the holder of y^1 learn the vector $\left(y^2, y^3, \ldots, y^{n-1}\right)$. r_1 knows s_2, but the holder of y^1 incorrectly infers $S_2 = \underline{x}$. In this case, the holder of y^2 exits at the price $h^R\left(y^2, \underline{x}, y^2, y^2, \ldots, y^{n-1}\right)$. See Milgrom and Weber (1982).

Suppressing the dependence on $\left(y^2, y^3, \ldots, y^{n-1}\right)$, it is convenient to define $V^R\left(y^1\right)$ by

$$V^R\left(y^1\right) = h^R\left(y^1, \underline{x}, y^1, y^2, \ldots, y^{n-1}\right).$$

The superscript, R, is used to distinguish the notation for the model with a ring from the similar notation for the model without the ring. $V^R\left(y^1\right)$ is the maximum willingness to pay by the holder of y^1 at the time when the holder of y^2 exits. This is simply the expectation of V, conditional on his own signal being y^1, the signals of the bidders who have exited being $(y^2, y^3, \ldots, y^{n-1})$, the signal of r_2 being $\underline{x}$, and the signal of r_1 being y^1.

Let $\hat{V}^R(s_1, s_2, y^1)$ be defined by the shorthand notation

$$\hat{V}^R(s_1, s_2, y^1) = h^R(s_1, s_2, y^1, y^2, \ldots, y^{n-1}). \tag{1}$$

We can interpret $\hat{V}^R(s_1, s_2, y^1)$ as the expectation of V for the ring, conditional on $(s_1, s_2, y^1, y^2, \ldots, y^{n-1})$.

The ring's problem, after r_2 exits at $\underline{x}$, is for r_1 to "pretend" to be a bidder with signal t. That is, the remaining ring bidder exits when the holder of y^1 is revealed to have signal $y^1 \geq t$. Notice that once r_1 has determined that winning the object cannot be profitable,[5] the exit point does not matter, as long as it occurs before the holder of y^2 exits. The equilibrium we select makes the standard assumption that r_1 exits at the first instant that continuing cannot be profitable. To be precise, suppose that the history of exits reveals $(y^j, \ldots, y^{n-1})$ and the current clock price reveals that remaining bidders have signals of at least t, $y^{j-1} \geq t$. Also suppose that for all $(y^1, \ldots, y^{j-1})$, we have $\hat{V}^R(s_1, s_2, y^1) \leq V^R(y^1)$. Then an active r_1 will exit at that point.

If r_1 has not exited by the time the holder of y^2 exits, the ring's payoff, given $(y^2, \ldots, y^{n-1}, s_1, s_2)$, is equal to the following expression.

$$\int_{y^2}^{t} \left[\hat{V}^R(s_1, s_2, y^1) - V^R(y^1)\right] dF_{Y^1|Y^2,\ldots,Y^{n-1},S_1,S_2}(y^1 \mid y^2, \ldots, y^{n-1}, s_1, s_2). \tag{2}$$

Thus, the optimal misbehavior, given $(y^2, \ldots, y^{n-1}, s_1, s_2)$, is found by choosing t^* to maximize (2).[6]

Under Assumption TS, we can give a complete dynamic characterization of the ring's strategy. For $j = 1, \ldots, n-1$, after the history with j bidders remaining, $(y^{j+1}, \ldots, y^{n-1})$, r_1 is the next to exit at exactly the price at which the profits would be zero if all the remaining bidders exit at that instant. That is, r_1 is the next to exit when type t^{*j} would exit, which depends on the history, satisfying

$$h^R(s_1, s_2, t^{*j}, \ldots, t^{*j}, y^{j+1}, \ldots, y^{n-1}) = h^R(t^{*j}, \underline{x}, t^{*j}, \ldots, t^{*j}, y^{j+1}, \ldots, y^{n-1}) \tag{3}$$

if a solution below $\bar{x}$ exists, and otherwise r_1 is the next to exit at the price

$$h^R(s_1, s_2, \bar{x}, \ldots, \bar{x}, y^{j+1}, \ldots, y^{n-1}). \tag{4}$$

[5] By "profit" for a bidder or for the ring receiving the object, we mean the payoff or net benefit, i.e., the value minus the price paid.

[6] For some signal realizations, it is optimal for r_1 to remain in the auction beyond the point at which a bidder with signal $\bar{x}$ would drop. In this case, off the equilibrium path, r_1 would remain in the auction until price $\hat{V}^R(s_1, s_2, \bar{x})$, believing that the deviating bidder(s) is type $\bar{x}$.

Notice that r_1 is sometimes willing to bid more than the price at which type $\bar{x}$ would exit, $h^R(\bar{x}, \underline{x}, \bar{x}, y^2, \ldots, y^{n-1})$. r_1 believes that a bidder who remains beyond the exit price of type $\bar{x}$ is type $\bar{x}$[7] and exits at the zero profit price, $h^R(s_1, s_2, \bar{x}, y^2, \ldots, y^{n-1})$.

PROPOSITION 1: *Suppose that Assumption TS holds. Then the optimal misbehavior of the ring, with signals* (s_1, s_2), *is characterized by* r_2 *pretending to be type* $\underline{x}$ *and* r_1 *exiting according to* (3) *or* (4).

Additionally, if x-separability is satisfied, we have the following corollary.

COROLLARY TO PROPOSITION 1: *Suppose that x-separability holds. Then either* r_1 *imitates the same type,* t^*, *independent of the history, or* r_1 *remains in the auction until after a bidder of type* $\bar{x}$ *would exit, independent of the history.*

B. *Common Knowledge of the Ring*

Proposition 2 below establishes that under Assumption TS, the strategies of the ring and the bidders are equilibrium strategies of the model in which the existence of the ring is common knowledge. It would seem that strategies of the bidders should change, because with a secret ring, the immediate exit by one of the ring members causes bidders to infer a signal of $\underline{x}$, but when the existence of the ring is common knowledge, it provides no inference to bidders about that ring member's signal. The correct intuition is based on the fact that the ring is prepared to compete up until the point of zero expected profits. If the bidder with signal y^1 wins the object, the price is $h^R(t^*, \underline{x}, t^*, y^2, \ldots, y^{n-1})$, and the value to this bidder is $h^R(s_1, s_2, y^1, y^2, \ldots, y^{n-1})$. However, since t^* is such that the value to the ring equals the clock price at the point of exiting, $h^R(s_1, s_2, t^*, y^2, \ldots, y^{n-1})$ equals $h^R(t^*, \underline{x}, t^*, y^2, \ldots, y^{n-1})$. Therefore, we can write the profits of the bidder with signal y^1 as

$$h^R(s_1, s_2, y^1, y^2, \ldots, y^{n-1}) - h^R(s_1, s_2, t^*, y^2, \ldots, y^{n-1}).$$

From this expression, we see that this bidder is best responding by treating the ring's behavior as if they were "ordinary" bidders.

Proposition 2 demonstrates that when Assumption TS is satisfied, the optimal ring strategy and the "sincere" strategies of the bidders constitutes a Bayesian Nash equilibrium (BNE) of the game in which the ring is common knowledge. We also show that the equilibrium is without regret.[8]

PROPOSITION 2: *Suppose that Assumption TS holds. The optimal strategy of the ring and the "sincere" strategies of the bidders as characterized earlier is an*

[7] The equilibrium with these out-of-equilibrium beliefs satisfies the intuitive criterion of Cho and Kreps (1987), because the type of bidder with the greatest incentive to deviate and remain in the auction is type $\bar{x}$.

[8] An equilibrium is without regret (i.e., an ex post equilibrium) when the players do not want to revise their actions even after learning the types of the other players.

equilibrium profile of the game in which the existence of the ring is common knowledge. Furthermore, the equilibrium is without regret.

III. Stability of the Ring

In this section, we suppose that the ring members cannot commit to sharing the payoff, in the event that they win the auction, until after they receive their signals. We show that the ring is stable, in the sense that potential ring members would want to join the ring and truthfully report their signals, by considering the following game. First, each of the bidders, including the potential ring members (bidders 1 and 2), observe their signal. Second, the potential ring members simultaneously decide whether to be part of the ring or not. If both members say "yes," then the ring forms, the ring members commit to share equally the profit from the auction, the ring members report their signals to each other (in an incentive compatible way, as incentives are aligned at this point), and the continuation of the game and the analysis of the previous section applies.[9] If any potential ring member says "no," then these decisions are communicated to each other (become common knowledge to the potential ring members), the ring does not form, and we proceed to the auction. We assume that when the ring does not form, each potential ring member can identify whether a bidder who exits is a bidder or the other potential ring member. However, the stability result also holds when the identity of those exiting is not observed (see Remark 1).[10]

A. *Stability of the Ring in the English Auction*

Consider the following strategy profile for the English auction:[11]

The two potential ring members say "yes," and along the equilibrium path, the game continues as analyzed in the previous section.

