Optimal Monopoly Mechanisms with Demand Uncertainty

James Peck and Jeevant Rampal*

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Abstract

This paper analyzes a monopoly firm's profit maximizing mechanism in the following context. There is a continuum of consumers with a unit demand for a good. The distribution of the consumers' valuations is given by one of two possible demand distributions/states: high demand or low demand. The consumers are uncertain about the demand state, and they update their beliefs after observing their own valuation for the good. The firm is uncertain about the demand state, but infers the demand state when the consumers report their valuations. The firm's problem is to maximize profits by choosing an optimal mechanism among the class of anonymous, deterministic, direct revelation mechanisms that satisfy interim incentive compatibility and ex-post individual rationality. We show that, under certain sufficient conditions, the firm's optimal mechanism is to set the monopoly price in each demand state. Under these conditions, Segal's (2003) optimal ex-post mechanism is robust to relaxing ex-post incentive compatibility to interim incentive compatibility.

Keywords: Monopoly mechanism; Correlated valuations; Bayesian incentive compatibility; Ex-post individual rationality.
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*James Peck: Department of Economics, The Ohio State University, 1945 N High Street, Columbus, OH 43210, USA; peck.33@osu.edu. Jeevant Rampal: Economics Area, Indian Institute of Management Ahmedabad, Wing 14-J, IIMA Old Campus, Ahmedabad, Gujarat-380015, India; jeevantr@iima.ac.in. We are grateful to Yaron Azrieli, Gabriel Carroll, PJ Healy, Dan Levin, Debasis Mishra, and Dan Vincent for helpful conversations or suggestions.

1 Introduction

Consider a monopoly firm that is trying to maximize profit in the presence of aggregate demand uncertainty. In each demand state, there is a distribution of consumer valuations, or a demand curve, where each consumer is negligible relative to the market and desires at most one unit of the good. It is well known that in the absence of demand uncertainty there is no scope for price discrimination and the monopolist's optimal mechanism is to charge the same optimal monopoly price to all consumers.[1]

On the other hand, when aggregate demand uncertainty is present, and therefore consumer valuations are correlated, Crémer and McLean (1985, 1988) show that, under certain conditions, the monopolist can extract the entire consumer surplus using a Bayesian incentive compatible (BIC) and interim individually rational (IIR) mechanism. These mechanisms involve consumers participating in side bets with the firm, where consumers must make/receive huge payments depending on the outcome of the bet. In particular, these payments can be far more than the consumers' valuation of the good being sold. However, in many monopoly situations, a consumer cannot be prevented from walking away from a deal when asked to pay more than his/her valuation of the good. Segal (2003) accounts for this by requiring ex-post individual rationality (EIR) along with ex-post incentive compatibility. Segal (2003) shows that the optimal mechanism that satisfies these conditions involves state-by-state monopoly pricing (SBSMP).[2]

In the private values setting, Segal's (2003) ex-post incentive compatibility is equivalent to dominant strategy incentive compatibility (DIC). Imposing DIC rules out many forms of price discrimination. For example, given a profile of reported types, DIC requires any anonymous and deterministic mechanism to charge the same price from any consumer receiving the good. Therefore, to capture the situation where consumers cannot be charged more than their valuation of the good but price discrimination is possible, we impose EIR like Segal (2003), but relax DIC to BIC.

We assume that there are two distributions, high and low, from which valuations

[1]Myerson (1981) solves the optimal auction problem, and Bulow and Roberts (1989) show that the monopoly problem is equivalent to the Myerson setting when consumer valuations are independent (in which case the demand curve would be known in a large economy). See also Harris and Raviv (1981) and Riley and Zeckhauser (1983).

[2]Segal (2003) analyzes the model with a finite number of consumers and several possible distributions from which valuations are independently drawn. He shows that, when the number of buyers is large, the optimal mechanism converges to SBSMP.

are independently drawn. Appealing to the law of large numbers, these distributions also correspond to the two possible realized demand curves. We first impose regularity conditions on the demand process. Under these assumptions, and three additional conditions, we show that SBSMP is optimal among all anonymous, deterministic, EIR, and BIC mechanisms. One of the three additional conditions is similar to a "single crossing" condition for beliefs. Peck and Rampal (2019) provide a counterexample to SBSMP when this single crossing property does not hold. The other two conditions are concavity and a restriction over beliefs.

When the conditions of the SBSMP Proposition are satisfied, it follows that Segal's characterization of SBSMP as the optimal ex-post mechanism is robust to relaxing DIC to BIC.[3] To our knowledge, this is the first BIC-DIC equivalence result for environments with correlated types and EIR. Crémer and McLean (1988) provide such a result under IIR and a spanning condition, which could require large payments from consumers.[4] Mookherjee and Reichelstein (1992), Manelli and Vincent (2010), Gershkov et al. (2013), and Kushnir and Liu (2019) all consider BIC-DIC equivalence with independent types.

Krähmer and Strausz (2015) and Bergemann et al. (2017) consider a sequential screening problem with one buyer, who first observes the distribution from which his valuation is drawn, and later observes his valuation. When EIR is imposed, the seller does not engage in screening if a monotonicity condition is satisfied, instead setting a take-it-or-leave-it price. Although the model is quite different from ours, the no-screening result is similar to SBSMP, in the sense that all consumers are offered the same deal.

This paper is related to the literature on product bundling (see Manelli and Vincent (2006, 2007)) since consumption of the same good in different states of nature can be interpreted as different commodities. The counterexample to SBSMP in Peck and Rampal (2019) is similar to an example in Carroll (2017). This paper is also related to Daskalakis et al. (2016), who model uncertainty about the "item type" and show that the optimal mechanism is equivalent to an optimal multi-item mechanism, where a buyer's valuation for item s is her valuation for the good in state s multiplied by the probability of state s (i.e. the valuation of state-s contingent consumption of

[3]Specifically, we are referring to the case in which the number of consumers approaches infinity, there are two possible demand distributions, and we consider deterministic mechanisms.

[4]See also Kushnir (2015).

the good). However, the proof of Theorem 1 in Daskalakis et al. (2016) relies on the assumption that the probability of each item type is independent of the buyer's type, an assumption that is not satisfied in our setting. Also, under the bundling interpretation of multiple state-contingent commodities, EIR is not satisfied if payments are made and the good is not consumed in all states. Moreover, aggregate uncertainty causes beliefs about the state to differ across consumer types, which breaks the interpretation of our monopolist's problem as a standard bundling problem.

In Section 2, the model is laid out and some preliminary analysis is conducted. Section 3 contains the main result about SBSMP. Section 4 contains some concluding remarks. Proofs are contained in the appendix.

2 Model

A risk neutral, profit-maximizing monopoly firm faces a continuum of consumers with a unit demand for a good. The firm has zero marginal cost of production. There are two demand states, low and high.[5] The probability of the low state is α_1 and the probability of the high state is $\alpha_2 = 1 - \alpha_1$. For $i = 1, 2$, consumers' valuations in state i are distributed over $V = [\underline{v}, \overline{v}]$ according to the demand distribution $D_i(\cdot)$. In particular, $D_i(v)$ is the measure of consumers with valuation greater than v in state i. Think of the following process. First, nature selects the demand state, according to the probabilities α_1 and α_2. Then, out of the measure of "potential" consumers, C, nature selects a consumer to be active in state i with probability $D_i(\underline{v})/C$. Finally, for the set of selected active consumers, nature independently selects valuations giving rise to the distribution, $D_i(v)$.[6] Consumers and the firm know the structure of demand, but not the realization.

Because there is aggregate demand uncertainty, a consumer's valuation provides her with significant information about the demand state. For a consumer whose valuation is v, her updated belief about the realized demand state is:

$$Pr(\text{Demand state is } i|\text{own valuation is } v) = \frac{\alpha_i(-D_i'(v))}{\alpha_1(-D_1'(v)) + \alpha_2(-D_2'(v))}. \quad (1)$$

[5] Conditions 1, 2, and 3 (below) imply that the monopoly price in state 1 is strictly less than the monopoly price in state 2, so we refer to state 1 as the low state and state 2 as the high state.

[6] See Peck (2017) for more details and the derivation of (1) according to Bayes' rule.

In what follows, we assume that demand is twice-continuously differentiable. The density of the downward sloping demand distribution at valuation v is denoted as $(-D_i'(\cdot))$ in state $i = 1, 2$. We will assume that $(-D_i'(\cdot)) > 0$ holds at all $v \in V$ for $i = 1, 2$.

We consider direct revelation mechanisms satisfying BIC and EIR. According to the revelation principle, consumers report truthfully, without loss of generality. We appeal to the law of large numbers to conclude that the firm is able to infer the demand state perfectly from the profile of reported types. We restrict attention to deterministic mechanisms that specify, for each state, which valuation types consume and the amount paid by each type that consumes.[7] The requirement that the payment scheme satisfy ex-post individually rationality implies that the firm is not allowed to charge more than the reported valuation in any demand state and that if a consumer is not given the good in some demand state, then the firm cannot elicit any positive payment from that consumer in that demand state. To summarize, the firm's problem is to maximize its expected revenue using an anonymous, deterministic, interim incentive compatible, and ex-post individually rational mechanism, when facing a continuum of consumers who update about the demand state based on their private valuations. We state this problem formally in the next sub-section.

2.1 The Monopoly Firm's Problem

Let $x_i(v)$ denote the probability with which the monopoly firm gives the good to valuation v in state i. As noted before, we will restrict ourselves to the case where the firm sells the good to v with probability 1 or 0. So, $x_i(v) \in \{0, 1\}$ for all v and $i = 1, 2$. Let $t_i(v)$ denote the payment required from v given that the demand state is i, conditional on v purchasing the good in state i. Thus, a mechanism offered by the monopoly firm is as follows:

[7] Thus, we require anonymous mechanisms and rule out randomized mechanisms that specify a probability of consuming in state i. We are unable to solve the model without this restriction, but it may limit the firm's profit opportunities, as shown by Peck and Rampal (2019). We also rule out introducing randomness indirectly, by allowing consumption to depend on features of the profile of reports other than the inferred state.

$$
\begin{aligned}
x_i(v) &\in \{0,1\},\ \forall v \in V,\ i = 1,2, \\
0 \le t_i(v) &\le v,\ \forall v \in V,\ i = 1,2.
\end{aligned} \tag{2}
$$

For a given mechanism offered by the monopoly firm, let V_i denote the subset of valuations of V who consume only in state i. That is, for $i = 1,2$ and $j \neq i$, $v \in V_i$ if and only if $x_i(v) = 1$ and $x_j(v) = 0$ hold. So, by the ex-post individual rationality (EIR) condition, for valuations in V_i, $t_i(v)$ can be positive but must be less than v, and $t_j(v)$ must be 0. Let V_{12} denote the subset of valuations of V who consume in both states, in which case $x_1(v) = 1$ and $x_2(v) = 1$ hold. So, by the EIR condition, for valuations in V_{12}, both $t_1(v)$ and $t_2(v)$ can be positive but must be less than v. Let $V_\varnothing$ denote the subset of valuations of V which are not give the good in either state. That is, $V_\varnothing = V - [V_1 \cup V_2 \cup V_{12}]$.

We can state the simplified firm's problem as follows. The firm chooses the sets V_1, V_2, and V_{12}, and the functions $t_i : V \to [0, \overline{v}]$, for $i = 1,2$ to solve:

$$
\begin{aligned}
\max \textstyle\int_{V_{12}} [t_1(v)\alpha_1(-D_1'(v)) + t_2(v)\alpha_2(-D_2'(v))]dv + \int_{V_1} t_1(v)\alpha_1(-D_1'(v))dv \\
+ \textstyle\int_{V_2} t_2(v)\alpha_2(-D_2'(v))dv.
\end{aligned} \tag{3}
$$

Subject to (i) ex-post individual rationality, (henceforth EIR, given by (2)); and (ii) interim/Bayesian incentive compatibility (henceforth BIC, given below) [8]

$$
\begin{aligned}
(v - t_1(v))x_1(v)\alpha_1(-D_1'(v)) + (v - t_2(v))x_2(v)\alpha_2(-D_2'(v)) \ge \\
(v - t_1(\widehat{v}))x_1(\widehat{v})\alpha_1(-D_1'(v)) + (v - t_2(\widehat{v}))x_2(\widehat{v})\alpha_2(-D_2'(v));\ \forall v,\ \widehat{v} \in V.
\end{aligned} \tag{4}
$$

2.2 Conditions and Preliminary Results

In this subsection we specify conditions on the demand process and establish preliminary results. We start with regularity conditions for the two demand states, i.e. the "maintained assumptions" about demand. Then Fact 1 follows from BIC.

