Optimal Monopoly Mechanisms with Demand Uncertainty

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Abstract

This paper analyzes a monopoly firm’s profit maximizing mechanism in the following context. There is a continuum of consumers with a unit demand for a good. The distribution of the consumers’ valuations is given by one of two possible demand distributions/states: high demand or low demand. The consumers are uncertain about the demand state, and they update their beliefs after observing their own valuation for the good. The firm is uncertain about the demand state, but infers the demand state when the consumers report their valuations. The firm’s problem is to maximize profits by choosing an optimal mechanism among the class of anonymous, deterministic, direct revelation mechanisms that satisfy interim incentive compatibility and ex-post individual rationality. We show that, under certain sufficient conditions, the firm’s optimal mechanism is to set the monopoly price in each demand state. Under these conditions, Segal’s (2003) optimal ex post mechanism is robust to relaxing ex post incentive compatibility to interim incentive compatibility. We also provide a counterexample when these conditions are not satisfied.

1 Introduction

Consider a monopolist who must decide how best to exploit its market in the presence of aggregate demand uncertainty. The demand curve represents the distribution of consumer

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valuations, where each consumer is negligible relative to the market and desires at most one unit of the good. Without aggregate demand uncertainty, it is well known that there is no scope for price discrimination, and that the best the monopolist can do is offer the good to all consumers at the monopoly price.¹

With aggregate demand uncertainty, consumer valuations are correlated. It follows from Crémer and McLean (1985, 1988) that there is an interim or Bayesian incentive compatible (BIC) and interim individually rational mechanism (IIR) that fully extracts all consumer surplus in this setting. These mechanisms can impose huge payments, either to or from the consumers, which seems unrealistic. In many monopoly situations, a consumer cannot be prevented from canceling an unfavorable deal when asked for payment. For this reason, Segal (2003) requires ex post incentive compatibility and ex post individual rationality (EIR), and shows that the optimal mechanism involves state-by-state monopoly pricing (SBSMP).²

Since we are in a private values setting, ex post incentive compatibility is equivalent to dominant strategy incentive compatibility (DIC).

Here, we follow Segal’s lead in exploring the implications of EIR. However, imposing DIC rules out, by assumption, many forms of price discrimination. For example, given a profile of reports, DIC requires any anonymous and deterministic mechanism to receive the same payment from (i.e., set the same price for) any consumer receiving the good. Therefore, to capture the situation in which price discrimination is possible but consumers can walk away from an unfavorable deal, we impose EIR and BIC.

We assume that there are two distributions, high and low, from which valuations are independently drawn. Appealing to the law of large numbers, these distributions also correspond to the two possible realized demand curves. We first impose strong regularity conditions on

¹Myerson (1981) solves the optimal auction problem, and Bulow and Roberts (1989) show that the monopoly problem is equivalent to the Myerson setting when consumer valuations are independent (in which case the demand curve would be known in a large economy). See also Harris and Raviv (1981) and Riley and Zeckhauser (1983).

²Segal (2003) analyzes the model with a finite number of consumers and several possible distributions from which valuations are independently drawn. He shows that, when the number of buyers is large, the optimal mechanism converges to SBSMP.
the demand process. Under these assumptions, we show that SBSMP is optimal among all anonymous, deterministic, EIR, and BIC mechanisms. Next we provide an example in which SBSMP is not optimal. In one of the two demand states, a lower valuation type consumes while a higher valuation type does not consume. The example also illustrates that the probability of consuming, conditional on valuation type, \( v \), need not be monotonic in \( v \).

Our main result, providing conditions under which SBSMP is optimal, shows that Segal’s characterization of SBSMP as the optimal ex post mechanism is robust to relaxing DIC to BIC.\(^3\) To our knowledge, this is the first BIC-DIC equivalence result for environments with correlated types and EIR. Crémer and McLean (1988) provide such a result under IIR and a spanning condition, which could require large payments from consumers.\(^4\) Mookherjee and Reichelstein (1992), Manelli and Vincent (2010), and Gershkov et al (2013) all consider BIC-DIC equivalence with independent types.