If one potential ring member (without loss of generality, r_1) says "yes" and the other potential ring member (r_2) says "no," then r_1 believes, and continues to believe throughout the auction, that r_2's signal is $\bar{x}$. In the auction, r_2 exits immediately, after every history in which exiting is not weakly dominated; if exiting is weakly dominated, r_2 remains. As a continuation strategy, r_1 exits when there are several remaining bidders if there are no signal realizations for the remaining bidders for which winning is profitable, given beliefs about r_2. If r_2 is the only remaining bidder with r_1 (so that y^1, $y^2, \ldots, y^{n-1}$ have been revealed), then r_1 remains in the auction until the price reaches r_1's willingness to pay, $\hat{V}^{NR}(s_1, y^1) \equiv h^R(s_1, \bar{x}, y^1, y^2, \ldots, y^{n-1})$. If r_2 has already exited at a price at which the remaining bidder infers $S_2 = x'$, then the remaining bidder's maximum willingness to pay is given by $V^{NR}(y^1) \equiv h^R(s_1, x', y^1, y^2, \ldots, y^{n-1})$.

[9] We do not model how the potential ring members arrive at an equal division of the profit or loss. Loertscher and Marx (2021) study collusive behavior in a private values setting in which a bidder is selected to be active, without exchange of private information, by a randomization device.

[10] Marshall and Marx (2009) show that whether or not the auctioneer releases the identities of registered bidders can affect the viability of collusion in private value auctions.

[11] We continue to assume that the bidders are oblivious to the presence of the (potential) ring. So, for all intents and purposes, this is a game being played by the two ring members.

Thus, r_1's problem is to "pretend" to be a bidder with signal $t^* > s_1$, where t^* is a maximizer of

$$(5)\quad \int_{\max\{x', y^2\}}^{t} \left[\hat{V}^{NR}(s_1, y^1) - V^{NR}(y^1)\right] dF_{Y^1|Y^2,\ldots,Y^{n-1},S_1,S_2}\left(y^1 \mid y^2,\ldots,y^{n-1},s_1,\bar{x}\right).$$

Finally, if both potential ring members say "no," then r_1 and r_2 each believe that the other bidder's signal is $\bar{x}$. r_1 exits when there are several remaining bidders if there are no signal realizations for the remaining bidders for which winning is profitable, given beliefs about r_2. Otherwise, if r_2 is the only remaining bidder, then r_1 will exit at the price $\hat{V}^{NR}(s_1, y^1)$. If, instead, r_2 has already exited, at a price such that the remaining bidder infers $S_2 = x'$, then r_1's problem is to "pretend" to be a bidder with signal $t^* > s_1$, where t^* is a maximizer of (5). *r_2 chooses the same continuation strategy.*[12]

PROPOSITION 3. *The strategy profile described above constitutes a PBE for the English auction. Furthermore, this equilibrium satisfies the intuitive criterion of Cho and Kreps* (1987).

PROOF:

If r_1 and r_2 both say "yes," then truthfully revealing their signals to each other and bidding as specified in the previous section is sequentially rational.

Suppose r_1 says "yes" and r_2 says "no." Certainly it is sequentially rational for r_1 to exit whenever remaining cannot be profitable and otherwise to remain until there is only one other remaining bidder. Consider the case in which the only remaining bidder besides r_1 is r_2. Given r_1's belief that r_2 is type $\bar{x}$, when the price is greater than $\hat{V}^{NR}(s_1, y^1)$, it is sequentially rational for r_1 to exit immediately. If the price is less than $\hat{V}^{NR}(s_1, y^1)$, it is sequentially rational for r_1 to remain until $\hat{V}^{NR}(s_1, y^1)$. Now consider the case in which the only remaining bidder besides r_1 is a bidder (i.e., not r_2). Then r_1's expected profit, from pretending to be type t, is (5), so his continuation strategy is sequentially rational.

Whenever r_2 has a continuation strategy that weakly dominates exiting, remaining in the auction is sequentially rational. Otherwise, it is sequentially rational for r_2 to exit immediately. To see this, suppose that r_2 wins the auction when r_1 is the last to exit. Then the profit received by r_2 is

$$h^R(s_1, s_2, y^1, y^2, \ldots, y^{n-1}) - h^R(s_1, \bar{x}, y^1, y^2, \ldots, y^{n-1}),$$

which is nonpositive (strictly negative unless we have $s_2 = \bar{x}$). Suppose that r_2 wins the auction when a bidder is the last to exit. Then the fact that r_1 exited implies that $h^R(s_1, \bar{x}, y^1, y^2, \ldots, y^{n-1})$ is less than the price at which r_1 exited. Therefore, the price paid by r_2 is more than $h^R(s_1, \bar{x}, y^1, y^2, \ldots, y^{n-1})$ and the value to r_2 is less than $h^R(s_1, \bar{x}, y^1, y^2, \ldots, y^{n-1})$, so winning cannot be profitable.

[12] That is, r_2's continuation strategy is the continuation strategy of r_1 described above, after we relabel r_2 as r_1 and relabel r_1 as r_2.

Suppose r_1 and r_2 both say "no." Given r_1's belief that r_2 is type $\bar{x}$, by the arguments given above, r_1's continuation strategy is sequentially rational. Since r_2 is adopting the same continuation strategy as r_1, r_2's continuation strategy is sequentially rational as well.

It is sequentially rational for r_1 and r_2 to say "yes." After saying "no," the best continuation play is to exit immediately, while saying "yes" leads to positive expected profit.

To demonstrate that the intuitive criterion is satisfied, consider a deviation to "no" by r_2, after which r_1 believes that r_2 is type $\bar{x}$ with probability one. An r_2 of type $\bar{x}$ could benefit from this deviation, as would be the case if type r_1 exited immediately in the subsequent auction (and this continuation by r_1 would be undominated). Thus, we cannot rule out these beliefs.[13] ∎

Remark 1: Consider the continuation of the game when r_1 says "yes" and r_2 says "no." If we were to extend the earlier equilibrium selection (for an established ring) of exiting when remaining in the auction cannot be profitable, then r_2 would exit immediately after every history. The construction in Proposition 3 guarantees that the equilibrium does not involve a potential ring member choosing a weakly dominated strategy. For Proposition 3, we consider the model in which the potential ring members observe the identities of the bidders exiting from the auction, so r_1 knows the point at which r_2 exits, and vice versa. A similar proposition could be proved for the model in which the identities of those exiting from the auction is not observed. In that case, r_1 believes that r_2 has signal $\bar{x}$ and that the first one to exit is r_2. With identities unobservable, r_2 is even less willing to compete with r_1 than when identities are observable. The reason is that if r_2 deviates to remain in the auction, then r_1 believes that the bidders are remaining in the auction longer, and revealing higher signals, than they actually are.

B. *Robustness to Larger Rings*

If the number of ring members is $m > 2$, the optimal bidding behavior would have all but one ring member exit immediately in the English auction. The optimal bid by the remaining ring member requires slightly modifying the analysis of Section II to take into account that there are multiple ring members mimicking type $\underline{x}$. For the question of stability, we consider the following game. First, each of the bidders and the potential ring members observe their signal. Second, the potential ring members simultaneously decide whether to be part of the ring or not. If a subset of at least two members say "yes," then a ring consisting of that subset forms and the ring members commit to share equally the profit from the auction. It is assumed that the makeup of the ring is common knowledge among the m potential ring members.

We now sketch the argument that the ring of size m is stable for the English auction. If one potential ring member deviates and says "no," then the $m - 1$ ring

[13] Indeed, the type of player r_2 with the *greatest* incentive to deviate to "no" is type $\bar{x}$.

members believe that the deviator is type $\bar{x}$. All but one of the ring members exit immediately, but the remaining ring member waits until the expected profit for the ring is zero. The deviator exits immediately. After the deviation, it is sequentially rational for the deviator to exit immediately, because outbidding the ring member guarantees nonpositive profits. Given that a lone deviator receives zero profits, everyone saying "yes" is sequentially rational. (If several potential ring members say "no," then they are all believed to be type $\bar{x}$. Choose any continuation strategies that are sequentially rational given the beliefs.)

It would be tempting to say that larger rings are more profitable for the members, but this is not necessarily the case. A larger ring provides a larger informational advantage and larger overall ring profits, but these profits must be divided by more ring members, so the effect per ring member is ambiguous.

IV. Misbehavior by the Auctioneer

Here we consider a model with n bidders and additionally one shill, who observes a signal but is secretly in league with the auctioneer. We assume that the bidders treat the auctions as having $n+1$ bidders. Let us use the previous notations for the bidders' signals, and let the signal of the auctioneer/shill be given by the random variable $S \equiv X_{n+1}$, with realization s. For $j = 1, \ldots, n$, we denote the j-th order statistic of the bidders as Y^j, with realization y^j.[14]

The analysis begins with the following observations.

Observation 1: Any bidder who holds signal y^j and remains active infers the vector of signals of the bidders who have exited, $\left(y^{j+1}, \ldots, y^n\right)$, from the exit prices.

Observation 2: Since the auctioneer can stay active in the auction and exit at the same clock price as the holder of the second-highest signal, y^2, she would never exit earlier. Exiting at a lower price is dominated by exiting at the same price as the holder of y^2, since it raises the payment without any risk of not selling. Thus, from the exit prices, both the auctioneer and the holder of y^1 learn the vector $\left(y^2, y^3, \ldots, y^n\right)$.