Condition 1 (*regularity*). (i) $D_1(v)$ and $D_2(v)$ are twice continuously differentiable.

[8]The BIC condition is stated after canceling $[\alpha_1(-D_1'(v)) + \alpha_2(-D_2'(v))]$ from the denominator on both sides of the inequality.

(ii) Demand is strictly downward sloping everywhere, i.e., $D_i'(v) < 0$ holds for all $v \in V$ and $i \in \{1, 2\}$.

Fact 1: *If the firm's mechanism satisfies BIC then $t_i(v) = t_i(\widehat{v})$ must hold for all v, $\widehat{v}$ in V_i, where $i = 1$, 2.*

Proof of Fact 1. For $i = 1, 2$ and $j \neq i$, if v, $\widehat{v} \in V_i$, then $x_i(v) = x_i(\widehat{v}) = 1$ and $x_j(v) = x_j(\widehat{v}) = 0$ hold. Thus, the BIC condition (4) implies

$$(v - t_i(v))\alpha_i(-D_i'(v)) \geq (v - t_i(\widehat{v}))\alpha_i(-D_i'(v)),$$

which implies $t_i(v) \leq t_i(\widehat{v})$. Similarly, the BIC condition for $\widehat{v}$ with respect to v implies $t_i(\widehat{v}) \leq t_i(v)$. So Fact 1 holds. ■

Lemma 1 provides a first step towards characterizing the firm's optimal mechanism. Let v_i^* denote the infimum valuation of the set V_i for $i \in \{1, 2, 12\}$.[9]

Lemma 1: *At the monopoly firm's optimal EIR and BIC mechanism, V_{12} is non-empty.*

The proof of Lemma 1 is given in the Appendix.

Condition 2 (*information effect*). (i) $Z(v) \equiv \frac{(-D_1'(v))}{(-D_2'(v))}$ is strictly decreasing in v for all $v \in V$. That is, $Z'(v) < 0$ holds for all $v \in V$.

Condition 2 specifies the information effect. Note that $\frac{\alpha_1}{\alpha_2}Z(v)$ is the ratio of the probability a type v assigns to state 1 to the probability assigned to state 2. Thus, Condition 2 says that, the greater the valuation of a consumer, the greater the probability she assigns to the high demand state.

The next step is to characterize the sets V_1, V_2 and V_{12}. In particular, the question is, given Conditions 1 and 2, whether the requirement that the firm's mechanism satisfy BIC and EIR constraints implies that the firm's mechanism must order and structure the sets V_1, V_2 and V_{12} in a particular manner. Lemma 2 addresses this question.

Lemma 2: *Let v_i denote the arbitrary valuation of the set V_i for $i \in \{1, 2, 12\}$. Given Conditions 1 and 2, if the firm's mechanism satisfies the BIC and EIR constraints*

[9] The infima of these sets are well defined because they are bounded subsets of $\mathcal{R}$.

and the appropriate sets are non-empty, then we must have (i) $v_1 < v_{12}$, and (ii) at the firm's optimal mechanism, $v_2^* < v_{12}^*$ must hold.
The proof of Lemma 2 is given in the Appendix.

Given Lemma 1, it follows that the monopoly firm chooses a mechanism from among the following possible types of mechanisms: (1) with only V_{12} non-empty; (2) with V_1 and V_{12} non-empty, and V_2 empty; (3) with V_2 and V_{12} non-empty, and V_1 empty; (4) with V_1, V_2, and V_{12} all non-empty. In general, when the profit maximizing monopoly price in the two demand states is different, it can be shown that the firm can improve upon a mechanism with just V_{12} non-empty (we show this in the proof of Proposition SBSMP). Thus, the main question is going to be: which among (2)-(4) is optimal for the firm?

Note that, if the firm's mechanism gives the good to valuation v in state i or state j or both, then BIC implies that valuations greater than v are also given the good in some state; because otherwise valuations greater than v can report their valuation as v, get the good in whichever state v gets the good, and make a payment less than v (because, by EIR, the firm cannot charge more than the reported valuation to v), and earn a strictly positive surplus. Fact 1 implies $t_i(v) = t_i(v_i^*)$ for $i = 1, 2$. Lemma 3 (below) further specifies the payment scheme.

Lemma 3: *If V_1 and V_2 are both non-empty, and $v_i^* < v_j^*$ holds; or, if only V_i and V_{12} are nonempty with $v_i^* < v_{12}^*$, then BIC, EIR and Conditions 1 and 2 imply that $t_i(v_i^*) = v_i^* = t_i(v_i)$ holds for all $v_i \in V_i$.*
Proof of Lemma 3. The fact that $t_i(v_i)$ is constant for all $v_i \in V_i$ has been established by Fact 1. So, if V_1 and V_2 are both non-empty, and $v_i^* < v_j^*$ holds; or, if only V_i and V_{12} are nonempty with $v_i^* < v_{12}^*$, then valuations less than v_i^* are not given the good in either state. To see why $t_i(v_i^*) = v_i^*$ holds, note that for any v such that $v < v_i^*$ holds, $t_i(v_i^*) \geq v$ must hold to satisfy BIC of v. So $t_i(v_i^*) \geq v_i^*$ must hold. The EIR constraint for v_i^* implies $t_i(v_i^*) \leq v_i^*$. Putting these two statements together, we have $t_i(v_i^*) = v_i^*$.■

3 The Optimal Mechanism

Lemma 4 (below) shows that in the firm's optimal mechanism, it cannot be the case that V_1, V_2, and V_{12} are all non-empty. However, to prove Lemma 4, we require that demand be concave in both states.

Condition 3 (*concave demand*). Demand is strictly concave, i.e., $D_i''(v) < 0$ holds for all $v \in V$ and $i \in \{1, 2\}$.

Concavity is used in Lemma 4 (below) to establish that the revenue function $pD_i(p)$ is concave for $i = 1, 2$, for which concavity is a sufficient condition, but not necessary. For $i = 1, 2$, let p_i^m be the profit maximizing monopoly price in demand state i. Concavity is also used as a sufficient condition (along with Conditions 1 and 2) to establish that p_1^m is lower than p_2^m (in Fact 2 below).

Fact 2: Conditions 1, 2, and 3 imply

$$\frac{D_2(v^*)}{(-D_2'(v^*))} > \frac{D_1(v^*)}{(-D_1'(v^*))} \tag{5}$$

for all $v^* \in V$. And (5) implies p_1^m, which solves $p_1^m = -\frac{D_1(p_1^m)}{D_1'(p_1^m)}$, is strictly lower than p_2^m, which solves $p_2^m = -\frac{D_2(p_2^m)}{D_2'(p_2^m)}$.
The proof of Fact 2 is given in the Appendix.

To show Lemma 4, we also require that after observing their respective valuations, all types agree (as per their beliefs) about which state is more likely.

Condition 4 (*agreement over the more likely state*). Either (i) $\frac{\alpha_1}{\alpha_2} Z(v) < 1$ holds for all $v \in V$ or (ii) $\frac{\alpha_1}{\alpha_2} Z(v) > 1$ holds for all $v \in V$.

Lemma 4: *Suppose Conditions 1-4 hold, then the optimal mechanism cannot have V_1, V_2, and V_{12} all non-empty.*
The proof for Lemma 4 is given in the Appendix.

Condition 4 is needed in the proof of Lemma 4 to show that the profit from any mechanism with V_1, V_2, and V_{12} non-empty is bounded above by a mechanism which may not satisfy BIC and EIR, but where V_1, V_2, and V_{12} are adjacent connected intervals; i.e. the bounding mechanism has $V_i = [v_i^*, v_j^*)$, $V_j = [v_j^*, v_{12}^*)$, and $V_{12} = [v_{12}^*, \overline{v}]$ for either $i = 1$ and $j = 2$, or $i = 2$ and $j = 1$. In particular, even with Conditions 1-4, there is no guarantee that all BIC and EIR mechanisms with V_1, V_2, and V_{12} non-empty will have the interval property. However, the bounding mechanism satisfies the interval property, which allows us to express the upper bound profit in terms of the infima v_1^*, v_2^*, and v_{12}^*. We then show that the upper bound profit can be strictly increased by switching to a mechanism with only V_1 and V_{12} non-empty, which also satisfies BIC and EIR.

Lemma 4 yields that under its conditions, in the optimal mechanism, either only V_{12} is non-empty, or only V_1 and V_{12} are non-empty, or only V_2 and V_{12} are non-empty. The SBSMP Proposition below establishes that under Conditions 1-4, only V_1 and V_{12} are non-empty in the optimal mechanism. This in-turn yields that the firm's optimal mechanism is to set the monopoly price in each demand state, i.e. the firm's optimal mechanism is state-by-state monopoly pricing (SBSMP).

Proposition (SBSMP): *If Conditions 1, 2, 3, and 4 hold, then within the class of deterministic BIC and EIR mechanisms, the monopoly firm's optimal mechanism is state-by-state monopoly pricing (SBSMP). The SBSMP mechanism is as follows:*

$$\begin{cases} t_1(v) = p_1^m, & \forall\, v \geq v_1^* \\ t_2(v) = p_2^m, & \forall\, v \geq v_{12}^* \\ x_1(v) = 1 \ \forall v \geq p_1^m, & x_1(v) = 0 \ \forall v < p_1^m, \\ x_2(v) = 1 \ \forall v \geq p_2^m, & x_2(v) = 0 \ \forall v < p_2^m. \end{cases} \tag{6}$$

Proof (summary). By Lemma 4, mechanisms with V_1, V_2, and V_{12} all non-empty are sub-optimal. The proof proceeds by ruling out the possibility that the optimal mechanism can be one with only V_{12} non-empty, or one with only V_2 and V_{12} non-empty. This proves that the optimal mechanism must have only V_1 and V_{12} non-empty. Finally, we show that the optimal mechanism among mechanisms with only V_1 and V_{12} non-empty is SBSMP. The detailed proof is provided in the Appendix.

4 Concluding Remarks

Under certain regularity conditions, we have shown that SBSMP is optimal among all anonymous, deterministic, ex-post IR (EIR), and interim IC (BIC) mechanisms. The result is far from obvious, as illustrated by the counterexample in Peck and Rampal (2019), when the regularity conditions are not satisfied. It would be nice to allow for randomized mechanisms, but much of the Myerson machinery is unavailable and very few results are available in the literature when types are correlated.