Since consumption in one state of nature can be interpreted as a different commodity from consumption in another state of nature, this paper is related to the literature on product bundling (see Manelli and Vincent (2006, 2007)). Indeed, our counterexample to SBSMP is similar to an example in Carroll (2017). Daskalakis et al (2017) model uncertainty about the “item type” and show that the optimal mechanism is equivalent to an optimal multi-item mechanism, where a buyer’s valuation for item \( s \) is defined as her valuation for a contract to receive consumption contingent on state \( s \) (i.e., her valuation for the good in state \( s \) multiplied by the probability of state \( s \) ). However, the proof of Theorem 1 in Daskalakis et al (2017) relies on the assumption that the probability of each item type is independent of the buyer’s type, an assumption that is not satisfied in our setting. This issue is discussed further in the concluding remarks.

In section 2, the model is laid out and some preliminary analysis is conducted. Section 3 contains the main result about SBSMP and the counterexample when regularity conditions

\(^3\)Specifically, we are referring to the case in which the number of consumers approaches infinity, there are two possible demand distributions, and we consider deterministic mechanisms.

\(^4\)See also Kushnir (2015).
are not assumed. Section 4 contains some concluding remarks. Proofs are contained in the appendix.

2 Model

A risk neutral, profit-maximizing monopoly firm faces a continuum of consumers with a unit demand for a good. The firm has zero marginal cost of production. There are two demand states, low and high. The probability of the low state is $\alpha_1$ and the probability of the high state is $\alpha_2 = 1 - \alpha_1$. For $i = 1, 2$, consumers' valuations in state $i$ are distributed over $V = [v, \bar{v}]$ according to the demand distribution $D_i(\cdot)$. In particular, $D_i(v)$ is the measure of consumers with valuation greater than $v$ in state $i$. Think of the following process. First, nature selects the demand state, according to the probabilities $\alpha_1$ and $\alpha_2$. Then, out of the measure of “potential” consumers, $C$, nature selects a consumer to be active in state $i$ with probability $D_i(v)/C$. Finally, for the set of selected active consumers, nature independently selects valuations giving rise to the distribution, $D_i(v)$.

Consumers and the firm know the structure of demand, but not the realization.

Because there is aggregate demand uncertainty, a consumer’s valuation provides her with significant information about the demand state. For a consumer whose valuation is $v$, her updated belief about the realized demand state is:

$$Pr(\text{Demand state is } i | \text{own valuation is } v) = \frac{\alpha_i(-D'_i(v))}{\alpha_1(-D'_1(v)) + \alpha_2(-D'_2(v))}. \quad (1)$$

In what follows, we assume that demand is twice-continuously differentiable. The density of the downward sloping demand distribution at valuation $v$ is denoted as $(-D'_i(\cdot))$ in state $i = 1, 2$. We will assume that $(-D'_i(\cdot)) > 0$ holds at all $v \in V$ for $i = 1, 2$.

We consider direct revelation mechanisms satisfying BIC and EIR. According to the revelation principle, consumers report truthfully, without loss of generality. We appeal to

\[ \text{See Peck (2017) for more details and the derivation of (1) below according to Bayes’ rule.} \]
the law of large numbers to conclude that the firm is able to infer the demand state perfectly from the profile of reported types. We restrict attention to deterministic mechanisms that specify, for each state, which valuation types consume and the amount paid by each type that consumes.\footnote{Thus, we require anonymous mechanisms and rule out randomized mechanisms that specify a probability of consuming in state $i$. We are unable to solve the model without this restriction, but it may limit the firm’s profit opportunities. We also rule out introducing randomness indirectly, by allowing consumption to depend on features of the profile of reports other than the inferred state.} The requirement that the payment scheme satisfy ex-post individually rationality implies that the firm is not allowed to charge more than the reported valuation in any demand state and that if a consumer is not given the good in some demand state, then the firm cannot elicit any positive payment from that consumer in that demand state. To summarize, the firm’s problem is to maximize its expected revenue using an anonymous, deterministic, interim incentive compatible, and ex-post individually rational mechanism, when facing a continuum of consumers who update about the demand state based on their private valuations. We state this problem formally in the next sub-section.