Observation 3: Since the auctioneer never exits before the holder of y^2, that bidder does not observe the signals, s or y^1.

Observation 4: The holder of y^2 exits at $d_2 = E\left[V \mid \left(Y^1 = y^2, S = y^2, Y^2 = y^2, Y^3 = y^3, \ldots, Y^n = y^n\right)\right]$; see Milgrom and Weber (1982). A bidder with signal x_j would exit at a price equal to the expected value of the object, assuming that all remaining active bidders hold the tied signal of x_j.

Suppressing the dependence on $\left(y^2, y^3, \ldots, y^n\right)$, define $V\left(y^1\right)$ by

$$V\left(y^1\right) = E\left[V \mid Y^1 = y^1, S = y^1, Y^2 = y^2, Y^3 = y^3, \ldots, Y^n = y^n\right]. \tag{6}$$

Since both the auctioneer and the holder of y^1 can infer the vector $\left(y^2, y^3, \ldots, y^n\right)$, and since the holder of y^1 does not know that the holder of s is a shill bidder, $V\left(y^1\right)$ is the maximum willingness to pay by the holder of y^1 at the time when the holder of y^2 exits.[15] Note that the holder of y^1 treats the shill as a bidder, so

[14] Here the joint distribution of the value and signals is given by $F(V, X) = F\left(V, X_1, \ldots, X_{n+1}\right)$.

[15] This is well known from the literature. See Matthews (1977) and Milgrom and Weber (1982).

(6) is simply the expectation of V, conditional on his own signal being y^1, the signals of the bidders who have exited being $\left(y^2, y^3, \ldots, y^n\right)$, and the signal of the remaining bidder (who happens to be the shill) being y^1.

Let $\hat{V}\left(s^m, s\right)$ be defined by

$$\hat{V}\left(s^m, s\right) = E\left[V \mid Y^1 = s^m, S = s, Y^2 = y^2, Y^3 = y^3, \ldots, Y^n = y^n\right],$$

where $s^m \equiv \max\left\{s, y^2\right\}$.[16] We can interpret $\hat{V}\left(s^m, s\right)$ as the maximum willingness to pay of a bidder who holds signal s and "happens" to still be active at the price d_2, where the holder of y^2 exits. Note that $s \lesseqgtr y^2$ holds as $\hat{V}\left(s^m, s\right) \lesseqgtr d_2$ holds.

We stated in Observation 2 that the auctioneer never exits before d_2. As we show below, it is very possible that she would exit strictly after y^2 (at a strictly higher price than d_2), **even** when $s \leq y^2$ holds. Consider an auctioneer who contemplates mimicking the exit price of a bidder who is type $t > y^2$. Then, given the behavior of the bidder holding y^1, the auctioneer wins the auction and retains the good if $y^1 < t$ holds, and the auctioneer succeeds in selling the good at a higher price if $y^1 > t$ holds. In the latter case, her revenue is equal to the exit price of a bidder who is type t, given by

$$d_t = E\left[V \mid Y^1 = t, S = t, Y^2 = y^2, Y^3 = y^3, \ldots, Y^n = y^n\right].$$

Suppressing the dependence on $\left(s, y^2, y^3, \ldots, y^n\right)$, define $G(t)$, and associated density $g(t)$, by

$$G(t) = F_{Y^1|S,Y^2\ldots Y^n}(t) = \Pr\left[Y^1 \leq t \mid S = s, Y^2 = y^2, Y^3 = y^3, \ldots, Y^n = y^n\right],$$

which is the probability that the first-order statistic is not higher than t, given the realizations of $\left(S = s, Y^2 = y^2, Y^3 = y^3, \ldots, Y^n = y^n\right)$.

We can now write the maximization problem of the auctioneer as

$$\max_{t \geq s^m} \int_{y^2}^{t} \hat{V}(\tau, s)\, dG(\tau) + \left[1 - G(t)\right] V(t). \tag{7}$$

In (7), the first term represents the event in which the auctioneer wins the item (by mimicking type t) and retains its value, which is $\hat{V}\left(Y^1, s\right)$. In the second term, $\left[1 - G(t)\right]$ is the probability that $Y^1 \geq t$ holds, resulting in the auctioneer losing the item and receiving the price, $V(t)$. Recall, $V(t)$ is the price that the auctioneer receives by mimicking a bidder of type t, which is not her own value at that point. Also note that the auctioneer is mimicking a type, $t \geq s^m$, although the integral runs from y^2 to t. The reason is that if $s > y^2$ holds, the auctioneer will never mimic a type less than s, while y^1 can be anything above y^2.

[16] This notation is used to cover the possibility that $s < y^2$ holds, although the auctioneer never drops before d_2.

Let t^* denote the optimal type to mimic, solving (7). The first-order condition for an interior solution is

$$0 = \hat{V}(t,s) - V(t) + \left[\frac{1 - G(t)}{g(t)}\right]\frac{\partial V(t)}{\partial t}. \tag{8}$$

The following proposition shows that an interior solution occurs if s is greater than, or even smaller than but close enough to, y^2. In that case, the auctioneer chooses $t^* > s^m$ and risks retaining the item at an expected loss. On the other hand, if s is very low relative to y^2, the auctioneer might exit immediately after the bidder with y^2 exits.

PROPOSITION 4: *For all values of s satisfying $s \geq y^2$, and for some values of s strictly less than y^2, the optimal auctioneer misbehavior in the English auction solves* (8), *and $t^* > s^m$ holds, so the auctioneer risks retaining the item at an expected loss.*

PROOF:

The right side of (8) is the derivative of the auctioneer's payoff with respect to t. Since $\left[\frac{1 - G(y^2)}{g(y^2)}\right]\frac{\partial V(y^2)}{\partial t}$ is strictly positive and $\hat{V}(t,s) \geq V(t)$ holds for $t \leq s$, it follows that the right side of (8) is strictly positive for $t \leq s$, so we must have $t^* > s$. Therefore, $s \geq y^2$ implies $t^* > \max\{s, y^2\}$. By continuity, there is an $\epsilon > 0$ such that for $s \geq y^2 - \epsilon$, the right side of (8) is strictly positive for all $t \leq y^2$. ∎

The intuition for why the auctioneer risks retaining the item at an expected loss is the same as the intuition for setting a reserve price strictly above her private value in independent private values (IPV) and other settings. Condition (8) bears a close resemblance to Myerson's (1981) optimal auction. In the IPV setting, the optimal reserve price equates the seller's use value to the virtual valuation of the remaining bidder. In (8), $\hat{V}(t,s)$ is the seller's expectation of the value if the remaining bidder's type equals the type the seller is mimicking, t, a sort of use value at the margin; $V(t) - \left[\frac{1 - G(t)}{g(t)}\right]\frac{\partial V(t)}{\partial t}$ is the virtual value of the remaining bidder, but where $V(t)$ reflects the remaining bidder's mistaken belief that the shill is a bidder.

Typically, the right side of (8) is strictly decreasing in t. This would be the case, for example, when $\hat{V}(t,s) - V(t)$ is decreasing in t; $V(t)$ is concave; and we have the standard condition, implied by affiliation, that $\left[\frac{1 - G(y^2)}{g(y^2)}\right]$ is strictly decreasing in t. If so, then the optimal auctioneer misbehavior involves a cutoff s, which depends on y^2, below which the auctioneer exits immediately after the bidder with y^2 exits, and above which the auctioneer exits strictly after the bidder with y^2 exits.[17]

[17] A previous version of this paper presented an explicit solution under Assumption AVG.

V. Misbehavior in Dynamic versus Static Settings

A. *A Misbehavior Principle*

There is a precise sense in which misbehavior is (weakly) more profitable in dynamic settings than static settings. Suppose that we have a dynamic indirect mechanism that yields the same equilibrium mapping from type realizations to outcomes as a direct mechanism. Now suppose that the utility functions change, either for one agent (e.g., because the agent is colluding with the auctioneer) or for a ring of agents (who are colluding with each other), but that the remaining agents are not aware of this change in incentives. Then the agent(s) with new incentives weakly prefer the dynamic mechanism to the direct mechanism. In the dynamic mechanism, they can mimic the behavior of the type they would have reported in the direct mechanism. However, the dynamic mechanism also allows tailoring behavior to information revealed during the play.[18]

Formalizing this principle requires some notation. Suppose that we have n agents and each agent's type, $t_i \in T_i$ for $i = 1, \ldots, n$, is private information. Let T denote the set of type profiles and Y the set of outcomes. Each agent is an expected utility maximizer with Bernoulli utility function, $u_i : T \times Y \to \mathbb{R}$. Denote the joint distribution of types by $F(t_1, \ldots, t_n)$. An incentive compatible direct mechnism is a function, $m : T \to Y$, satisfying the usual incentive compatibility constraint. An indirect mechanism is an extensive form Bayesian game, in which each player i, after observing his/her type, chooses a strategy $\sigma_i \in \Sigma_i$. Thus, σ_i specifies an action by agent i at every decision node at which agent i is called upon to act. This allows for the possibility that agent i's action can depend on the actions chosen by other agents. Denote the set of strategy profiles of the underlying game by Σ. A strategy of the indirect mechanism is a mapping from T_i to Σ_i, which allows σ_i to depend on t_i. The outcome is given by $\Gamma : \Sigma \to Y$. Denote the profile of equilibrium strategies of the indirect mechanism by $(\sigma_1^*, \ldots, \sigma_n^*)$.[19]

We define an incentive compatible direct mechanism and an indirect mechanism to be *outcome equivalent* if, for all type realizations, we have

$$m(t_1, \ldots, t_n) = \Gamma(\sigma_1^*(t_1), \ldots, \sigma_n^*(t_n)).$$

PROPOSITION 5 (Misbehavior Principle): *Suppose that agents* 1 *and* 2 *are able to collude, without the knowledge of the other agents or the mechanism designer, and that the colluding agents choose reports* (*in the direct mechanism*) *and strategies* (*in the indirect mechanism*) *in order to maximize a "social welfare" function,* $u_{1,2} : T \times Y \to \mathbb{R}$.[20] *If the direct mechanism and the indirect mechanism are outcome equivalent, then the*

[18] As an application of this principle, in the private values setting, a misbehaving auctioneer (or bidding ring) weakly prefers the English auction to the second-price auction. For the IPV case, it turns out that the misbehaving auctioneer is indifferent between the two formats, but with affiliated private values, English is strictly preferred. A bidding ring is indifferent between the two formats.