5 Appendix

Proof of Lemma 1. We will prove Lemma 1 by contradiction. Suppose at the firm's optimal EIR and BIC mechanism, V_{12} is empty. We will argue that each of the alternatives yields strictly lower profits than profits from an EIR and BIC mechanism with V_{12} non-empty. The alternatives are:
(i) Only V_i non-empty, for $i \in \{1, 2\}$. By BIC and EIR, the firm's profit in this case is bounded above by $\alpha_i v_i^* D_i(v_i^*)$. If instead all $v_i \in V_i$ were given the good in both states (i.e. $V_{12} = V_i$) at price v_i^*, the profit would be $\alpha_i v_i^* D_i(v_i^*) + \alpha_j v_i^* D_j(v_i^*)$, for $i, j = 1, 2$ and $i \neq j$, which is strictly greater than $\alpha_i v_i^* D_i(v_i^*)$, and all EIR and BIC constraints would still be satisfied.
(ii) Only V_i and V_j non-empty, for $i, j = 1, 2$ and $i \neq j$. Without loss of generality, let $v_i^* \leq v_j^*$ hold. By BIC and EIR, using Fact 1, the firm's profit in this case is strictly lower than $\alpha_i v_i^* D_i(v_i^*) + \alpha_j v_j^* D_j(v_j^*)$ since V_i and V_j are disjoint sets by definition. However $\alpha_i v_i^* D_i(v_i^*) + \alpha_j v_j^* D_j(v_j^*)$ is exactly the profit if instead of only V_i and V_j non-empty, a different mechanism is used; one where only V_i and V_{12} are non-empty, and (i) V_i equals $[v_i^*, v_j^*)$, which is empty if $v_j^* = v_i^*$ holds, and $t_i(v_i) = v_i^*$ holds for all $v_i \in V_i$, (ii) V_{12} equals $[v_{12}^*, \overline{v}]$ with $v_{12}^* = v_j^*$, and $t_i(v_{12}) = v_i^*$, $t_j(v_{12}) = v_{12}^*$ hold for all $v_{12} \in V_{12}$. Further, it is straightforward to verify that such a mechanism satisfies all EIR and BIC constraints. ■

Proof of Lemma 2. Proof of (i). First, we will show that $v_1 < v_{12}$ must hold. Suppose not; that is, let $v_1 > v_{12}$ hold (note that we cannot have v_1 equal to v_{12} because a valuation cannot belong to both V_1 and V_{12}). Consider the BIC constraints of v_{12} with respect to v_1 and of v_1 with respect to v_{12}:

$$(v_{12}-t_1(v_{12}))\alpha_1(-D_1'(v_{12}))+(v_{12}-t_2(v_{12}))\alpha_2(-D_2'(v_{12})) \geq (v_{12}-t_1(v_1))\alpha_1(-D_1'(v_{12}));$$

and

$$(v_1 - t_1(v_1))\alpha_1(-D_1'(v_1)) \geq (v_1 - t_1(v_{12}))\alpha_1(-D_1'(v_1)) + (v_1 - t_2(v_{12}))\alpha_2(-D_2'(v_1)).$$

These can be rewritten as

$$(v_{12} - t_2(v_{12}))\frac{\alpha_2}{\alpha_1 Z(v_{12})} \geq t_1(v_{12}) - t_1(v_1), \tag{7}$$

and

$$t_1(v_{12}) - t_1(v_1) \geq (v_1 - t_2(v_{12}))\frac{\alpha_2}{\alpha_1 Z(v_1)}. \tag{8}$$

Together, (7) and (8) imply:

$$(v_{12} - t_2(v_{12}))\frac{\alpha_2}{\alpha_1 Z(v_{12})} \geq (v_1 - t_2(v_{12}))\frac{\alpha_2}{\alpha_1 Z(v_1)}. \tag{9}$$

Next, note that $v_1 > t_2(v_{12})$ must hold. This is because, by EIR , $v_{12} \geq t_2(v_{12})$ must hold; further, $v_1 > v_{12}$ holds by assumption. Thus, $v_1 > v_{12} \geq t_2(v_{12})$ holds. Note that $v_1 > v_{12}$ implies $(v_1 - t_2(v_{12})) > (v_{12} - t_2(v_{12}))$. Further, by Condition 2, $\frac{1}{Z(v)}$ is increasing for all v. Thus, $\frac{\alpha_2}{\alpha_1 Z(v_1)} > \frac{\alpha_2}{\alpha_1 Z(v_{12})}$ holds, which implies,

$$(v_{12} - t_2(v_{12}))\frac{\alpha_2}{\alpha_1 Z(v_{12})} < (v_1 - t_2(v_{12}))\frac{\alpha_2}{\alpha_1 Z(v_1)},$$

which is a contradiction of (9). Thus it must be the case that $v_{12} > v_1$ holds.

Proof of (ii). The aim is to show that $v_2^* < v_{12}^*$ must hold in the firm's optimal mechanism. The proof is by contradiction, that is, suppose $v_{12}^* \leq v_2^*$ holds. Given Lemma 1 and Lemma 2(i), if $v_{12}^* \leq v_2^*$ holds then either (a) $v_1^* < v_{12}^* \leq v_2^*$ holds and V_1, V_2, and V_{12} are all non-empty, or (b) only V_2 and V_{12} are non-empty with $v_{12}^* \leq v_2^*$. We will rule out both cases.

Ruling out (a): Suppose mechanism A is an arbitrary mechanism with $v_1^* < v_{12}^* \leq v_2^*$ and V_1, V_2, and V_{12} all non-empty. In mechanism A, by the BIC of $v < v_1^*$ with respect to v_1^* (which implies $t_1(v_1^*) \geq v_1^*$) and by the EIR of v_1^* (which implies $t_1(v_1^*) \leq v_1^*$), we must have $t_1(v_1^*) = v_1^*$. Second, by Fact 1, all $v_1 \in V_1$ and $v_2 \in V_2$ must be charged v_1^* and $t_2(v_2^*)$, respectively. Third, by the BIC of $v \in V_{12}$ with respect to v_{12}^* we have

$$\begin{gathered}(v - t_1(v))\alpha_1(-D_1'(v)) + (v - t_2(v))\alpha_2(-D_2'(v)) \leq \\ (v - t_1(v_{12}^*))\alpha_1(-D_1'(v)) + (v - t_2(v_{12}^*))\alpha_2(-D_2'(v)),\end{gathered}$$

which can be rearranged to

$$t_1(v)\alpha_1(-D_1'(v)) + t_2(v)\alpha_2(-D_2'(v)) \leq t_1(v_{12}^*)\alpha_1(-D_1'(v)) + t_2(v_{12}^*)\alpha_2(-D_2'(v)). \tag{10}$$

The implication of (10) is that the profit from $v \in V_{12}$, given on the left side of (10) (see (3)), is bounded above by charging v the same payment scheme as offered to v_{12}^* (this follows from (10)). Fourth, by Lemma 2(i), $V_1 = [v_1^*, v_{12}^*)$ holds since no type with valuation above v_{12}^* can be in V_1. Given these four features of mechanism A, the profit from mechanism A is bounded above by

$$\pi_A = \begin{array}{c} \alpha_1[D_1(v_1^*) - D_1(v_{12}^*)]v_1^* + \alpha_1[D_1(v_{12}^*) - D_1(v_2^*)]t_1(v_{12}^*) \\ +\alpha_2[D_2(v_{12}^*) - D_2(v_2^*)]t_2(v_{12}^*) + \alpha_1 t_1(v_{12}^*) \int_{[v_2^*,\overline{v}]-V_2}(-D_1'(v))dv \\ +\alpha_2 t_2(v_{12}^*) \int_{[v_2^*,\overline{v}]-V_2}(-D_2'(v))dv + \alpha_2 t_2(v_2^*) \int_{V_2}(-D_2'(v))dv. \end{array} \tag{11}$$

In (11) we must have $t_2(v_2^*) \leq t_2(v_{12}^*)$. To see this, note that the BIC of v_2^* with respect to v_{12}^* yields:

$$\begin{array}{c} (v_2^* - t_2(v_2^*))\alpha_2(-D_2'(v_2^*)) \geq \\ (v_2^* - t_1(v_{12}^*))\alpha_1(-D_2'(v_2^*)) + (v_2^* - t_2(v_{12}^*))\alpha_2(-D_2'(v_2^*)), \end{array} \tag{12}$$

and since $t_1(v_{12}^*) \leq v_{12}^* \leq v_2^*$ holds (first inequality holds by EIR and second holds by assumption of mechanism A), (12) implies $t_2(v_2^*) \leq t_2(v_{12}^*)$.

Now consider an alternative mechanism, labeled mechanism B, where only the following changes are made to mechanism A: V_2 is set to be empty, all valuations in V_2 are allocated to V_{12}, and all valuations in V_{12} are charged $t_i(v_{12}^*)$ in state-i for $i = 1, 2$ (the payment scheme offered to v_{12}^* in mechanism A). In mechanism B the profit is

$$\pi_B = \alpha_1[D_1(v_1^*) - D_1(v_{12}^*)]v_1^* + \alpha_1 D_1(v_{12}^*)t_1(v_1^*) + \alpha_2 D_2(v_{12}^*)t_2(v_{12}^*), \tag{13}$$

which is clearly greater than π_A since valuations in V_2, which are in V_{12} for mechanism B, are charged a weakly greater amount in state 2 and the firm gets strictly positive revenue from them in state 1 as well. To complete the argument note that if BIC and EIR were satisfied in mechanism A, then they continue to hold in mechanism B. We first check that the BIC of all types in V_{12} with respect to types in V_1 is satisfied. Since the BIC of v_{12}^* with respect to v_1^* is satisfied in mechanism A, we have

$$\begin{array}{c} (v_{12}^* - t_1(v_{12}^*))\alpha_1(-D_1'(v_{12}^*)) + (v_{12}^* - t_2(v_{12}^*))\alpha_2(-D_2'(v_{12}^*)) \geq \\ (v_{12}^* - v_1^*)\alpha_1(-D_1'(v_{12}^*)), \text{ or} \end{array}$$

$$v_{12}^* \geq (t_1(v_{12}^*) - v_1^*)\tfrac{\alpha_1}{\alpha_2} Z(v_{12}^*) + t_2(v_{12}^*) \tag{14}$$

holds. Replacing v_{12}^* with any $v > v_{12}^*$ in (14) yields

$$v > (t_1(v) - v_1^*)\tfrac{\alpha_1}{\alpha_2} Z(v) + t_2(v), \tag{15}$$

since $t_1(v)$ and $t_2(v)$ are identical to $t_1(v_{12}^*)$ and $t_2(v_{12}^*)$ in mechanism B, and since $Z(v) < Z(v_{12}^*)$ holds because of $v > v_{12}^*$ and Condition 2. (15) can be rearranged to show that the BIC of v with respect to v_1^*, and thereby all types in V_1, is satisfied. The BIC of all $v \in V_1$ with respect to V_{12} continues to hold in mechanism B since all valuations in V_{12} are offered the same payment scheme.

Ruling out (b): Suppose only V_2 and V_{12} are non-empty with $v_{12}^* \leq v_2^*$. We will first rule out $v_{12}^* = v_2^* = v^*$ being compatible with an optimal mechanism. By EIR $t_2(v_2^*) \leq v_2^*$ must hold, and by Fact 1, $t_2(v) = t_2(v_2^*)$ holds for all $v \in V_2$. Further, valuations in V_2 pay 0 in state 1. Following the arguments around (10), under BIC, the firm cannot extract more profit from V_{12} than by charging the same payment scheme to all $v \in V_{12}$. Further, by EIR $t_i(v) \leq v_{12}^*$ holds for $i = 1, 2$. Thus, profit from a mechanism with V_2 and V_{12} non-empty, and $v_{12}^* = v_2^* = v^*$ is bounded above by

$$\int_{[v^*,\overline{v}]-V_2} v^*\{\alpha_1(-D_1'(v)) + \alpha_2(-D_2'(v))\}dv + \int_{V_2} v^*\alpha_2(-D_2'(v))dv, \tag{16}$$

where V_2 is a subset of $[v^*, \overline{v}]$. On the other hand, the BIC and EIR mechanism with only $V_{12} = [v_{12}^*, \overline{v}]$ non-empty, $t_i(v) = v_{12}^* = v^*$ for $i = 1, 2$ and all $v \in V_{12}$, yields profit equal to

$$\int_{[v^*,\overline{v}]} v^*\{\alpha_1(-D_1'(v)) + \alpha_2(-D_2'(v))\}dv,$$

which is clearly greater than (16). Thus mechanisms with only V_2 and V_{12} non-empty with $v_{12}^* = v_2^*$ are ruled out.