2.1 The Monopoly Firm’s Problem

Let $x_i(v)$ denote the probability with which the monopoly firm gives the good to valuation $v$ in state $i$. As noted before, we will restrict ourselves to the case where the firm sells the good to $v$ with probability 1 or 0. So, $x_i(v) \in \{0, 1\}$ for all $v$ and $i = 1, 2$. Let $t_i(v)$ denote the payment required from $v$ given that the demand state is $i$, conditional on $v$ purchasing the good in state $i$. Thus, a mechanism offered by the monopoly firm is as follows:

$$x_i(v) \in \{0, 1\}, \; \forall v \in V, \; i = 1, 2,$$

$$0 \leq t_i(v) \leq v, \; \forall v \in V, \; i = 1, 2. \; \tag{2}$$

For a given mechanism offered by the monopoly firm, let $V_i$ denote the subset of valuations of $V$ who consume only in state $i$. That is, for $i = 1, 2$ and $j \neq i$, $v \in V_i$ if and only if
\(x_i(v) = 1\) and \(x_j(v) = 0\) hold. So, by the ex-post individual rationality (EIR) condition, for valuations in \(V_i\), \(t_i(v)\) can be positive but must be less than \(v\), and \(t_j(v)\) must be 0. Let \(V_{12}\) denote the subset of valuations of \(V\) who consume in both states, in which case \(x_1(v) = 1\) and \(x_2(v) = 1\) hold. So, by the EIR condition, for valuations in \(V_{12}\), both \(t_1(v)\) and \(t_2(v)\) can be positive but must be less than \(v\). Let \(V_∅\) denote the subset of valuations of \(V\) which are not give the good in either state. That is, \(V_∅ = V - [V_1 \cup V_2 \cup V_{12}]\).

We can state the simplified firm’s problem as follows. The firm chooses the sets \(V_1\), \(V_2\), and \(V_{12}\), and the functions \(t_i : V \to [0, \tau]\), for \(i = 1, 2\) to solve:

\[
\max \int_{V_{12}} [t_1(v)\alpha_1(-D'_1(v)) + t_2(v)\alpha_2(-D'_2(v))]dv + \int_{V_1} t_1(v)\alpha_1(-D'_1(v))dv \\
+ \int_{V_2} t_2(v)\alpha_2(-D'_2(v))dv.
\] (3)

Subject to (i) ex-post individual rationality, (henceforth EIR, (2)); and (ii) interim/Bayesian incentive compatibility (henceforth BIC, given below)\(^7\)

\[
(v - t_1(v))x_1(v)\alpha_1(-D'_1(v)) + (v - t_2(v))x_2(v)\alpha_2(-D'_2(v)) \geq \\
(v - t_1(\hat{v}))x_1(\hat{v})\alpha_1(-D'_1(v)) + (v - t_2(\hat{v}))x_2(\hat{v})\alpha_2(-D'_2(v)); \forall v, \hat{v} \in V.
\] (4)

### 2.2 Conditions

In this subsection we specify conditions on the demand process. We will then show that these conditions, in addition to the EIR and BIC conditions (2 and 4), imply that the sets \(V_1\), \(V_2\) and \(V_{12}\) are intervals. First, note the following implication of the BIC condition.

**Fact 1**: If the firm’s mechanism satisfies BIC then \(t_i(v) = t_i(\hat{v})\) must hold for all \(v, \hat{v}\) in \(V_i\), where \(i = 1, 2\).

**Proof of Fact 1**. For \(i = 1, 2\) and \(j \neq i\), if \(v, \hat{v} \in V_i\), then \(x_i(v) = x_i(\hat{v}) = 1\) and

\(^7\)The BIC condition is stated after canceling \([\alpha_1(-D'_1(v)) + \alpha_2(-D'_2(v))]\) from the denominator on both sides of the inequality.
\(x_j(v) = x_j(\hat{v}) = 0\) hold. Thus, the BIC condition (4) implies

\[(v - t_i(v))\alpha_i(-D'_i(v)) \geq (v - t_i(\hat{v}))\alpha_i(-D'_i(v)),\]

which implies \(t_i(v) \leq t_i(\hat{v})\). Similarly, the BIC condition for \(\hat{v}\) implies \(t_i(\hat{v}) \leq t_i(v)\). So the statement of Fact 1 holds. ■

Next, regularity conditions for the two demand states. We start with “maintained assumptions” about demand.