[19] The solution concept can be PBE, subgame perfect Nash equilibrium, or anything else. In case there are multiple equilibria, an equilibrium selection is implicit in $(\sigma_1^*(t_1), \ldots, \sigma_n^*(t_n))$.

[20] For example, the agents may agree on how to split the profits from colluding and seek to maximize those profits.

welfare of the colluding agents is weakly higher in the indirect mechanism than in the direct mechanism.

PROOF:

Consider a reporting strategy in the direct mechanism, mapping types into reports, denoted by $(\tilde{t}_1(t_1,t_2),\tilde{t}_2(t_1,t_2))$. Because the direct mechanism and the indirect mechanism are outcome equivalent, the mapping from types into strategies in Γ, given by

$$(\tilde{\sigma}_1(t_1,t_2),\tilde{\sigma}_2(t_1,t_2)) = (\sigma_1^*(\tilde{t}_1(t_1,t_2)),\sigma_2^*(\tilde{t}_2(t_1,t_2))),$$

yields the same outcome for the colluding agents as in the direct mechanism, for every realization of types $(t_1,\ldots,t_n)$. ∎

Note that the same argument applies if a single agent (say, agent 1) has a new utility function without the knowledge of the other agents. For example, perhaps agent 1 has agreed to be a shill agent for the mechanism designer. To demonstrate this, if agent 1 chooses a reporting strategy $\tilde{t}_1(t_1)$ in the direct mechanism, the strategy $\sigma_1^*(\tilde{t}_1(t_1))$ in the indirect mechanism yields the same outcome for every type realization.

B. *The Sophi Auction*

We now consider a static mechanism, the Sophi auction, which in the absence of misbehavior is outcome equivalent to the dynamic English auction. Our contribution here is the characterization of the optimal way the ring or the shill exploit this tailoring opportunity and the result that under x-separability, the dynamic English auction provides the ring no advantage over the static Sophi auction. That is, the misbehavior principle need not hold strictly.

The Sophi (Sophisticated) Auction: In the Sophi auction, each bidder i submits a bid, $\tilde{x}_i$, and the bidder placing the highest bid wins. Any bid below $\underline{x}$ is converted to $\underline{x}$. Then, if all bids (except possibly the highest bid) are less than or equal to $\bar{x}$, then the highest bidder (say bidder i) pays

$$P^S(\tilde{x}) = E[V \mid Y^1 = \tilde{y}^2, Y^2 = \tilde{y}^2, Y^3 = \tilde{y}^3, \ldots, Y^n = \tilde{y}^n]. \tag{9}$$

That is, the payment is the expectation of V, conditional on the signals of the other bidders equaling their bids, $\tilde{x}_{-i}$, and the signal of bidder i equaling the highest bid of the other bidders. All other bidders pay nothing. Notice that bids are not in the same space as prices; bidders understand how the bids affect prices through the payment rule. Although it does not arise in equilibrium, we must specify the payment rule when two or more bids exceed $\bar{x}$, in which case (9) is not well defined. If two or more bids exceed $\bar{x}$, and if $\tilde{x}_i$ is the highest bid, the payment made by bidder i is determined as follows: increase the lowest bid by $(\tilde{x}_i - \bar{x})$, then reduce all bids exceeding $\bar{x}$ to $\bar{x}$, then apply pricing rule (9). Ausubel (1999) considers a model with multiple objects and interdependent values, but for the special case of one object and common values, the generalized Vickrey auction simplifies to a version of Sophi in which bids cannot exceed $\bar{x}$.

In equilibrium, we have $P^S(\tilde{x}) = E[V \mid y^2, y^2, y^3, \ldots, y^n]$. Levin and Reiss (2020) proved that the bidder who wins the object must pay according to the Sophi rule in any direct mechanism that satisfies the properties that (i) the bidder with the highest signal receives the object, (ii) a bidder who does not receive the object pays nothing, and (iii) truthful reporting is an ex post (no-regret) equilibrium.[21] Thus, the Sophi auction is interesting in its own right. In equilibrium, Sophi "asks" bidders to simply bid their signals, because the price rule corrects for the adverse selection. In practice, Sophi may help bidders who typically ignore or undercorrect the adverse selection and overbid, thereby avoiding or mitigating the winner's curse.

The Sophi auction is not currently practiced, but the average bid mechanism (ABM) similarly attempts to reduce the winner's curse and defaults, and it is used in practice.[22] However, ABM has no symmetric and strictly monotone equilibrium (see Chang, Chen, and Salmon 2014), so it is difficult to play or use for predictions. In these regards, Sophi is superior to ABM.[23]

Comparison between the English and Sophi Auctions without Misbehavior.— The simple equilibrium bidding rule for the Sophi auction sharply contrasts with the complicated equilibrium strategies in common value auctions such as first-price and English auctions. In the symmetric equilibrium of a first-price auction, each bidder ought to bid as if holding the highest signal and then use a complicated Bayesian calculation to correct his estimation of the common value and also decide on the proper (optimal) shading. In English auctions, each bidder must continuously update his estimation of the common value after each exit. Levin and Reiss (2020) show that in both the Sophi and English auctions, the highest signal holder wins the object and pays $E[V \mid y^2, y^2, y^3, \ldots, y^n]$. Thus, the Sophi auction and the English auction are allocation and price equivalent.

The equilibrium bidding rule under Sophi is as simple as can be to bid one's signal, so no informational requirements are placed on the bidders.[24] Under Assumption AVG, where the average of the n private signals is a sufficient statistic for $E[V]$, the payment rule when bidder 1 makes the largest bid and bidder 2 makes the second-largest bid is

$$(10) \quad P^S(\tilde{x}) = \frac{1}{n}(\tilde{x}_2 + \tilde{x}_2 + \tilde{x}_3 + \ldots + \tilde{x}_n) = \frac{1}{n}\left[\left(\sum_{i=1}^{n} \tilde{x}_i\right) - (\tilde{x}_1 - \tilde{x}_2)\right].$$

[21] We post the proof from Levin and Reiss (2020) in the Appendix for the convenience of the readers.

[22] The ABM has been used by the government of Taiwan and by the Florida Department of Transportation. A variation in which the high and low bids are dropped before taking the average has been used in Peru, in Switzerland, and by the State of New York. See Chang, Chen, and Salmon (2014).

[23] There is a rich theoretical and experimental literature that inspired new auction designs. Examples include work on combinatorial (or package) auctions with complementaries, used in spectrum auctions (see Kagel, Lien, and Milgrom 2010, 2014; Kwasnica et al. 2005; Milgrom 2007; and references therein); dynamic auctions (see Ausubel 2004); and position auctions (see Varian 2007 and references therein).

[24] Sophi has an additional normative appeal. Instructing bidders to "just bid your estimate" does not only sound simple, it is also optimal (in equilibrium). It is much harder to see what bidding advice one could instruct bidders in other commonly used auctions.

This payment rule is simple and imposes no informational requirements on the auctioneer. Under Assumption AVG, Sophi is immune to the Wilson critique (although not in general).

In contrast, the English auction places heavy informational requirements on the bidders, even under Assumption AVG: they must know the joint distribution, $F(V,X)$, and perform a complicated Bayesian updating to infer the other bidders' types after each exit. In fact, experimental evidence (see Levin, Kagel, and Richard 1996 and Kagel and Levin 2002) shows that although the English format mitigates the winner's curse relative to the first-price auction, the winner's curse still exists (even under Z). Thus, in actual environments in which Assumption AVG only holds approximately, a mechanism using pricing rule (10) "à la Sophi" might yield outcomes closer to the equilibrium predictions of the English auction than the English auction mechanism itself.