Now suppose only V_2 and V_{12} are non-empty with $v_{12}^* < v_2^*$. The BIC of v_2^* with respect to v_{12}^*, i.e. (12), yields $t_2(v_2^*) < v_{12}^*$. This is because the first term on the right side of (12) is strictly positive since $(v_2^* - v_{12}^*) > 0$ holds by assumption, $\alpha_1(-D_1'(v_2^*)) > 0$ holds by Condition 1, and $t_1(v_{12}^*) \leq v_{12}^*$ holds by EIR. But $t_2(v_2^*) < v_{12}^*$ contradicts the BIC of types with valuation slightly lower than v_{12}^* with respect to v_2^*. These types receive 0 surplus since they don't consume in either state, but if they misreport their valuation as v_2^*, then they get the good in state 2 at a price lower than their valuation, which yields a positive surplus.■

Proof of Fact 2. To see why Conditions 1-3 imply (5), first rewrite (5) as:

$$\frac{\int_{v^*}^{\bar{v}}(-D_2'(v))dv}{\int_{v^*}^{\bar{v}}(-D_1'(v))dv} > \frac{(-D_2'(v^*))}{(-D_1'(v^*))}. \tag{17}$$

Note that the left side of (17) is equal to

$$\frac{\int_{v^*}^{\bar{v}}\frac{(-D_1'(v))}{Z(v)}dv}{\int_{v^*}^{\bar{v}}(-D_1'(v))dv}.$$

$Z(v)$ and $(-D_1'(v))$ are non-negative and strictly decreasing due to Conditions 1-3. Thus, it follows from Wang (1993, Lemma 2)[10] that we have

$$\frac{\int_{v^*}^{\bar{v}}\frac{(-D_1'(v))}{Z(v)}dv}{\int_{v^*}^{\bar{v}}(-D_1'(v))dv} > \frac{\int_{v^*}^{\bar{v}}\frac{1}{Z(v)}dv}{\int_{v^*}^{\bar{v}}dv}. \tag{18}$$

Because $Z(v)$ is strictly decreasing and we are considering $v \geq v^*$, it follows that the right side of (18) exceeds $\frac{1}{Z(v^*)}$. Therefore, we have

$$\frac{\int_{v^*}^{\bar{v}}\frac{(-D_1'(v))}{Z(v)}dv}{\int_{v^*}^{\bar{v}}(-D_1'(v))dv} > \frac{1}{Z(v^*)},$$

which implies (17), and its equivalent, (5).■

Proof of Lemma 4: Ruling out V_1, V_2, and V_{12} all non-empty.

Proof. The proof of Lemma 4 relies on Lemmas 5-8 detailed below. First we show Claim 1.

Claim 1: Suppose V_1 and V_2 are non-empty and BIC, EIR and Conditions 1-3 hold. Then:
(a) Condition 4(i), i.e. $\frac{\alpha_1}{\alpha_2}Z(v) < 1$ for all v, implies $v_1^* < v_2^*$.
(b) Condition 4(ii), i.e. $\frac{\alpha_1}{\alpha_2}Z(v) > 1$ for all v, implies $v_2^* < v_1^*$.

Proof: To begin with, we argue that $v_1^* = v_2^*$ is impossible under Condition 4. This is because: first, $v_1^* = v_2^* = v^*$, for some $v^* \in V$, implies (by Fact 1, EIR of v^*, and

[10] In Wang's notation, $x(\phi) = 1$, $y(\phi) = (-D_1'(v))$, and $z(\phi) = \frac{1}{Z(v)}$.

the BIC of valuations less than v^* with respect to v^*) that price for consumption must be v^* for all types in V_1 and V_2, i.e. $t_1(v_1) = t_2(v_2) = v^*$ holds for all $v_1 \in V_1$ and $v_2 \in V_2$; and second, Condition 4 implies that given their updated beliefs, either all types consider state 2 more likely (Condition 4(i)) or all types consider state 1 more likely (Condition 4(ii)). Thus, under Condition 4(i) all types prefer reporting a valuation in V_2 (so V_1 has to be empty), and similarly under Condition 4(ii) V_2 must be empty.

Now, we consider strict contradictions of Claim 1(a) and 1(b).

(a) By contradiction, suppose $v_2^* < v_1^*$ holds. By Lemma 2(i) and Lemma 3, $t_2(v_2^*) = v_2^*$ holds. The BIC of v_1^* with respect to v_2^* yields

$$v_1^* - t_1(v_1^*) \geq (v_1^* - v_2^*)\frac{\alpha_2}{\alpha_1 Z(v_1^*)}, \text{ or}$$

$$t_1(v_1^*) \leq v_1^* - (v_1^* - v_2^*)\frac{\alpha_2}{\alpha_1 Z(v_1^*)} < v_2^*, \tag{19}$$

where the last inequality holds because $\frac{\alpha_1}{\alpha_2} Z(v_1^*) < 1$ holds by Condition 4(i). But having $t_2(v_2^*) = v_2^*$ and $t_1(v_1^*) < v_2^*$ violates the BIC of v_2^* with respect to v_1^*.

(b) By contradiction, suppose $v_1^* < v_2^*$ holds. By Lemma 2(i) and Lemma 3, $t_1(v_1^*) = v_1^*$ holds. The BIC of v_2^* with respect to v_1^* yields

$$v_2^* - t_2(v_2^*) \geq (v_2^* - v_1^*)\frac{\alpha_1}{\alpha_2} Z(v_2^*), \text{ or}$$

$$t_2(v_2^*) \leq v_2^* - (v_2^* - v_1^*)\frac{\alpha_1}{\alpha_2} Z(v_2^*) < v_1^*, \tag{20}$$

where the last inequality holds because $\frac{\alpha_1}{\alpha_2} Z(v_2^*) > 1$ holds by Condition 4(ii). But having $t_1(v_1^*) = v_1^*$ and $t_2(v_2^*) < v_1^*$ violates the BIC of v_1^* with respect to v_2^*.■

The sketch of the proof of Lemma 4 is as follows. By Lemma 2 and Claim 1(a) (respectively 1(b)), under Condition 4(i) (respectively 4(ii)), if V_1, V_2, and V_{12} are all non-empty in the optimal mechanism, then $v_1^* < v_2^* < v_{12}^*$ (respectively $v_2^* < v_1^* < v_{12}^*$) holds. For any choice of v_1^*, v_2^*, v_{12}^*, Lemma 5 (Lemma 7) specifies a mechanism that provides an upper bound for profits when $v_1^* < v_2^* < v_{12}^*$ (respectively $v_2^* < v_1^* < v_{12}^*$) holds. Lemma 6 (Lemma 8) then demonstrates that this upper bound is increasing as V_2 is "shrunk" until V_2 is empty, at which point the upper bound is also achievable using a mechanism that satisfies all BIC and EIR conditions.

Lemma 5: *Consider the class of BIC and EIR mechanisms where V_1, V_2, and V_{12}*

are all non-empty, and v_1^*, v_2^*, and v_{12}^* are given. If Conditions 1-3 and 4(i) hold, then the profit under a mechanism in this class can be no greater than the profit that would result if all consumers report truthfully in the following mechanism:

$$\begin{cases} V_1 = & [v_1^*, v_2^*) \\ V_2 = & [v_2^*, v_{12}^*) \\ V_{12} = & [v_{12}^*, \overline{v}] \\ t_1(v_1) = v_1^* & \forall v_1 \in V_1 \\ t_2(v_2) = v_2^* - (v_2^* - v_1^*)\frac{\alpha_1}{\alpha_2} Z(v_2^*) & \forall v_2 \in V_2 \\ t_1(v_{12}) = v_1^* & \forall v_{12} \in V_{12} \\ t_2(v_{12}) = t_2(v_2^*) + (v_{12}^* - v_1^*)\frac{\alpha_1}{\alpha_2} Z(v_{12}^*) & \forall v_{12} \in V_{12} \end{cases} \tag{21}$$

Proof of Lemma 5. By Claim 1(a), $v_1^* < v_2^*$ holds. Further, for the optimal mechanism with V_1, V_2, and V_{12} non-empty, by Lemma 2(ii), $v_2^* < v_{12}^*$ must also hold. So, consider an arbitrary BIC and EIR mechanism with V_1, V_2, and V_{12} all non-empty, and v_1^*, v_2^*, and v_{12}^* given such that $v_1^* < v_2^* < v_{12}^*$ holds. Label this mechanism as mechanism C. We will argue that the profit from mechanism C is weakly lower than the profit from mechanism (21) with v_1^*, v_2^*, and v_{12}^* the same as in mechanism C. Note that we do not require mechanism (21) to satisfy BIC and EIR at this point in the argument.

Mechanism C must have the following features: By assumption $v_1^* < v_2^* < v_{12}^*$ holds. By Lemma 3, $t_1(v_1) = v_1^*$ must hold for all $v_1 \in V_1$. By Fact 1, $t_2(v_2) = t_2(v_2^*)$ holds for all $v_2 \in V_2$. By BIC, v_2^* must be indifferent with respect to reporting v_1^* since if instead type v_2^* strictly prefers reporting v_2^* over reporting v_1^*, then by continuity, for a valuation $v_1 \in V_1$ less than v_2^*, but close enough to v_2^*, we will have that v_1 also strictly prefers reporting v_2^* rather than v_1, which contradicts either the BIC of v_1 or the definition of v_2^* as the infimum of V_2. Thus, the following holds:

$$(v_2^* - t_2(v_2^*))\alpha_2(-D_2'(v_2^*)) = (v_2^* - v_1^*)\alpha_1(-D_1'(v_2^*)), \text{ or} \tag{22}$$

$$t_2(v_2^*) = v_2^* - (v_2^* - v_1^*)\frac{\alpha_1}{\alpha_2} Z(v_2^*). \tag{23}$$

Next, we argue that mechanism C must have $V_1 = [v_1^*, v_2^*)$; to show this, we need Claim 2.

Claim 2. For mechanism C, all types with valuation greater than v_2^* strictly prefer reporting v_2^* over reporting v_1^*.

Proof: By Lemma 3, $t_1(v_1) = v_1^*$ holds for all $v_1 \in V_1$. By Fact 1, $t_2(v_2) = t_2(v_2^*)$ holds for all $v_2 \in V_2$. By previous arguments, $t_2(v_2^*)$ is such that v_2^* is indifferent between reporting truthfully and reporting v_1^*. To prove Claim 2, we will show that for all types v such that $v > v_2^*$ holds, v strictly prefers reporting v_2^* over reporting v_1^*. Rewriting (22), the binding BIC of v_2^* with respect to v_1^*, yields

$$\frac{(v_2^* - t_2(v_2^*))}{(v_2^* - v_1^*)} = \frac{\alpha_1}{\alpha_2} Z(v_2^*). \tag{24}$$

Replacing v_2^* with v strictly greater than v_2^* in (24) yields

$$\frac{(v - t_2(v_2^*))}{(v - v_1^*)} \geq \frac{(v_2^* - t_2(v_2^*))}{(v_2^* - v_1^*)} = \frac{\alpha_1}{\alpha_2} Z(v_2^*) > \frac{\alpha_1}{\alpha_2} Z(v),$$

where the first inequality follows because $t_2(v_2^*) \geq v_1^*$ holds (by BIC), and the last inequality follows because $Z(v)$ is strictly decreasing (by Condition 2) and $v > v_2^*$ holds by assumption. Thus,

$$\frac{(v - t_2(v_2^*))}{(v - v_1^*)} > \frac{\alpha_1}{\alpha_2} Z(v) \text{ holds.}$$

Cross-multiplying yields

$$(v - t_2(v_2^*))\alpha_2(-D_2'(v)) > (v - v_1^*)\alpha_1(-D_1'(v)).$$

Thus, for any type v such that $v > v_2^*$ holds, v strictly prefers reporting v_2^* over reporting v_1^*.

Given Claim 2, we have that mechanism C must have $V_1 = [v_1^*, v_2^*)$, $V_2 \subset [v_2^*, \overline{v}]$, $t_1(v_1) = v_1^*$ for all $v_1 \in V_1$, and $t_2(v_2) = t_2(v_2^*)$ for all $v_2 \in V_2$, where $t_2(v_2^*)$ is given by (23). Note that mechanism (21) also has these same properties. In addition, mechanism (21) specifies: (a) that $V_2 = [v_2^*, v_{12}^*)$ and $V_{12} = [v_{12}^*, \overline{v}]$ hold, and (b) the payment scheme over V_{12}. To finish the proof of Lemma 5, we must argue that the features (a) and (b) of mechanism (21) don't reduce its profit relative to the profit from mechanism C.