**Condition 1 (regularity).**
(i) \(D_1(v)\) and \(D_2(v)\) are twice continuously differentiable.
(ii) Demand is strictly downward sloping everywhere, i.e., \(D'_i(v) < 0\) holds for all \(v \in V\) and \(i \in \{1, 2\}\).
(iii) Demand is strictly concave, i.e., \(D''_i(v) < 0\) holds for all \(v \in V\) and \(i \in \{1, 2\}\).

**Condition 2 (information effect).**
(i) \(Z(v) \equiv \frac{(-D'_1(v))}{(-D'_2(v))}\) is strictly decreasing in \(v\) for all \(v \in V\). That is, \(Z'(v) < 0\) holds for all \(v \in V\).

Condition 2 specifies the information effect. Note that \(\frac{\alpha_1}{\alpha_2}Z(v)\) is the ratio of the probability a type \(v\) assigns to state 1 to the probability assigned to state 2. Thus, Condition 2 says that, the greater the valuation of a consumer, the greater the probability she assigns to the high demand state.

**Condition 3 (information effect is not “too strong”).** \(vZ(v) = v\frac{(-D'_1(v))}{(-D'_2(v))}\) is strictly increasing in \(v\) for all \(v \in V\).

Condition 3 ensures that the information effect is not too strong, in the following sense.
When considering the overall effect of $v$ on $vZ(v)$, the direct effect of increasing $v$ outweighs the information effect due to $Z(v)$ decreasing. Condition 3 is equivalent to the condition, $|Z'(v)| < \frac{Z(v)}{v}$ for all $v$.

The next step is to characterize the sets $V_1$, $V_2$ and $V_{12}$. In particular, the question is whether the requirement that the firm’s mechanism satisfy BIC, EIR, and Conditions 1-3, implies that the firm’s mechanism must order the sets $V_1$, $V_2$ and $V_{12}$ in a particular manner and whether $V_1$, $V_2$, and $V_{12}$ must have a particular structure. Lemma 1 below describes an implication regarding the order among these sets. Lemma 1 states that the valuations in $V_{12}$ must be greater than the valuations in the sets $V_1$ and $V_2$ if BIC, EIR and Conditions 1-3 are satisfied.

**Lemma 1:** Consider an arbitrary element from each of $V_1$, $V_2$, and $V_{12}$: $v_1 \in V_1$, $v_2 \in V_2$, and $v_{12} \in V_{12}$. Given the BIC and EIR constraints, and Conditions 1-3, we must have (i) $v_1 < v_{12}$, and (ii) $v_2 < v_{12}$.

The proof of Lemma 1 is given in the Appendix.

In what follows, for a given payment scheme, i.e., $t_i : V \rightarrow [0, v]$ for $i \in \{1, 2\}$, and a given good-allocation scheme, i.e. $x_i(v) \in \{0, 1\}$ for $i \in \{1, 2\}$, let $v_1^*, v_2^*$, and $v_{12}^*$ be the infimum valuations which find it BIC and EIR to be in the set $V_1$, $V_2$, and $V_{12}$ respectively (assuming that the appropriate set is non-empty).\(^8\)

The monopoly firm chooses a mechanism from among the following possible types of mechanisms: (1) Only one of the sets $V_1$, $V_2$ or $V_{12}$ is non-empty; (2) $V_1$ and $V_2$ are non-empty, and $V_{12}$ is empty; (3) $V_1$ and $V_{12}$ are non-empty, and $V_2$ is empty; (4) $V_2$ and $V_{12}$ are non-empty, and $V_1$ is empty; (5) $V_1$, $V_2$ and $V_{12}$ are all non-empty.

Note that, if the firm’s mechanism gives the good to valuation $v$ in state $i$ or state $j$ or both, then BIC implies that valuations greater than $v$ are also given the good in some state; because otherwise valuations greater than $v$ can report their valuation as $v$, get the good in

\(^8\)The infima of these sets are well defined because they are bounded subsets of $\mathcal{R}$. 

8
whichever state \( v \) gets the good, and make a payment less than \( v \) (because, by EIR, the firm cannot charged more than the reported valuation to \( v \)), and earn a strictly positive surplus.