Comparison between the English and Sophi Auctions with a Ring.—To characterize the optimal ring misbehavior in the Sophi auction, first notice that one of the ring members (say, r_2) will bid $\underline{x}$ in order to keep the price as low as possible. We can write the ring's payoff, when r_1 bids t, as

$$(11)\quad \int_{y^2,y^3,\ldots,y^{n-1}} \left[\int_{y^2}^{t} \Delta\left(s_1,s_2,y^1\right) dF_{Y^1|Y^2,\ldots,Y^{n-1},S_1,S_2}\left(y^1 \,|\, y^2,\ldots,y^{n-1},s_1,s_2\right)\right] dF_{Y^2,\ldots,Y^{n-1}|S_1,S_2}\left(y^2,y^3,\ldots,y^{n-1} \,|\, s_1,s_2\right),$$

where $\Delta\left(s_1,s_2,y^1\right)$ is defined by

$$\Delta\left(s_1,s_2,y^1\right) \;=\; \hat{V}^R\left(s_1,s_2,y^1\right) - V^R\left(y^1\right).$$

Notice that when $t > y^1$ holds and the ring wins the auction, $\hat{V}^R\left(s_1,s_2,y^1\right)$ is the expectation of V and $V^R\left(y^1\right)$ is the price the ring pays, according to the Sophi pricing rule and the bidders' truthful bidding.

The optimal misbehavior for the ring in the Sophi auction is for r_1 to bid t to maximize (11). Sometimes a solution is to bid $\bar{x}$, in which case bids greater than $\bar{x}$ are also optimal.

It follows from Proposition 5 that the ring is weakly better off under the English auction than under the Sophi auction. Proposition 6(i) shows that when the result holds strictly, and Proposition 6(ii) shows that when x-separability holds, then the two auction formats yield the same outcomes for the ring.

PROPOSITION 6: (*i*) *The optimal misbehavior of the ring with signals* (s_1,s_2) *yields strictly higher payoff in the English auction than in the Sophi auction, except when the optimal misbehavior can be achieved with a* t^* *that does not depend on* $\left(y^2,\ldots,y^{n-1}\right)$. (*ii*) *Under x-separability, the optimal misbehavior of the ring yields the same outcome in the English auction and the Sophi auction, and the optimal misbehavior is without regret.*

In an online Appendix, we provide analogs of Propositions 2 and 3 for the Sophi auction. Proposition A1 provides conditions under which the optimal strategy of the ring and the sincere strategies of the bidders in the Sophi auction is an equilibrium of the game in which the existence of the ring is common knowledge. Proposition A2 provides conditions under which the optimal misbehavior is robust to allowing potential ring members to decide whether or not to join the ring.

Comparison between the English and Sophi Auctions with a Misbehaving Auctioneer.—Without misbehavior, the English and Sophi auctions lead to the same equilibrium outcomes for any realization of signals. However, the auctioneer's ability to manipulate the auction secretly, with a shill bidder, differs across the two formats. We show that the auctioneer always receives a higher expected payoff, conditional on her signal, in the English auction than in the Sophi auction, as long as her signal is less than $\bar{x}$.[25]

To characterize the optimal auctioneer misbehavior in the Sophi auction, first notice that for *given* $\left(y^2, y^3, \ldots, y^n\right)$, the auctioneer's payoff as a function of the bid t is given by

$$\int_{y^2}^{t} \hat{V}(\tau, s)\, dG(\tau) + \left[1 - G(t)\right] V(t),$$

which follows from the rules for the Sophi payments and the sincere bidding of the bidders. The overall payoff of an auctioneer with signal s but bidding t is given by

$$(12)\; \int_{y^2, y^3, \ldots, y^n} \left\{ \int_{y^2}^{t} \hat{V}(\tau, s)\, dG(\tau) + \left[1 - G(t)\right] V(t) \right\} dF_{Y^2, \ldots, Y^n | S}\left(y^2, y^3, \ldots, y^n \,|\, s\right).$$

The optimal auctioneer misbehavior, given signal s, is to submit a bid $t > s$ that maximizes (12).

Here we show that the misbehaving auctioneer strictly prefers the English auction format over the Sophi auction format.

PROPOSITION 7: *For all $s < \bar{x}$, a misbehaving auctioneer strictly prefers the English auction over the Sophi auction.*

The idea behind the proof of Proposition 7 is that there is always a chance that y^2 is very high. In the Sophi auction, the auctioneer does not observe y^2 and ends up bidding $t^{*Sophi} < y^2$. In the English auction, this would correspond to the auctioneer exiting before the bidder with y^2, which cannot be optimal. More generally, we would expect that t^* should depend nontrivially on $\left(y^2, y^3, \ldots, y^n\right)$, in which case the English auction yields a higher payoff.

[25] Rothkopf and Harstad (1995) compare the misbehavior of the auctioneer in the private values Vickrey auction to the first-price auction, but they assume that the auctioneer can observe all bids before submitting her bid, whereas we assume that in the Sophi auction, the auctioneer submits her bid before she observes the remaining bids.

VI. Concluding Remarks

We analyze, separately, the misbehavior of a ring of bidders and the misbehavior of an auctioneer who uses shill bidding within a common value ascending-price English auction. For the case of the bidding ring, one ring member exits at the first opportunity. We characterize when and how the other ring member mimics a type strictly lower than the highest possible type as well as when the other ring member guarantees a win on the equilibrium path by remaining in the auction beyond the price at which the highest possible type should exit. Under Assumption TS, bidders would not change their strategies even if the existence of the ring was common knowledge. We also augmented the game to include a stage that allows the potential ring members to decide, after observing their private signals, whether to join the ring or not. We then show that the potential members prefer to join the ring and truthfully report their signals no matter what their signals are.

For the case of the auctioneer using a shill bidder, we characterize when the auctioneer exits immediately after the bidder with the second-highest signal exits and when the auctioneer remains longer, thereby risking keeping the object.

We also compare the scope for misbehavior in dynamic versus static mechanisms, by providing a general "misbehavior principle." We compare the dynamic English auction to the Sophi auction, a static mechanism that is equivalent to the English auction in the absence of misbehavior. In general, misbehavior is more profitable under the dynamic mechanism. However, and somewhat surprisingly, we recognize an important special case (x-separability) where, in spite of the additional information revealed to the ring from the exit prices in the English auction, the ring cannot do better under English than under Sophi.

The work of Li (2017) on obviously dominant strategies suggests that dynamic auctions may have features that bring behavior closer to the BNE norm than their static counterparts. In the private values setting, the dominant strategy of exiting at one's value is obvious in the dynamic English auction, but not so in the static second-price auction (see Kagel, Harstad, and Levin 1987). With multiple-unit demands, Kagel and Levin (2001) show that the dynamic Vickrey auction performs better than the static uniform price auction. In the common values setting, Levin, Peck, and Isanov (2016) show that behavior in the dynamic Dutch auction is closer to BNE than in the strategically equivalent static first-price auction. The dynamic format of the ticking price clock gives subjects a hint about adverse selection, since no one else has stopped the clock. In the present paper, in contrast, we find that the static Sophi format is actually better for bidders than the English format when unexpected misbehavior by the auctioneer is present.

Appendix

PROOF THAT *X*-SEPARABILITY IMPLIES ASSUMPTION TS:

Suppose not, so there exist realizations of signals for which we have

$$h\left(\tilde{y},\ldots,\tilde{y},y_a^{j+1},\ldots,y_a^n\right) = h\left(\tilde{y},\ldots,\tilde{y},y_b^{j+1},\ldots,y_b^n\right) \tag{A1}$$

and

(A2) $$h\left(\hat{y},\ldots,\hat{y},y_a^{j+1},\ldots,y_a^n\right) \neq h\left(\hat{y},\ldots,\hat{y},y_b^{j+1},\ldots,y_b^n\right),$$

where $\hat{y} > \tilde{y}$ holds. There are two cases.

If $j < n-1$ holds, then there are at least two signals in $\left(y_a^{j+1},\ldots,y_a^n\right)$ and $\left(y_b^{j+1},\ldots,y_b^n\right)$. Select $\left(x_i,x_j\right)$ and $\left(x_i,x_j\right)'$ according to $\left(x_i,x_j\right) = \left(y_a^{n-1},y_a^n\right)$ and $\left(x_i,x_j\right)' = \left(y_b^{n-1},y_b^n\right)$. By x-separability, (A1) implies

$$h\left(\hat{y},\ldots,\hat{y},y_a^{j+1},\ldots,y_a^n\right) = h\left(\hat{y},\ldots,\hat{y},y_b^{j+1},\ldots,y_b^n\right),$$

contradicting (A2).