Note that BIC of v_i^* with respect to $v_{12} \in V_{12}$ implies $t_i(v_{12}) \geq t_i(v_i^*)$ for $i = 1, 2$

and all $v_{12} \in V_{12}$, which means assigning any type with valuation greater than v_{12}^* to V_2 instead of V_{12} will only reduce profit. Thus, setting $V_1 = [v_1^*, v_2^*)$, $V_2 = [v_2^*, v_{12}^*)$, and $V_{12} = [v_{12}^*, \overline{v}]$ as in mechanism (21) yields greater profit than from mechanism C if the payment scheme of mechanism (21) also yields greater profit from V_{12} than the payment scheme of mechanism C.

So consider the payment scheme over the set $V_{12} = [v_{12}^*, \overline{v}]$. To finish the proof of Lemma 5, Claim 3 demonstrates that the payment scheme in mechanism (21) maximizes the firm's expected profit from V_{12} subject to a subset BIC and EIR constraints. This means adding all the missing BIC and EIR constraints, as must be done for mechanism C, can only reduce profit from V_{12}.

Claim 3: Suppose Conditions 1-3, 4(i), and $v_1^* < v_2^* < v_{12}^*$ hold with $V_{12} = [v_{12}^*, \overline{v}]$. Given $t_1(v_1^*) = v_1^*$, and $t_2(v_2^*)$ according to (23), the payment scheme

$$t_1(v_{12}) = v_1^* \;\; \forall v_{12} \in V_{12},$$

$$t_2(v_{12}) = t_2(v_2^*) + (v_{12}^* - v_1^*)\frac{\alpha_1}{\alpha_2} Z(v_{12}^*) \;\; \forall v_{12} \in V_{12},$$

maximizes profits from V_{12}, subject to: (i) the BIC constraint of v_{12}^* with respect to v_2^*, (ii) the BIC constraint of types $v \in [V_{12} - \{v_{12}^*\}]$ with respect to v_{12}^*, (iii) the EIR constraint of v_{12}^*, (iv) the BIC constraint of v_1^* with respect to v_{12}^*, i.e. $t_1(v_{12}^*) \geq v_1^*$, and (v) the BIC constraint of v_2^* with respect to v_{12}^*, i.e. $t_2(v_{12}^*) \geq t_2(v_2^*)$.

Proof of Claim 3. The BIC of types $v \in [V_{12} - \{v_{12}^*\}]$ with respect to v_{12}^* can be rewritten as

$$\begin{aligned} &t_1(v)\alpha_1(-D_1'(v)) + t_2(v)\alpha_2(-D_2'(v)) \leq \\ &t_1(v_{12}^*)\alpha_1(-D_1'(v)) + t_2(v_{12}^*)\alpha_2(-D_2'(v)). \end{aligned} \tag{25}$$

The term on the left side of (25) is the contribution of v towards the firm's profit in (3). Thus (25) shows that, for any given V_{12}, the firm cannot increase profits by charging different payment schemes to different types in V_{12}. Further, if the payment and good-allocation scheme is the same for all types within V_{12}, the BIC constraints of types $v \in [V_{12} - \{v_{12}^*\}]$ with respect to v_{12}^* are satisfied. So, the maximization problem detailed in Claim 3 can be stated as follows.

$$\max_{t_1(v_{12}^*), t_2(v_{12}^*)} \quad t_1(v_{12}^*)\alpha_1 \int_{v_{12}^*}^{\overline{v}} (-D_1'(v))dv + t_2(v_{12}^*)\alpha_2 \int_{v_{12}^*}^{\overline{v}} (-D_2'(v))dv. \tag{26}$$

Subject to:
The BIC constraint of v_{12}^* with respect to v_2^*:

$$\begin{gathered}(v_{12}^* - t_1(v_{12}^*))\alpha_1(-D_1'(v_{12}^*)) + (v_{12}^* - t_2(v_{12}^*))\alpha_2(-D_2'(v_{12}^*)) \geq \\ (v_{12}^* - t_2(v_2^*))\alpha_2(-D_2'(v_{12}^*)).\end{gathered} \tag{27}$$

The EIR constraints of v_{12}^* and the BIC constraints of v_1^* and v_2^* with respect to v_{12}^*:

$$t_1(v_{12}^*) \leq v_{12}^*; \;\; t_2(v_{12}^*) \leq v_{12}^*; \;\; t_1(v_{12}^*) \geq v_1^*; \;\; t_2(v_{12}^*) \geq t_2(v_2^*). \tag{28}$$

Rearranging (27) yields:

$$\begin{gathered}t_1(v_{12}^*)\alpha_1(-D_1'(v_{12}^*)) + t_2(v_{12}^*)\alpha_2(-D_2'(v_{12}^*)) \leq \\ v_{12}^*\alpha_1(-D_1'(v_{12}^*)) + t_2(v_2^*)\alpha_2(-D_2'(v_{12}^*)).\end{gathered} \tag{29}$$

At the optimum, (29) will bind. Further, by Fact 2 we have $\frac{D_2(v_{12}^*)}{(-D_2'(v_{12}^*))} > \frac{D_1(v_{12}^*)}{(-D_1'(v_{12}^*))}$, or

$$\frac{\int_{v_{12}^*}^{\bar{v}}(-D_2'(v))dv}{(-D_2'(v_{12}^*))} > \frac{\int_{v_{12}^*}^{\bar{v}}(-D_1'(v))dv}{(-D_1'(v_{12}^*))}. \tag{30}$$

Due to the linearity of the maximand (26) and the constraint (29) in $t_1(v_{12}^*)$ and $t_2(v_{12}^*)$, it follows from (30) that the solution is to set $t_1(v_{12}^*)$ as low as possible and $t_2(v_{12}^*)$ as high as possible, subject to (29), $t_1(v_{12}^*) \geq v_1^*$ and $t_2(v_{12}^*) \leq v_{12}^*$. We claim that, $t_1(v_{12}) = v_1^*$ and $t_2(v_{12}) = t_2(v_2^*) + (v_{12}^* - v_1^*)\frac{\alpha_1}{\alpha_2}Z(v_{12}^*)$ for all $v_{12} \in V_{12}$ is optimal (where $t_2(v_{12})$ is derived using $t_1(v_{12}^*) = v_1^*$ in (29)).

To verify this claim, we show that setting $t_2(v_{12}^*) = t_2(v_2^*) + (v_{12}^* - v_1^*)\frac{\alpha_1}{\alpha_2}Z(v_{12}^*)$ satisfies $t_2(v_{12}^*) \leq v_{12}^*$. Using (23), we can express $(v_{12}^* - t_2(v_{12}^*))$ as

$$(v_{12}^* - v_2^*) + (v_2^* - v_1^*)\frac{\alpha_1}{\alpha_2}Z(v_2^*) - (v_{12}^* - v_1^*)\frac{\alpha_1}{\alpha_2}Z(v_{12}^*). \tag{31}$$

Evaluated at $v_{12}^* = v_2^*$, expression (31) is zero, so we will be done with this claim if we show that the expression is non-decreasing in v_{12}^*. Differentiating with respect to v_{12}^* yields

$$1 - (v_{12}^* - v_1^*)\frac{\alpha_1}{\alpha_2}Z'(v_{12}^*) - \frac{\alpha_1}{\alpha_2}Z(v_{12}^*). \tag{32}$$

Since $Z'(v_{12}^*) < 0$ and $(1 - \frac{\alpha_1}{\alpha_2}Z(v_{12}^*)) \geq 0$ (Condition 4(i)) hold, the expression (32) is non-decreasing. This completes the proof of Claim 3 and Lemma 5.■

Lemma 6: *If Conditions 1-3, and 4(i) hold, then at the firm's optimal mechanism within the class of BIC and EIR mechanisms, it cannot be the case that the sets* V_1, V_2, *and* V_{12} *are all non-empty.*

Proof of Lemma 6. By Lemma 2 and Claim 1(a), under Conditions 1-3 and 4(i), if the firm's optimal BIC and EIR mechanism has V_1, V_2, and V_{12} all non-empty, then $v_1^* < v_2^* < v_{12}^*$ must hold; which means (by Lemma 5) that mechanism (21) with the same v_1^*, v_2^*, v_{12}^* yields weakly greater profit than the optimal BIC and EIR mechanism under Conditions 1-3 and 4(i). However, mechanism (21) may not satisfy BIC. To prove Lemma 6, we will consider the profit from an arbitrary mechanism (21) with $v_1^* < v_2^* < v_{12}^*$, and show that this profit strictly increases by appropriately reducing the gap between v_2^* and v_{12}^*, thereby making $V_2 = [v_2^*, v_{12}^*)$ smaller. Ultimately, when V_2 is empty and $V_1 = [v_1^*, v_{12}^*)$, $V_{12} = [v_{12}^*, \overline{v}]$, the resulting (21) mechanism also satisfies all BIC and EIR conditions. This will rule out the possibility that, under Conditions 1-3 and 4(i), the firm's optimal BIC and EIR mechanism has V_1, V_2, and V_{12} all non-empty.

The profit from the mechanism given in (21) is:

$$\begin{gathered}\pi(v_1^*, v_2^*, v_{12}^*) = \alpha_1 v_1^*[D_1(v_1^*) - D_1(v_2^*) + D_1(v_{12}^*)] \\ +\alpha_2[(v_{12}^* - v_1^*)\tfrac{\alpha_1}{\alpha_2}Z(v_{12}^*)]D_2(v_{12}^*) + \alpha_2 D_2(v_2^*)(v_2^* - (v_2^* - v_1^*)\tfrac{\alpha_1}{\alpha_2}Z(v_2^*)).\end{gathered}$$

It will be convenient to write the last term in this expression, i.e. $\alpha_2 D_2(v_2^*)(v_2^* - (v_2^* - v_1^*)\frac{\alpha_1}{\alpha_2}Z(v_2^*))$, as

$$\alpha_2 v_2^* D_2(v_2^*)(1 - \frac{\alpha_1}{\alpha_2}Z(v_2^*)) + \alpha_1 v_1^* Z(v_2^*) D_2(v_2^*).$$

Thus, we have

$$\begin{gathered}\pi(v_1^*, v_2^*, v_{12}^*) = \alpha_1 v_1^*[D_1(v_1^*) - D_1(v_2^*) + D_1(v_{12}^*)] \\ +\alpha_2[(v_{12}^* - v_1^*)\tfrac{\alpha_1}{\alpha_2}Z(v_{12}^*)]D_2(v_{12}^*) + \alpha_2 v_2^* D_2(v_2^*)(1 - \tfrac{\alpha_1}{\alpha_2}Z(v_2^*)) \\ +\alpha_1 v_1^* Z(v_2^*) D_2(v_2^*).\end{gathered} \tag{33}$$

Taking the derivative of the profit expression in (33) with respect to v_2^* yields

$$\frac{\partial\pi}{\partial v_2^*} = -\alpha_1 v_1^* D_1'(v_2^*) + \alpha_2 \frac{\partial(v_2^* D_2(v_2^*))}{\partial v_2^*}(1 - \frac{\alpha_1}{\alpha_2} Z(v_2^*)) - \alpha_1(v_2^* - v_1^*) D_2(v_2^*) Z'(v_2^*)$$
$$+\alpha_1 v_1^* Z(v_2^*) D_2'(v_2^*).$$

Now consider the case where v_2^* is strictly less than the monopoly price in state 2 (henceforth p_2^m), i.e. $v_2^* < p_2^m$ holds. By simplifying the derivative, for the case where $v_2^* < p_2^m$ holds, we can see that

$$\frac{\partial\pi}{\partial v_2^*} = \alpha_2 \frac{\partial(v_2^* D_2(v_2^*))}{\partial v_2^*}(1 - \frac{\alpha_1}{\alpha_2} Z(v_2^*)) - \alpha_1(v_2^* - v_1^*) D_2(v_2^*) Z'(v_2^*) > 0$$

holds. Where the last inequality follows because $\frac{\partial(v_2^* D_2(v_2^*))}{\partial v_2^*} > 0$ holds due to Condition 3 and because we are considering the case where $v_2^* < p_2^m$ holds; further, $\frac{\alpha_1}{\alpha_2} Z(v_2^*) \leq 1$ holds by Condition 4(i), $Z'(v_2^*) < 0$ holds by Condition 2, and $v_2^* > v_1^*$ holds by Claim 1(a). The implication of $\frac{\partial\pi}{\partial v_2^*} > 0$ is that whenever $v_1^* < v_2^* < v_{12}^*$ and $v_2^* < p_2^m$ hold, the firm can strictly increase profits from mechanism (21) by increasing v_2^* towards v_{12}^* so that $V_2 = [v_2^*, v_{12}^*)$ shrinks.

Next, consider the case where $p_2^m \leq v_2^*$ holds. So, by definition of mechanism (21), $p_2^m \leq v_2^* < v_{12}^*$ must hold. Consider the derivative of profit (33) with respect to v_{12}^*.

$$\frac{\partial\pi}{\partial v_{12}^*} = \alpha_1 v_1^* D_1'(v_{12}^*) + \alpha_1 \frac{\partial(v_{12}^* D_2(v_{12}^*))}{\partial v_{12}^*} Z(v_{12}^*)$$
$$+\alpha_1(v_{12}^* - v_1^*) Z'(v_{12}^*) D_2(v_{12}^*) - \alpha_1 v_1^* Z(v_{12}^*) D_2'(v_{12}^*)$$
$$= \alpha_1 \frac{\partial(v_{12}^* D_2(v_{12}^*))}{\partial v_{12}^*} Z(v_{12}^*) + \alpha_1(v_{12}^* - v_1^*) Z'(v_{12}^*) D_2(v_{12}^*) < 0.$$

The last inequality holds because $Z'(v_{12}^*) < 0$ and $\frac{\partial(v_{12}^* D_2(v_{12}^*))}{\partial v_{12}^*} < 0$ hold; the former by Condition 2, and the latter follows from concavity (Condition 3) and $v_{12}^* > p_2^m$. The implication of $\frac{\partial\pi}{\partial v_{12}^*} < 0$ is that whenever $v_1^* < v_2^* < v_{12}^*$ and $v_2^* \geq p_2^m$ hold, the firm can strictly increase profits from mechanism (21) by decreasing v_{12}^* toward v_2^*, so that $V_2 = [v_2^*, v_{12}^*)$ becomes smaller.

To summarize, in either case, $v_2^* < p_m^2$ or $v_2^* \geq p_m^2$, the firm strictly increases profit from mechanism (21) by either increasing v_2^* or decreasing v_{12}^*, thereby making the interval $V_2 = [v_2^*, v_{12}^*)$ smaller by reducing $(v_{12}^* - v_2^*)$. Therefore, for appropriately chosen values of v_1^* and v_{12}^*, the mechanism (21) with $(v_{12}^* - v_2^*) = 0$ and V_2 empty, i.e. the mechanism given as follows

$$\begin{cases} V_1 = & [v_1^*, v_{12}^*) \\ V_{12} = & [v_{12}^*, \overline{v}] \\ V_2 = & \emptyset \\ t_1(v) = v_1^* & \forall v \geq v_1^* \\ t_2(v) = v_{12}^* & \forall v \geq v_{12}^* \end{cases} \tag{34}$$

yields higher profits than any mechanism (21) with $v_1^* < v_2^* < v_{12}^*$ and V_1, V_2, and V_{12} all non-empty.

To finish the proof it is straightforward to verify that mechanism (34) satisfies all BIC and EIR constraints. Therefore, under Conditions 1-3 and 4(i), it cannot be the case that in the firm's optimal mechanism V_1, V_2, and V_{12} are all non-empty.■

Lemma 7: *Consider the class of BIC and EIR mechanisms where V_1, V_2, and V_{12} are all non-empty, and v_1^*, v_2^*, and v_{12}^* are given. If Conditions 1-3 and 4(ii) hold, then the profit of a mechanism in this class can be no greater than the profit that would result if all consumers report truthfully in the following mechanism:*

$$\begin{cases} V_2 = & [v_2^*, v_1^*) \\ V_1 = & [v_1^*, v_{12}^*) \\ V_{12} = & [v_{12}^*, \overline{v}] \\ t_2(v_2) = v_2^* & \forall v_2 \in V_2 \\ t_1(v_1) = v_1^* - (v_1^* - v_2^*)\frac{\alpha_2}{\alpha_1 Z(v_1^*)} & \forall v_1 \in V_1 \\ t_1(v_{12}) = v_1^* - (v_1^* - v_2^*)\frac{\alpha_2}{\alpha_1 Z(v_1^*)} & \forall v_{12} \in V_{12} \\ t_2(v_{12}) = v_{12}^* & \forall v_{12} \in V_{12} \end{cases} \tag{35}$$

Proof of Lemma 7. Suppose Conditions 1-3 and 4(ii) hold. Consider an arbitrary BIC and EIR mechanism with V_1, V_2, and V_{12} all non-empty, and v_1^*, v_2^*, and v_{12}^* given; label this mechanism as mechanism D. We will argue that the profit from mechanism D is weakly lower than the profit from the mechanism (35), where v_1^*, v_2^*, and v_{12}^* are the same as in mechanism D. Note that mechanism D must satisfy BIC and EIR, however mechanism (35) is not required to satisfy either at this point in the argument.

Mechanism D must have the following features: By Claim 1(b) and Lemma 2(i) $v_2^* < v_1^* < v_{12}^*$ must hold. By Lemma 3, $t_2(v_2) = v_2^*$ must hold for all $v_2 \in V_2$. By Fact 1, $t_1(v_1) = t_1(v_1^*)$ must hold for all $v_1 \in V_1$. By BIC, v_1^* must be indifferent between reporting v_2^* and reporting v_1^*, since if instead type v_1^* strictly prefers reporting v_1^* over reporting v_2^*, then by continuity, for a valuation v_2 less than v_1^*, but close enough to v_1^*, we will have that v_2 also strictly prefers reporting v_1^* rather than v_2^*, which contradicts either the BIC of v_2 or the definition of v_1^* as the infimum of V_1. Thus, we have

$$\begin{gathered}(v_1^* - t_1(v_1^*))\alpha_1(-D_1'(v_1^*)) = (v_1^* - v_2^*)\alpha_2(-D_2'(v_1^*)), \text{ or} \\ t_1(v_1^*) = v_1^* - (v_1^* - v_2^*)\tfrac{\alpha_2}{\alpha_1 Z(v_1^*)}.\end{gathered} \tag{36}$$

Therefore, mechanism D's payment scheme over V_1 and V_2 is determined by BIC. Note that mechanism (35) has the same payment scheme over V_1 and V_2.

Now we argue that setting V_2, V_1, and V_{12} as in mechanism (35) indeed yields greater profit than from mechanism D. From (36), notice that $t_1(v_1^*) > v_2^*$ holds by Condition 4(ii). Since Condition 4(ii) and $t_1(v_1^*) > v_2^*$ hold, we have:

$$\frac{\alpha_1}{\alpha_2} Z(v) t_1(v_1^*) > v_2^* \ \forall v, \text{ or}$$

$$\alpha_1 t_1(v_1^*)(-D_1'(v)) > \alpha_2 v_2^*(-D_2'(v)) \ \forall v. \tag{37}$$

Inequality (37) implies that for any type with valuation greater than v_1^*, the firm earns more profit from that type if it is in V_1 rather than in V_2. Further, by BIC (Lemma 2(i)), no valuation greater than v_{12}^* can be in V_1. Last, BIC implies $t_i(v_{12}) \geq t_i(v_i^*)$ for $i = 1, 2$ and all $v_{12} \in V_{12}$, which means assigning any type with valuation greater than v_{12}^* to V_2 rather than V_{12} will only reduce profit. Thus, the profit received from types below v_{12}^* is weakly higher in mechanism (35), with $V_2 = [v_2^*, v_1^*)$, $V_1 = [v_1^*, v_{12}^*)$, and $V_{12} = [v_1^*, \overline{v}]$ than in mechanism D.

To show that mechanism (35) yields higher overall profits than mechanism D, Claim 4 demonstrates that the payment scheme of mechanism (35) maximizes the firm's expected profit from V_{12}, subject to a subset of BIC and EIR constraints. Thus, adding the missing BIC and EIR constraints, as must be done in mechanism D, can only reduce profit from V_{12} relative to mechanism (35).

Claim 4: Suppose Conditions 1-3, and $v_2^* < v_1^* < v_{12}^*$ hold with $V_{12} = [v_{12}^*, \overline{v}]$. Given

$t_2(v_2^*) = v_2^*$, and $t_1(v_1^*)$ according to (36), the payment scheme

$$t_1(v_{12}) = t_1(v_1^*) = v_1^* - (v_1^* - v_2^*)\frac{\alpha_2}{\alpha_1 Z(v_1^*)} \ \ \forall v_{12} \in V_{12},$$
$$t_2(v_{12}) = v_{12}^* \ \ \forall v_{12} \in V_{12},$$

maximizes profits from V_{12}, subject to: (i) the BIC constraint of v_{12}^* with respect to v_1^*, (ii) the BIC constraint of types $v \in [V_{12} - \{v_{12}^*\}]$ with respect to v_{12}^*, (iii) the EIR constraint of v_{12}^*, (iv) the BIC constraint of v_2^* with respect to v_{12}^*, i.e. $t_2(v_{12}^*) \geq v_2^*$, and (v) the BIC constraint of v_1^* with respect to v_{12}^*, i.e. $t_1(v_{12}^*) \geq t_1(v_1^*)$.

Proof of Claim 4. Repeating the arguments in Claim 3, (25) shows that, given V_1, V_2, and V_{12}, the firm cannot increase profits from V_{12} by charging different payment schemes to different types in V_{12}. Further, if the payment and good-allocation scheme is the same for all types within V_{12}, the BIC constraints of types $v \in [V_{12} - \{v_{12}^*\}]$ with respect to v_{12}^* are satisfied. So, the maximization problem detailed in Claim 4 can be stated as follows.

$$\max_{t_1(v_{12}^*), t_2(v_{12}^*)} \quad t_1(v_{12}^*)\alpha_1 \int_{v_{12}^*}^{\bar{v}} (-D_1'(v))dv + t_2(v_{12}^*)\alpha_2 \int_{v_{12}^*}^{\bar{v}} (-D_2'(v))dv. \tag{38}$$

Subject to:

The BIC constraint of v_{12}^* with respect to v_1^*:

$$\begin{gathered}(v_{12}^* - t_1(v_{12}^*))\alpha_1(-D_1'(v_{12}^*)) + (v_{12}^* - t_2(v_{12}^*))\alpha_2(-D_2'(v_{12}^*)) \geq \\ (v_{12}^* - t_1(v_1^*))\alpha_1(-D_1'(v_{12}^*)).\end{gathered} \tag{39}$$

The EIR constraints of v_{12}^* and the BIC constraints of v_1^* and v_2^* with respect to v_{12}^*:

$$t_1(v_{12}^*) \leq v_{12}^*; \ \ t_2(v_{12}^*) \leq v_{12}^*; \ \ t_1(v_{12}^*) \geq t_1(v_1^*); \ \ t_2(v_{12}^*) \geq v_2^*. \tag{40}$$

Rearranging (39) yields:

$$\begin{gathered}t_1(v_{12}^*)\alpha_1(-D_1'(v_{12}^*)) + t_2(v_{12}^*)\alpha_2(-D_2'(v_{12}^*)) \leq \\ t_1(v_1^*)\alpha_1(-D_1'(v_{12}^*)) + v_{12}^*\alpha_2(-D_2'(v_{12}^*)).\end{gathered} \tag{41}$$

At the optimum, (41) will bind. Fact 2 yields

$$\frac{\int_{v_{12}^*}^{\bar{v}}(-D_2'(v))dv}{(-D_2'(v_{12}^*))} > \frac{\int_{v_{12}^*}^{\bar{v}}(-D_1'(v))dv}{(-D_1'(v_{12}^*))}. \tag{42}$$

Due to the linearity of the maximand, (38), and the constraint, (41), in the choice variable $t_1(v_{12}^*)$ and $t_2(v_{12}^*)$, it follows from (42) that the solution is to set $t_1(v_{12}^*)$ as low as possible and $t_2(v_{12}^*)$ as high as possible, subject to (41), $t_1(v_{12}^*) \geq t_1(v_1^*)$, and $t_2(v_{12}^*) \leq v_{12}^*$. Thus, $t_1(v) = t_1(v_1^*)$ and $t_2(v) = v_{12}^*$ for all $v \in V_{12}$ is optimal. It is straightforward to check that all constraints imposed in Claim 4 are satisfied.