If $j = n-1$ holds, (A1) can be written as

(A3) $$h\left(\tilde{y},\ldots,\tilde{y},y_a^n\right) = h\left(\tilde{y},\ldots,\tilde{y},y_b^n\right),$$

and (A2) can be written as

(A4) $$h\left(\hat{y},\ldots,\hat{y},y_a^n\right) \neq h\left(\hat{y},\ldots,\hat{y},y_b^n\right).$$

Select $\left(x_i,x_j\right)$ and $\left(x_i,x_j\right)'$ according to $\left(x_i,x_j\right) = \left(\tilde{y},y_a^n\right)$ and $\left(x_i,x_j\right)' = \left(\hat{y},y_b^n\right)$. By x-separability, (A3) implies

(A5) $$h\left(\hat{y},\ldots,\hat{y},\tilde{y},y_a^n\right) = h\left(\hat{y},\ldots,\hat{y},\tilde{y},y_b^n\right).$$

Now we use x-separability again, where we select $\left(x_i,x_j\right)$ and $\left(x_i,x_j\right)'$ according to $\left(x_i,x_j\right) = \left(\hat{y},y_a^n\right)$ and $\left(x_i,x_j\right)' = \left(\hat{y},y_b^n\right)$. By x-separability, (A5) implies

$$h\left(\hat{y},\ldots,\hat{y},\hat{y},y_a^n\right) = h\left(\hat{y},\ldots,\hat{y},\hat{y},y_b^n\right),$$

contradicting (A4). ∎

Example of $V = a_1 y^1 + a_2 y^2 + a_3 y^3$.—The fact that Assumption TS holds for all parameters is trivial. We now verify that x-separability holds only for the case where $a_1 = a_2 = a_3$ holds. Using the notation in the definition of x-separability, suppose that we have $x_i > x_j > x^a, x_i' > x_j' > x^a, x_j > x_j'$ and

(A6) $$a_1 x_i + a_2 x_j + a_3 x^a = a_1 x_i' + a_2 x_j' + a_3 x^a.$$

Now consider x^b satisfying $x_j' < x^b < x_j$. Then x-separability requires

$$a_1 x_i + a_2 x_j + a_3 x^b = a_1 x_i' + a_2 x^b + a_3 x_j'.$$

Combining these equations yields $a_2 = a_3$. Now consider x^b satisfying $x_j' < x_j < x_i < x^b < x_i'$. Then x-separability and the previous finding, that $a_2 = a_3$ holds, requires

$$a_1 x^b + a_2 x_i + a_2 x_j = a_1 x_i' + a_2 x^b + a_2 x_j'.$$

From this equation and equation (18), simple algebra yields the conclusion that $a_1 = a_2$.

PROOF OF PROPOSITION 1:

We first show in steps 1 and 2 that the ring's strategy given by the indifference condition (3) is well defined.

Step 1: Consider the history in which no one has exited, and suppose, without loss of generality, that $s_1 \geq s_2$ holds. With $n-1$ active bidders, r_1 will be the next to exit by imitating a bidder with type t^{*n-1}, satisfying the indifference condition

$$h^R\left(s_1, s_2, t^{*n-1}, \ldots, t^{*n-1}\right) = h^R\left(t^{*n-1}, \underline{x}, t^{*n-1}, \ldots, t^{*n-1}\right), \tag{A7}$$

or if the left side exceeds the right side for all types, we have

$$h^R\left(s_1, s_2, \bar{x}, \ldots, \bar{x}\right) - h^R\left(\bar{x}, \underline{x}, \bar{x}, \ldots, \bar{x}\right) > 0. \tag{A8}$$

Now suppose a bidder with signal y^{n-1} exits first. We want to show that the ring's profits from winning at that instant are nonnegative.

For the case of (A7), $y^{n-1} < t^{*n-1}$ must hold. r_1's profits from winning at that instant are

$$h^R\left(s_1, s_2, y^{n-1}, \ldots, y^{n-1}\right) - h^R\left(y^{n-1}, \underline{x}, y^{n-1}, \ldots, y^{n-1}\right). \tag{A9}$$

Suppose, by way of contradiction, that (A9) is negative. Then there exists t' satisfying $s_1 \leq t' < y^{n-1}$ and

$$h^R\left(t', s_2, y^{n-1}, \ldots, y^{n-1}\right) = h^R\left(y^{n-1}, \underline{x}, y^{n-1}, \ldots, y^{n-1}\right). \tag{A10}$$

By Assumption TS, the fact that y^{n-1} is the largest signal in (A10), and $y^{n-1} \leq t^{*n-1}$, we have

$$h^R\left(t', s_2, t^{*n-1}, \ldots, t^{*n-1}\right) = h^R\left(y^{n-1}, \underline{x}, t^{*n-1}, \ldots, t^{*n-1}\right). \tag{A11}$$

Since $s_1 \leq t'$ holds, we have

$$h^R\left(s_1, s_2, t^{*n-1}, \ldots, t^{*n-1}\right) \leq h^R\left(t', s_2, t^{*n-1}, \ldots, t^{*n-1}\right). \tag{A12}$$

Using (A7) to substitute for the left side of (A12) and (A11) to substitute for the right side of (A12), we have

$$\text{(A13)} \qquad h^R\left(t^{*n-1}, \underline{x}, t^{*n-1}, \ldots, t^{*n-1}\right) \leq h^R\left(y^{n-1}, \underline{x}, t^{*n-1}, \ldots, t^{*n-1}\right).$$

Our maintained assumption that h^R is strictly increasing in its largest signal and $y^{n-1} < t^{*n-1}$ contradicts (A13).

For the case of (A8), we suppose, by way of contradiction, that (A9) is negative. Then there exists t' satisfying $s_1 \leq t' < y^{n-1}$ and (A10). By Assumption TS, the fact that y^{n-1} is the largest signal in (A10), we have

$$\text{(A14)} \qquad h^R\left(t', s_2, \bar{x}, \ldots, \bar{x}\right) = h^R\left(y^{n-1}, \underline{x}, \bar{x}, \ldots, \bar{x}\right).$$

Since $s_1 \leq t'$ holds, we have

$$\text{(A15)} \qquad h^R\left(s_1, s_2, \bar{x}, \ldots, \bar{x}\right) \leq h^R\left(t', s_2, \bar{x}, \ldots, \bar{x}\right).$$

From (A8), we have

$$\text{(A16)} \qquad h^R\left(\bar{x}, \underline{x}, \bar{x}, \ldots, \bar{x}\right) < h^R\left(s_1, s_2, \bar{x}, \ldots, \bar{x}\right).$$

Substituting the right side of (A14) into (A15) and combining with (A16), we have

$$h^R\left(\bar{x}, \underline{x}, \bar{x}, \ldots, \bar{x}\right) < h^R\left(y^{n-1}, \underline{x}, \bar{x}, \ldots, \bar{x}\right),$$

a contradiction.

Step 2: Proceeding inductively, consider the history, $\left(y^{j'+1}, \ldots, y^{n-1}\right)$, and suppose that (3) or (4) has characterized the ring's behavior for $j = j' + 1$ (i.e., before bidder $y^{j'+1}$ exited). There are two cases for what the ring was planning before $y^{j'+1}$ exited. Either there is $t^{*j'+1}$ solving the indifference condition,

$$\text{(A17)} \qquad h^R\left(s_1, s_2, t^{*j'+1}, \ldots, t^{*j'+1}, y^{j'+2}, \ldots, y^{n-1}\right)$$
$$= h^R\left(t^{*j'+1}, \underline{x}, t^{*j'+1}, \ldots, t^{*j'+1}, y^{j'+2}, \ldots, y^{n-1}\right),$$

or if the left side exceeds the right side for all types, we have

$$\text{(A18)} \ h^R\left(s_1, s_2, \bar{x}, \ldots, \bar{x}, y^{j'+2}, \ldots, y^{n-1}\right) - h^R\left(\bar{x}, \underline{x}, \bar{x}, \ldots, \bar{x}, y^{j'+2}, \ldots, y^{n-1}\right) > 0.$$

We want to show that the ring's profits from winning are nonnegative at the instant that the bidder with signal $y^{j'+1}$ exits.

For the case of (A17), $y^{j'+1} < t^{*j'+1}$ must hold. r_1's profits from winning at that instant are

$$\text{(A19)}\ h^R\left(s_1, s_2, y^{j'+1}, \ldots, y^{j'+1}, \ldots, y^{n-1}\right) - h^R\left(y^{j'+1}, \underline{x}, y^{j'+1}, \ldots, y^{j'+1}, \ldots, y^{n-1}\right).$$

Suppose, by way of contradiction, that (A19) is negative. Then there exists t' satisfying $s_1 \leq t' < y^{j'+1}$ and

$$\text{(A20)}\ h^R\left(t', s_2, y^{j'+1}, \ldots, y^{j'+1}, \ldots, y^{n-1}\right) = h^R\left(y^{j'+1}, \underline{x}, y^{j'+1}, \ldots, y^{j'+1}, \ldots, y^{n-1}\right).$$

By Assumption TS, the fact that $y^{j'+1}$ is the largest signal in (A20), and $y^{j'+1} < t^{*j'+1}$, we have

$$\text{(A21)}\quad h^R\left(t', s_2, t^{*j'+1}, \ldots, t^{*j'+1}, y^{j'+2}, \ldots, y^{n-1}\right) = h^R\left(y^{j'+1}, \underline{x}, t^{*j'+1}, \ldots, t^{*j'+1}, y^{j'+2}, \ldots, y^{n-1}\right).$$

Since $s_1 \leq t'$ holds, we have

$$\text{(A22)}\quad h^R\left(s_1, s_2, t^{*j'+1}, \ldots, t^{*j'+1}, y^{j'+2}, \ldots, y^{n-1}\right) \leq h^R\left(t', s_2, t^{*j'+1}, \ldots, t^{*j'+1}, y^{j'+2}, \ldots, y^{n-1}\right).$$

Using (A17) to substitute for the left side of (A22) and (A21) to substitute for the right side of (A22), we have

$$\text{(A23)}\quad h^R\left(t^{*j'+1}, \underline{x}, t^{*j'+1}, \ldots, t^{*j'+1}, y^{j'+2}, \ldots, y^{n-1}\right) \leq h^R\left(y^{j'+1}, \underline{x}, t^{*j'+1}, \ldots, t^{*j'+1}, y^{j'+2}, \ldots, y^{n-1}\right).$$

Our maintained assumption that h^R is strictly increasing in its largest signal and $y^{j'+1} < t^{*j'+1}$ contradicts (A13).