Therefore, under Conditions 1-3 and 4(ii), mechanism (35) yields greater profit than any BIC and EIR mechanism with V_1, V_2, and V_{12} all non-empty (represented by the arbitrary mechanism D in this proof).■

Lemma 8: *If Conditions 1-3, and 4(ii) hold, then at the firm's optimal mechanism within the class of BIC and EIR mechanisms, it cannot be the case that the sets V_1, V_2, and V_{12} are all non-empty.*

Proof of Lemma 8. Suppose Conditions 1-3 and 4(ii) hold and the firm's optimal BIC and EIR mechanism has V_1, V_2, and V_{12} all non-empty. By Lemma 2 and Claim 1(b), BIC implies $v_2^* < v_1^* < v_{12}^*$. By Lemma 7, the mechanism (35) with the same v_2^*, v_1^*, v_{12}^* yields weakly greater profit than the optimal BIC and EIR mechanism with V_1, V_2, and V_{12} all non-empty. However, mechanism (35) may not satisfy BIC. To prove Lemma 8, we will consider the profit from an arbitrary mechanism (35) with $v_2^* < v_1^* < v_{12}^*$, and show that this profit strictly increases by appropriately reducing the gap between v_2^* and v_1^*, thereby making $V_2 = [v_2^*, v_1^*)$ smaller. Ultimately, when V_2 is empty and $V_1 = [v_1^*, v_{12}^*)$, $V_{12} = [v_{12}^*, \overline{v}]$, the resulting (35) mechanism also satisfies all BIC and EIR conditions. This will rule out the possibility that, under Conditions 1-3 and 4(ii), the firm's optimal BIC and EIR mechanism has V_1, V_2, and V_{12} all non-empty.

The profit from mechanism (35) is:

$$\begin{aligned}\pi(v_2^*, v_1^*, v_{12}^*) &= \alpha_1 D_1(v_1^*)t_1(v_1^*) + \alpha_2[D_2(v_2^*) - D_2(v_1^*)]v_2^* + \alpha_2 D_2(v_{12}^*)v_{12}^* \\ &= \alpha_1 D_1(v_1^*)[v_1^* - (v_1^* - v_2^*)\frac{\alpha_2}{\alpha_1 Z(v_1^*)}] + \alpha_2[D_2(v_2^*) - D_2(v_1^*)]v_2^* + \alpha_2 D_2(v_{12}^*)v_{12}^*. \end{aligned} \tag{43}$$

Consider the derivative of the profit in (43) with respect to v_1^*. We have

$$\begin{aligned}\tfrac{\partial \pi(v_2^*, v_1^*, v_{12}^*)}{\partial v_1^*} &= \alpha_1 D_1(v_1^*)[1 - \tfrac{\alpha_2}{\alpha_1}(\tfrac{Z(v_1^*) - (v_1^* - v_2^*)Z'(v_1^*)}{Z(v_1^*)^2})] \\ &+ \alpha_1 v_1^* D_1'(v_1^*) - \alpha_2 D_1'(v_1^*)\tfrac{(v_1^* - v_2^*)}{Z(v_1^*)} - \alpha_2 v_2^* D_2'(v_1^*).\end{aligned} \tag{44}$$

Substituting $\frac{D_1'(v_1^*)}{Z(v_1^*)} = D_2'(v_1^*)$ into (44) and simplifying yields

$$\frac{\partial \pi(v_2^*, v_1^*, v_{12}^*)}{\partial v_1^*} = \alpha_1 D_1(v_1^*)[1 - \frac{\alpha_2}{\alpha_1}(\frac{Z(v_1^*) - (v_1^* - v_2^*)Z'(v_1^*)}{Z(v_1^*)^2})] + \alpha_1 v_1^* D_1'(v_1^*) - \alpha_2 v_1^* D_2'(v_1^*).$$

Rearranging terms, we have

$$\begin{aligned}\tfrac{\partial \pi(v_2^*, v_1^*, v_{12}^*)}{\partial v_1^*} &= \alpha_1[D_1(v_1^*) + D_1'(v_1^*)v_1^*] - \alpha_2[D_2'(v_1^*)v_1^* + \tfrac{D_1(v_1^*)}{Z(v_1^*)}] \\ &+ \tfrac{\alpha_2 D_1(v_1^*)(v_1^* - v_2^*)Z'(v_1^*)}{Z(v_1^*)^2}.\end{aligned} \tag{45}$$

Substituting $\frac{D_1'(v_1^*)}{Z(v_1^*)} = D_2'(v_1^*)$ into (45) and combining/rearranging terms, we have

$$\frac{\partial \pi(v_2^*, v_1^*, v_{12}^*)}{\partial v_1^*} = \alpha_1[D_1(v_1^*) + D_1'(v_1^*)v_1^*][1 - \frac{\alpha_2}{\alpha_1 Z(v_1^*)}] + \frac{\alpha_2 D_1(v_1^*)(v_1^* - v_2^*)Z'(v_1^*)}{Z(v_1^*)^2}. \tag{46}$$

Since $v_1^* > v_2^*$ and $Z'(v_1^*) < 0$ hold, the last term in (46) is negative. By Condition 4(ii), $[1 - \frac{\alpha_2}{\alpha_1 Z(v_1^*)}] > 0$ holds. Thus, if $v_1^* \geq p_1^m$ holds, then the right side of (46) is negative, which implies that the firm can strictly increase profit from mechanism (35) (given in (43)) by decreasing v_1^*, thereby reducing $(v_1^* - v_2^*)$ and shrinking V_2.

Next, suppose $v_1^* < p_1^m$ holds. Now let us compute

$$\frac{\frac{\partial \pi(v_2^*, v_1^*, v_{12}^*)}{\partial v_2^*}}{\alpha_2} = \frac{D_1(v_1^*)}{Z(v_1^*)} - D_2(v_1^*) + D_2(v_2^*) + v_2^* D_2'(v_2^*). \tag{47}$$

When $\frac{\partial \pi(v_2^*, v_1^*, v_{12}^*)}{\partial v_2^*}$ is evaluated at $v_2^* = v_1^*$, the right side of (47) becomes

$$\frac{D_1(v_1^*)}{Z(v_1^*)} - D_2(v_1^*) + D_2(v_1^*) + v_1^* D_2'(v_1^*), \text{or} \tag{48}$$

$$D_2'(v_1^*)[\frac{D_1(v_1^*)}{D_1'(v_1^*)} + v_1^*], \text{ or}$$

$$Z(v_1^*)[D_1(v_1^*) + D_1'(v_1^*)v_1^*]. \tag{49}$$

Note that (49) is strictly positive since $Z(v_1^*) > 0$ holds and since $[D_1(v_1^*)+D_1'(v_1^*)v_1^*]$, the marginal revenue in state 1, is strictly greater than 0 because of Condition 3 and our supposition: $v_1^* < p_1^m$. From Condition 3, marginal revenue in state 2 is decreasing in v, so from (47), $\frac{\partial\pi(v_2^*,v_1^*,v_{12}^*)}{\partial v_2^*}$ is decreasing in v_2^*. Since we have shown that $\frac{\partial\pi(v_2^*,v_1^*,v_{12}^*)}{\partial v_2^*}$ is strictly positive when evaluated at $v_2^* = v_1^*$, it follows that $\frac{\partial\pi(v_2^*,v_1^*,v_{12}^*)}{\partial v_2^*}$ is strictly positive for all v_2^* strictly lower than v_1^*. In other words, when $v_1^* < p_1^m$ holds, the profit from mechanism (35) (given in (43)) strictly increases as v_2^* is increased and thereby $(v_1^* - v_2^*)$ is reduced and V_2 is shrunk.

To summarize, in either case, $v_1^* < p_m^1$ or $v_1^* \geq p_m^1$, the profit from mechanism (35) can be strictly increased by either increasing v_2^* or decreasing v_1^*, thereby making the interval $V_2 = [v_2^*, v_1^*)$ smaller by reducing $(v_1^* - v_2^*)$. Therefore, for appropriately chosen values of v_1^* and v_{12}^*, the mechanism (35) with $(v_{12}^* - v_2^*) = 0$ and V_2 empty, i.e. the mechanism given as follows

$$\begin{cases} V_1 = & [v_1^*, v_{12}^*) \\ V_{12} = & [v_{12}^*, \overline{v}] \\ V_2 = & \emptyset \\ t_1(v) = v_1^* & \forall v \geq v_1^* \\ t_2(v) = v_{12}^* & \forall v \geq v_{12}^* \end{cases} \tag{50}$$

yields higher profits than any mechanism (35) with $v_2^* < v_1^* < v_{12}^*$ and V_1, V_2, and V_{12} all non-empty.

To finish the proof it is straightforward to verify that mechanism (50) satisfies all BIC and EIR constraints. Therefore, under Conditions 1-3 and 4(ii), it cannot be the case that in the firm's optimal BIC and EIR mechanism V_1, V_2, and V_{12} are all non-empty.■

Thus, Lemmas 5-8 demonstrate that given Conditions 1-4, BIC, and EIR, it will not be optimal for the firm to choose a mechanism where V_1, V_2 and V_{12} are all non-empty, since even mechanisms that yield an upper bound over profit from BIC and EIR mechanisms with V_1, V_2 and V_{12} all non-empty can be improved upon by a mechanism where V_2 is empty, and this mechanism also satisfies all BIC and EIR conditions. This concludes the proof of Lemma 4.■

Proof of the SBSMP Proposition

To prove the SBSMP proposition, we first show that the optimal mechanism cannot have only V_{12} non-empty or only V_2 and V_{12} nonempty. We finish by showing that the optimal mechanism with only V_1 and V_{12} nonempty is the SBSMP mechanism, which satisfies all BIC and EIR constraints.

Ruling out mechanisms with only V_{12} ***non-empty.*** Suppose the firm chooses a mechanism such that only the set $V_{12} = [v_{12}^*, \overline{v}]$ is non-empty. The EIR constraint of v_{12}^* and the BIC constraint of valuations less than v_{12}^* imply $t_1(v_{12}^*) = t_2(v_{12}^*) = v_{12}^*$. From (3) it follows that the firm's profit from any valuation $v \in V_{12}$ is

$$\alpha_1 t_1(v)(-D_1'(v)) + \alpha_2 t_2(v)(-D_2'(v)).$$

But rearranging the BIC of v with respect to v_{12}^* yields

$$\alpha_1 t_1(v)(-D_1'(v)) + \alpha_2 t_2(v)(-D_2'(v)) \leq \alpha_1 t_1(v_{12}^*)(-D_1'(v)) + \alpha_2 t_2(v_{12}^*)(-D_2'(v)).$$

So setting $t_1(v) = t_2(v) = v_{12}^*$ for all $v \in V_{12}$ achieves the maximum profit the firm can make from V_{12} under BIC and EIR. Thus, the profit for the case of only V_{12} non-empty is $\alpha_1 v_{12}^* D_1(v_{12}^*) + \alpha_2 v_{12}^* D_2(v_{12}^*)$, which, for all $v_{12}^* \in V$, is strictly less than the profit from the SBSMP mechanism, $\alpha_1 p_1^m D_1(p_1^m) + \alpha_2 p_2^m D_2(p_2^m)$, since $p_1^m \neq p_2^m$ holds by Fact 2.

Ruling out mechanisms with only V_2 ***and*** V_{12} ***non-empty.***
By Lemma 2(ii), $v_2^* < v_{12}^*$ must hold. By Lemma 3, $t_2(v_2) = v_2^*$ holds for all $v_2 \in V_2$. We first argue that mechanisms with only V_2 and V_{12} non-empty are sub-optimal when Conditions 1-3 and 4(i) hold.