For the case of (A18), we suppose, by way of contradiction, that (A19) is negative. Then there exists t' satisfying $s_1 \leq t' < y^{j'+1}$ and (A20). By Assumption TS, the fact that $y^{j'+1}$ is the largest signal in (A20), we have

$$\text{(A24)}\quad h^R\left(t', s_2, \bar{x}, \ldots, \bar{x}, y^{j'+2}, \ldots, y^{n-1}\right) = h^R\left(y^{j'+1}, \underline{x}, \bar{x}, \ldots, \bar{x}, y^{j'+2}, \ldots, y^{n-1}\right).$$

Since $s_1 \leq t'$ holds, we have

$$\text{(A25)}\quad h^R\left(s_1, s_2, \bar{x}, \ldots, \bar{x}, y^{j'+2}, \ldots, y^{n-1}\right) \leq h^R\left(t', s_2, \bar{x}, \ldots, \bar{x}, y^{j'+2}, \ldots, y^{n-1}\right).$$

From (A18), we have

$$\text{(A26)}\quad h^R\left(\bar{x}, \underline{x}, \bar{x}, \ldots, \bar{x}, y^{j'+2}, \ldots, y^{n-1}\right) < h^R\left(s_1, s_2, \bar{x}, \ldots, \bar{x}, y^{j'+2}, \ldots, y^{n-1}\right).$$

Substituting the right side of (A24) into (A25) and combining with (A26), we have

$$h^R\big(\bar{x},\underline{x},\bar{x},\ldots,\bar{x},y^{j'+2},\ldots,y^{n-1}\big) \;<\; h^R\big(y^{j'+1},\underline{x},\bar{x},\ldots,\bar{x},y^{j'+2},\ldots,y^{n-1}\big),$$

a contradiction.

Step 3: The ring's strategy is a best response to the bidders, ex post, for all signal realizations. When the ring does not win the auction, a deviation to win would lead to profits of

$$\text{(A27)} \qquad h^R\big(s_1,s_2,y^1,\ldots,y^{n-1}\big) - h^R\big(y^1,\underline{x},y^1,\ldots,y^{n-1}\big).$$

r_1 exited with j remaining bidders at some t^{*j} satisfying

$$h^R\big(s_1,s_2,t^{*j},\ldots,t^{*j},y^{j+1},\ldots,y^{n-1}\big) \;=\; h^R\big(t^{*j},\underline{x},t^{*j},\ldots,t^{*j},y^{j+1},\ldots,y^{n-1}\big).$$

Since $t^{*j} \;<\; y^j$ holds, we can apply Assumption TS, yielding

$$h^R\big(s_1,s_2,y^j,\ldots,y^j,y^{j+1},\ldots,y^{n-1}\big) \;=\; h^R\big(t^{*j},\underline{x},y^j,\ldots,y^j,y^{j+1},\ldots,y^{n-1}\big).$$

Since $y^j \;\le\; y^{j-1} \;\le\; \ldots \;\le\; y^1$ holds, we can apply Assumption TS repeatedly, yielding

$$\text{(A28)} \qquad h^R\big(s_1,s_2,y^1,\ldots,y^{n-1}\big) \;=\; h^R\big(t^{*j},\underline{x},y^1,\ldots,y^{n-1}\big).$$

Since $t^{*j} \;<\; y^1$ holds, the right side of (A28) is strictly less than $h^R\big(y^1,\underline{x},y^1,\ldots, y^{n-1}\big)$,which implies that (A27) is negative and the deviation cannot be profitable for any signal realizations.

When the ring wins the auction, there are two cases. In the first case, we have $t^{*1} \;\ge\; y^1$ satisfying the indifference condition,

$$\text{(A29)} \qquad h^R\big(s_1,s_2,t^{*1},y^2,\ldots,y^{n-1}\big) \;=\; h^R\big(t^{*1},\underline{x},t^{*1},y^2,\ldots,y^{n-1}\big).$$

We claim that t^{*1} is the unique solution to (A29). If there were another solution, t', Assumption TS and (A29) implies

$$h^R\big(s_1,s_2,t',y^2,\ldots,y^{n-1}\big) \;=\; h^R\big(t^{*1},\underline{x},t',y^2,\ldots,y^{n-1}\big).$$

However, monotonicity in the highest signal would then imply

$$h^R\big(s_1,s_2,t',y^2,\ldots,y^{n-1}\big) \;<\; h^R\big(t',\underline{x},t',y^2,\ldots,y^{n-1}\big),$$

a contradiction. Since t^{*1} is the unique solution to (A29), then for $t \;<\; t^{*1}$ we have

$$h^R\big(s_1,s_2,t,y^2,\ldots,y^{n-1}\big) \;>\; h^R\big(t,\underline{x},t,y^2,\ldots,y^{n-1}\big).$$

Therefore,

$$h^R\big(s_1, s_2, y^1, y^2, \ldots, y^{n-1}\big) \geq h^R\big(y^1, \underline{x}, y^1, y^2, \ldots, y^{n-1}\big)$$

holds (strictly for $t^{*1} > y^1$), which implies that the profits from winning are nonnegative (positive for $t^{*1} > y^1$), so there is no profitable deviation.

In the second case, we have

$$h^R\big(s_1, s_2, \bar{x}, y^2, \ldots, y^{n-1}\big) > h^R\big(\bar{x}, \underline{x}, \bar{x}, y^2, \ldots, y^{n-1}\big).$$

Because there was no solution for t^{*1} solving (A29), it follows that

$$h^R\big(s_1, s_2, y^1, y^2, \ldots, y^{n-1}\big) > h^R\big(y^1, \underline{x}, y^1, y^2, \ldots, y^{n-1}\big)$$

holds. Therefore, the profits from winning are positive, and there is no profitable deviation. ∎

PROOF OF COROLLARY TO PROPOSITION 1:

We can apply Proposition 1. Suppose that after some history, $\big(\tilde{y}^{j'+1}, \ldots, \tilde{y}^{n-1}\big)$, the optimal strategy is for r_1 to be the next to exit according to the indifference condition,

$$h^R\big(s_1, s_2, t^{*j'}, \ldots, t^{*j'}, \tilde{y}^{j'+1}, \ldots, \tilde{y}^{n-1}\big) = h^R\big(t^{*j'}, \underline{x}, t^{*j'}, \ldots, t^{*j'}, \tilde{y}^{j'+1}, \ldots, \tilde{y}^{n-1}\big).$$

Then, for any history, $\big(y^j, \ldots, y^{n-1}\big)$, x-separability implies

$$h^R\big(s_1, s_2, t^{*j'}, \ldots, t^{*j'}, y^j, \ldots, y^{n-1}\big) = h^R\big(t^{*j'}, \underline{x}, t^{*j'}, \ldots, t^{*j'}, y^j, \ldots, y^{n-1}\big).$$

In other words, the same $t^{*j'}$ satisfies the indifference condition following the history, $\big(y^j, \ldots, y^{n-1}\big)$.

Suppose instead that after the history, $\big(\tilde{y}^{j'+1}, \ldots, \tilde{y}^{n-1}\big)$, the optimal strategy is for r_1 to remain in the auction until after a bidder of type $\bar{x}$ would exit. Thus, we have

$$h^R\big(s_1, s_2, \bar{x}, \ldots, \bar{x}, \tilde{y}^{j'+1}, \ldots, \tilde{y}^{n-1}\big) > h^R\big(\bar{x}, \underline{x}, \bar{x}, \ldots, \bar{x}, \tilde{y}^{j'+1}, \ldots, \tilde{y}^{n-1}\big).$$

Then, for any history, $\big(y^j, \ldots, y^{n-1}\big)$, x-separability implies

$$h^R\big(s_1, s_2, \bar{x}, \ldots, \bar{x}, y^j, \ldots, y^{n-1}\big) > h^R\big(\bar{x}, \underline{x}, \bar{x}, \ldots, \bar{x}, y^j, \ldots, y^{n-1}\big). \blacksquare$$

PROOF OF PROPOSITION 2.

Suppose that the existence of the ring is common knowledge.

We have already shown in Proposition 1 that if the bidders are choosing the same strategy that they choose with the secret ring, the ring's strategy is a best response ex post for every signal realization. To show that we have an equilibrium that is without regret, we will now show that the bidders do not have a profitable deviation for any realization of the signals. All signal realizations fall into one of the following cases.

Case 1: The ring wins and r_1 is pretending to be a bidder of type t^{*1} solving the indifference condition,

$$h^R\big(s_1, s_2, t^{*1}, y^2, \ldots, y^{n-1}\big) = h^R\big(t^{*1}, \underline{x}, t^{*1}, y^2, \ldots, y^{n-1}\big). \tag{A30}$$

If a bidder were to deviate to win the auction, depending on who is the deviator, the price paid would be at least as large as the right side of (A30), and therefore at least

$$h^R\big(s_1, s_2, t^{*1}, y^2, \ldots, y^{n-1}\big).$$

Since $t^{*1} \geq y^1$ holds, this expression is weakly less than $h^R\big(s_1, s_2, y^1, y^2, \ldots, y^{n-1}\big)$, which is the value of the object. Therefore, the deviation cannot be profitable.

Case 2: The ring wins and r_1 is willing to bid up to $h^R\big(s_1, s_2, \bar{x}, y^2, \ldots, y^{n-1}\big)$. If a bidder were to deviate to win the auction, depending on who is the deviator, the price paid would be at least $h^R\big(s_1, s_2, \bar{x}, y^2, \ldots, y^{n-1}\big)$, which is weakly greater than the value of the object, $h^R\big(s_1, s_2, y^1, y^2, \ldots, y^{n-1}\big)$.

Case 3: The ring exits with one bidder remaining. Then the price paid by the holder of signal y^1 is $h^R\big(t^{*1}, \underline{x}, t^{*1}, y^2, \ldots, y^{n-1}\big)$, where t^{*1} solves (A30). Since $y^1 \geq t^{*1}$ must hold, the left side of (A30) is weakly less than the value of the object, $h^R\big(s_1, s_2, y^1, y^2, \ldots, y^{n-1}\big)$, so the the holder of signal y^1 does not want to deviate. If another bidder were to deviate to win the auction, depending on who is the deviator, the price would be at least $h^R\big(t^{*1}, \underline{x}, y^1, y^1, y^3, \ldots, y^{n-1}\big)$, which, by Assumption TS, equals

$$h^R\big(s_1, s_2, y^1, y^1, y^3, \ldots, y^{n-1}\big). \tag{A31}$$

Expression (A31) is weakly greater than the value of the object, $h^R\big(s_1, s_2, y^1, y^2, \ldots, y^{n-1}\big)$, so the deviation is not profitable.

Case 4: The ring exits after some history, $\big(y^{j+1}, \ldots, y^{n-1}\big)$, where we have $j > 2$. Then r_1 exits at the price, $h^R\big(t^{*j}, \underline{x}, t^{*j}, \ldots, t^{*j}, y^{j+1}, \ldots, y^{n-1}\big)$, where t^{*j} solves (3). The holder of signal y^1 pays the price, $h^R\big(t^{*1}, \underline{x}, y^2, y^2, \ldots, y^{n-1}\big)$. Since we have $y^2 \geq t^{*j}$, Assumption TS and (3) imply

$$h^R\big(t^{*j}, \underline{x}, y^2, y^2, \ldots, y^{n-1}\big) = h^R\big(s_1, s_2, y^2, y^2, \ldots, y^{n-1}\big). \tag{A32}$$

Therefore, the price is weakly less than the value of the object, so the winner does not want to deviate. If another bidder were to deviate to win, the price would be at least $h^R\big(t^{*j}, \underline{x}, y^1, y^1, y^3, \ldots, y^{n-1}\big)$, which by Assumption TS and (A32) is equal to $h^R\big(s_1, s_2, y^1, y^1, y^3, \ldots, y^{n-1}\big)$. Therefore, the price is weakly greater than the value, so the deviation is not profitable. ∎

PROPOSITION (Levin and Reiss 2020, with permission): *Consider a direct mechanism that satisfies the following properties:*

(1) *If there is a unique highest signal, then its holder wins the object.*[26]

(2) *A bidder who doesn't get the object pays nothing.*

(3) *Truthful reporting is an ex post equilibrium.*

Then, the bidder who wins the object pays according to the "Sophi" rule.

PROOF:

Fix an agent i and a vector of signals of all other agents x_{-i}, and until the end of the proof, simplify by having $T = T_{-i} = \max\{x_{-i}\}$, the highest signal of all agents other than i. Consider two different realizations of i's signal, $x_i > x_i'$, where $x_i' > T$. Property number (1) implies that agent i wins the object in both realizations. It follows from property number (3) that i pays the same price in both cases, since otherwise she would have an incentive to misreport her signal to pay the lower price of the two. Thus, as long as $x_i > T$, the price that i pays is constant in x_i, and we denote it by c. (Obviously, c may depend on the vector x_{-i}, but since we fixed it for the entire proof, we omit it from the notation.) Now, if $x_i > T$ and i reports truthfully, then she wins the object and her utility is $E[V | x_i, x_{-i}] - c$. If instead i lies and reports some $\tilde{x}_i < T$, then she doesn't win the object (by property (1)), and by property (2), her utility is 0. Thus, by property (3) we must have $E[V | x_i, x_{-i}] - c \geq 0$ for every $x_i > T$. Suppose now that $x_i < T$ and reporting truthfully results in losing and earning 0. If instead i misreports with $\tilde{x}_i > T$, then her utility is $E[V | x_i, x_{-i}] - c$. Again from property (3) it must be the case that $0 \geq E[V | x_i, x_{-i}] - c$. We established that $\forall \epsilon > 0$, $E[V | x_i = T + \epsilon, x_{-i}] \geq c \geq E[V | x_i = T - \epsilon, x_{-i}]$. Thus, in the limit as $\epsilon \to 0$ yields, $c = E[V | x_i = T, x_{-i}]$, the proof is complete. ∎

PROOF OF PROPOSITION 6:

(i) For *given* $(y^2, \ldots, y^{n-1})$, the payoff in the Sophi auction for the ring with signals (s_1, s_2) is given by

$$\int_{y^2}^{t} \left[\hat{V}^R(s_1, s_2, y^1) - V^R(y^1)\right] dF_{Y^1|Y^2,\ldots,Y^{n-1},S_1,S_2}\left(y^1 | y^2, \ldots, y^{n-1}, s_1, s_2\right).$$

This is exactly the payoff received in the English auction, given $(y^2, \ldots, y^{n-1}, s_1, s_2)$, if the ring exits when the holder of y^1 is revealed to have signal $y^1 \geq t$. Thus, if the optimal misbehavior for the ring in the Sophi auction, with signals (s_1, s_2), is to bid $t^{*SophiR}$, the same payoff can be achieved in the English auction by exiting out when the holder of y^1 is revealed to have signal $y^1 \geq t^{*SophiR}$ for each realization of $(y^2, \ldots, y^{n-1})$. It follows that the ring receives a weakly higher payoff in the English auction than in the Sophi auction.

[26] Extending to allow for random tiebreaking is simple but does not add much.

If, for the ring with signals (s_1, s_2), the optimal misbehavior can be achieved with a constant t^* that does not depend on $(y^2, \ldots, y^{n-1})$, then bidding t^* in the Sophi auction allows the ring to receive the same payoff as in the English auction for each realization of $(y^2, \ldots, y^{n-1})$. On the other hand, if t^* depends nontrivially on $(y^2, \ldots, y^{n-1})$, then requiring the ring in the English auction to choose a constant t, independent of $(y^2, \ldots, y^{n-1})$, leads to some suboptimal choices and a strict reduction in payoff.

(ii) From the Corollary to Proposition 1, the optimal misbehavior of the ring in the English auction is for r_1 to pretend to be type t^*, where t^* depends on (s_1, s_2) but not on the history. From the proof of part (i), it follows that $t^* = t^{*SophiR}$ holds, so the outcomes are the same in the English auction and the Sophi auction. Since misbehavior is without regret in the English auction, it is without regret in the Sophi auction. ∎

PROOF OF PROPOSITION 7:

Comparing (7) and (12), it is clear that the auctioneer in the English auction can achieve the Sophi payoff by setting t^*, for each realization of $(y^2, y^3, \ldots, y^n)$, equal to the auctioneer's Sophi bid, which we denote as t^{*Sophi}. To show that the English is strictly preferred, it suffices to show that there are some realizations of $(y^2, y^3, \ldots, y^n)$, occurring with positive probability, such that t^{*Sophi} is not a solution to (7).

First, we show that $t^{*Sophi} < \bar{x}$ holds. Differentiating (12) with respect to t, and evaluating at $t = \bar{x}$, yields

$$\text{(A33)} \qquad \int_{y^2, y^3, \ldots, y^n} \left[\hat{V}(\bar{x}, s) - V(\bar{x})\right] dF_{Y^2, \ldots, Y^n | S}\left(y^2, y^3, \ldots, y^n \,|\, s\right).$$

Since for $s < \bar{x}$ the term in brackets in (A33) is negative for each $(y^2, y^3, \ldots, y^n)$, the auctioneer strictly increases profits by reducing t below $\bar{x}$, so $t^{*Sophi} < \bar{x}$ holds.

Thus, there must be a positive probability that $y^2 > t^{*Sophi}$ holds. In all such cases, however, t^{*Sophi} is not a solution to (7), because $t^* \geq y^2$ must hold. Intuitively, the auctioneer should not exit before the bidder with y^2 exits. ∎

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