So, let Conditions 1-3 and 4(i) hold, and consider the optimal BIC and EIR mechanism with only V_2 and V_{12} non-empty. Suppose the optimal v_2^* and v_{12}^* in this mechanism are $v_2^* = v_2^o$ and $v_{12}^* = v_{12}^o$. We first argue that the profit from such a mechanism has to be weakly lower than

$$\pi_u = \alpha_1 D_1(v_{12}^o) v_2^o + \alpha_2 D_2(v_{12}^o)[v_2^o + (v_{12}^o - v_2^o)\frac{\alpha_1}{\alpha_2} Z(v_{12}^o)] + \alpha_2 [D_2(v_2^o) - D_2(v_{12}^o)] v_2^o.$$

To see why, note that: (i) in π_u, the maximum possible amount, given EIR, is charged from the set V_2 because $t_2(v) = t_2(v_2^o) \leq v_2^o$ must hold for all $v \in V_2$, and we have set $t_2(v_2^o) = v_2^o$; (ii) by rearranging the BIC of valuations in V_{12} with respect to v_{12}^o, we obtain (25), from which it is clear that the firm cannot improve upon profits from V_{12} by charging different payment schemes to different valuations in V_{12}; (iii) by the arguments in the proof of Claim 3, due to Fact 2, the firm maximizes profits from V_{12}, subject to a subset of BIC and EIR constraints, by charging the highest price possible in state 2 and the lowest price possible in state 1; (iv) by EIR and BIC, $t_2(v_{12}^o) \leq v_{12}^o$ and $t_1(v_{12}^o) \geq v_2^o$, respectively, must hold, and we have set $t_1(v_{12}^o) = v_2^o$, and consequently, the binding BIC of v_{12}^o with respect to v_2^o yields

$$t_2(v_{12}^o) = v_2^o + (v_{12}^o - v_2^o)\frac{\alpha_1}{\alpha_2}Z(v_{12}^o);$$

and finally (v) we have set $V_{12} = [v_{12}^o, \overline{v}]$, in particular, we have not allowed any type with valuation greater than v_{12}^o to belong to V_2, which supports π_u being the upper bound because, by BIC, $t_1(v) \geq v_2^o$ and $t_2(v) \geq v_2^o$ hold for all v_{12} in V_{12}.

When Conditions 1-3 and 4(i) hold, the proof of Lemma 6 shows that the profits from an arbitrary mechanism (21) can be strictly increased by either increasing v_2^* or decreasing v_{12}^*, thereby shrinking V_2 and making $(v_{12}^* - v_2^*)$ smaller. If we set $v_2^* = v_2^o$, $v_{12}^* = v_{12}^o$, and $v_1^* = v_2^o - \epsilon$ in mechanism (21), then for any $\delta > 0$, we can find $\epsilon > 0$ small enough such that the difference between π_u and the profit from the resulting mechanism (21) is no more than δ. Notice that the gap between v_{12}^* and v_2^* does not depend on δ.

The proof of Lemma 6 showed that, relative to the mechanism (21), there is a profit advantage of shrinking the gap between v_{12}^* and v_2^* to zero. For the $v_2^* < p_2^m$ case, the profit advantage of increasing v_2^* is

$$\frac{\partial \pi}{\partial v_2^*} = \alpha_2 \frac{\partial(v_2^* D_2(v_2^*))}{\partial v_2^*}\left(1 - \frac{\alpha_1}{\alpha_2}Z(v_2^*)\right) - \alpha_1(v_2^* - v_1^*)D_2(v_2^*)Z'(v_2^*) > 0,$$

which is strictly positive even if $\epsilon = 0$. On the other hand, for the $v_2^* \geq p_2^m$ case, the profit advantage of decreasing v_{12}^* is:

$$\frac{\partial \pi}{\partial v_{12}^*} = \alpha_1 \frac{\partial(v_{12}^* D_2(v_{12}^*))}{\partial v_{12}^*}Z(v_{12}^*) + \alpha_1(v_{12}^* - v_1^*)Z'(v_{12}^*)D_2(v_{12}^*) < 0.$$

That is, in both cases, the profit advantage is proportional to $v_{12}^* - v_2^*$, but this profit advantage is bounded above zero and does not depend on δ. Thus, the profit from mechanism (34) with V_2 empty, $v_1^* = v_2^o - \epsilon$, and v_{12}^* appropriately chosen, is strictly greater than the profit π_u. Further it is straightforward to check that this mechanism satisfies all EIR and BIC conditions.

Next, Claim 5 deals with case where Conditions 1-3 and 4(ii) hold.

Claim 5: Under Conditions 1-3 and 4(ii), the profit from the optimal BIC and EIR mechanism with only V_2 and V_{12} non-empty (and V_1 empty), is lower than the profit from the optimal BIC and EIR mechanism with only V_1 and V_{12} non-empty (and V_2 empty).

Proof of Claim 5: The mechanisms with only V_1 and V_{12} non-empty that we will consider is given by (50). It is straightforward to verify that mechanism (50) satisfies all BIC and EIR conditions. To prove Claim 5, we will show that, under Conditions 1-3 and 4(ii), for appropriately chosen v_1^* and v_{12}^*, mechanism (50) also yields greater profit than the optimal BIC and EIR mechanism with only V_2 and V_{12} non-empty (and V_1 empty), which we denote by mechanism E. Mechanism E must have the following features. First, we must have $v_2^* < v_{12}^*$ (by Lemma 2(ii)) and $t_2(v_2) = v_2^*$ for all $v_2 \in V_2$ (by Lemma 3). Second, the profit from mechanism E is bounded above by the profit from an alternate mechanism, described below and denoted by mechanism F, that is only required to satisfy all the BIC and EIR constraints listed in Claim 3.

Mechanism F is identical to mechanism E except: (a) all types with valuation strictly greater than v_{12}^* who were in V_2 in mechanism E are instead allocated to V_{12}, and (b) these types are charged v_2^* in state 2 (as they were in mechanism E) and they are charged $t_1(v_{12}^*)$ in state 1. It is straightforward to verify that mechanism F satisfies all the constraints listed in Claim 3. Mechanism F has $V_{12} = [v_{12}^*, \overline{v}]$, and it yields greater profit than mechanism E, since if any type with valuation strictly greater than v_{12}^* belongs to V_2 in mechanism E, then such a type only yields v_2^* in state 2, while mechanism F assigns this type to V_{12}, elicits the same amount in state 2 but also elicits $t_1(v_{12}^*)$ from this type in state 1. Thus, the profit from mechanism

E is less than the profit from mechanism F, given as follows

$$\begin{cases} V_2 = & [v_2^*, v_{12}^*) \\ V_{12} = & [v_{12}^*, \overline{v}] \\ V_1 = & \emptyset \\ t_2(v_2) = v_2^* & \forall v_2 \in V_2 \\ t_1(v_{12}), t_2(v_{12}) & \text{for } v_{12} \in V_{12}, \end{cases} \tag{51}$$

where v_2^*, v_{12}^* are identical to mechanism E, and the payment scheme over V_{12} is chosen as described earlier. To prove Claim 5, we will show that for appropriately chosen v_1^* and v_{12}^*, the mechanism given in (50) yields strictly greater profit than mechanism F, given in (51). The profit from mechanism F can be no more than

$$\pi_2 = \alpha_1 D_1(v_{12}^*)v_2^* + \alpha_2 D_2(v_{12}^*)v_{12}^* + \alpha_2[D_2(v_2^*) - D_2(v_{12}^*)]v_2^*. \tag{52}$$

To see why, note that: (i) in π_2, by (51), $t_2(v_2) = v_2^*$ holds for all $v_2 \in V_2$; (ii) by the BIC of valuations in $V_{12} - \{v_{12}^*\}$ with respect to v_{12}^*, rearranged to (25), it is clear that the firm cannot improve upon profits from $V_{12} = [v_{12}^*, \overline{v}]$ by charging different payment schemes to different valuations in V_{12}; (iii) by the arguments in the proof of Claim 3, due to Fact 2, the firm maximizes profits from V_{12}, subject to the constraints in Claim 3 (which are satisfied by mechanism F) by charging the highest price possible in state 2 and the lowest price possible in state 1; and finally (iv) by the third and fifth constraints of Claim 3, $t_2(v_{12}^*) \leq v_{12}^*$ (EIR of v_{12}^*) and $t_1(v_{12}^*) \geq v_2^*$ (BIC of v_2^* with respect to v_{12}^*), respectively, must hold, and we have set $t_2(v_{12}^*) = v_{12}^*$ and $t_1(v_{12}^*) = v_2^*$ to calculate π_2.
Now consider the mechanism (50) where (a) we set v_1^* equal to v_2^* from mechanism F, and (b) we set the value of v_{12}^* in mechanism (50) equal to the value of v_{12}^* in mechanism F. The profit from mechanism (50) with these values of v_1^* and v_{12}^* is given by:

$$\pi_1 = \alpha_1 D_1(v_{12}^*)v_2^* + \alpha_2 D_2(v_{12}^*)v_{12}^* + \alpha_1[D_1(v_2^*) - D_1(v_{12}^*)]v_2^*. \tag{53}$$

Note that π_1 is strictly greater than π_2 because we have

$$\alpha_1[D_1(v_2^*) - D_1(v_{12}^*)] > \alpha_2[D_2(v_2^*) - D_2(v_{12}^*)], \text{ or}$$

$$\int_{v_2^*}^{v_{12}^*} \alpha_1(-D_1'(v))dv > \int_{v_2^*}^{v_{12}^*} \alpha_2(-D_2'(v))dv,$$

since, by Condition 4(ii), $\alpha_1(-D_1'(v)) > \alpha_2(-D_2'(v))$ or $\frac{\alpha_1}{\alpha_2}Z(v) > 1$ holds for all v.■

Mechanism with only V_1 ***and*** V_{12} ***non-empty.*** Consider the firm's optimal mechanism where only V_1 and V_{12} are non-empty. By Lemma 2 it follows that $v_1^* < v_{12}^*$, and $V_1 = [v_1^*, v_{12}^*)$, $V_{12} = [v_{12}^*, \overline{v}]$ hold. By Lemma 3, $t_1(v_1^*) = v_1^*$ must hold. The question is what payment scheme should be charged from V_{12}. This is answered in Claim P (below).

Claim P: Suppose only V_1 and V_{12} are non-empty with $V_1 = [v_1^*, v_{12}^*)$ and $V_{12} = [v_{12}^*, \overline{v}]$. Given Conditions 1, 2, and 3, and given $t_1(v_1^*) = v_1^*$, the payment scheme

$$t_1(v_{12}) = v_1^* \ \ \forall v_{12} \in V_{12},$$

$$t_2(v_{12}) = v_{12}^* \ \ \forall v_{12} \in V_{12},$$

maximizes profits from V_{12}, subject to all EIR and BIC constraints.

Proof: Note that $t_1(v_1) = v_1^*$ for all $v_1 \in V_1$ is implied by Lemma 3. To see why the payment scheme in Claim P maximizes profits from V_{12}, note that Claim 4 also solves the same maximization problem except with a different value for $t_1(v_1^*)$, and with additional constraints with respect to V_2, which don't bind there. Thus, it suffices to replace $t_1(v_1^*) = v_1^*$ in Claim 4, and to verify that the mechanism in Claim P satisfies all BIC and EIR constraints, which is straightforward.■

Claim P implies that the optimal profit in the case of only V_1 and V_{12} non-empty is

$$\pi = \alpha_1 D_1(v_1^*)v_1^* + \alpha_2 D_2(v_{12}^*)v_{12}^*,$$

where v_1^* and v_{12}^* should be chosen to maximize π. By Condition 1 and 3, the first-order conditions with respect to v_1^* and v_{12}^* yield the unique profit maximizing values: $v_1^* = p_1^m$ and $v_{12}^* = p_2^m$. Note that this mechanism satisfies all BIC and EIR constraints and yields the maximized profit equal to

$$\pi^* = \alpha_1 D_1(p_1^m)p_1^m + \alpha_2 D_2(p_2^m)p_2^m.$$

This completes the proof of the SBSMP Proposition.■

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