Fact 1 implies \( t_i(v) = t_i(v^\ast_i) \) for \( i = 1, 2 \). So, for mechanisms with only one of \( V_{12}, V_1 \), or \( V_2 \) non-empty, the profit expression is either

\[
\int_{v^\ast_{i2}}^{v^\ast_i} [t_1(v^\ast_i)\alpha_1(-D'_1(v)) + t_2(v)\alpha_2(-D'_2(v))]dv, \quad \text{or,} \tag{5}
\]

\[
\int_{v^\ast_i}^{v^\ast_1} t_1(v^\ast_i)\alpha_1(-D'_1(v))dv, \quad \text{or,} \tag{6}
\]

\[
\int_{v^\ast_2}^{v^\ast} t_2(v^\ast_2)\alpha_2(-D'_2(v))dv, \tag{7}
\]

respectively. Further, Lemma 1 implies that for mechanisms with only \( V_1 \) and \( V_{12} \) non-empty, the profit expression is

\[
\int_{v^\ast_{12}}^{v^\ast_1} t_1(v^\ast_1)\alpha_1(-D'_1(v))dv + \int_{v^\ast_{12}}^{v^\ast} [t_1(v)\alpha_1(-D'_1(v)) + t_2(v)\alpha_2(-D'_2(v))]dv. \tag{8}
\]

Lemma 1 also implies that for mechanisms with only \( V_2 \) and \( V_{12} \) non-empty, the profit expression is

\[
\int_{v^\ast_{12}}^{v^\ast_2} t_2(v^\ast_2)\alpha_2(-D'_2(v))dv + \int_{v^\ast_{12}}^{v^\ast} [t_1(v)\alpha_1(-D'_1(v)) + t_2(v)\alpha_2(-D'_2(v))]dv. \tag{9}
\]

We will argue later that the firm will not choose a mechanism with only \( V_1 \) and \( V_2 \) non-empty.

**Lemma 2:** If \( V_1 \) and \( V_2 \) are both non-empty, and \( v^\ast_i < v^\ast_j \) holds; or, if only \( V_i \) is nonempty among \( V_1 \) and \( V_2 \), then BIC, EIR and Conditions 1-3 imply that \( t_i(v^\ast_i) = v^\ast_i = t_i(v) \) holds for all \( v \in V_i \).

**Proof of Lemma 2.** The fact that \( t_i(v) \) is constant for all \( v \in V_i \) has been established by Fact 1. Further, given BIC, EIR and Conditions 1-3, Lemma 1 implies \( v^\ast_i < v^\ast_{12} \). So, if \( V_1 \) and \( V_2 \) are both non-empty, and \( v^\ast_i < v^\ast_j \) holds; or, if only \( V_i \) is nonempty among \( V_1 \) and \( V_2 \),
then valuations less than \( v^*_i \) are not given the good in either state. To see why \( t_i(v^*_i) = v^*_i \) holds, note that for any \( v \) such that \( v < v^*_i \) holds, \( t_i(v^*_i) \geq v \) must hold to satisfy BIC of \( v \). So \( t_i(v^*_i) \geq v^*_i \) must hold. The EIR condition for \( v^*_i \) implies \( t_i(v^*_i) \leq v^*_i \). Putting these two statements together, we have \( t_i(v^*_i) = v^*_i \). ■

An implication of Lemma 2 is that \( t_1(v^*_1) \) can be replaced by \( v^*_i \) in (6) and (8), and \( t_2(v^*_2) \) can be replaced by \( v^*_2 \) in (7) and (9).

Consider a mechanism such that \( V_1, V_2, \) and \( V_{12} \) are all non-empty. For an arbitrary payment scheme, and an arbitrary good-allocation scheme, Conditions 1-3, and the BIC and EIR constraints do not imply any particular order among the sets \( V_1 \) and \( V_2 \). In what follows, we will evaluate the monopoly profit from mechanisms such that either \( v^*_1 < v^*_2 \) holds, or \( v^*_2 < v^*_1 \) holds.

Condition 4 (below) is used to analyze the case where the firm chooses a mechanism such that the sets \( V_1, V_2, \) and \( V_{12} \) are all non-empty. Since \( \frac{\alpha_1}{\alpha_2} Z(v) \) is the ratio of the probability a type \( v \) assigns to state 1 to the probability assigned to state 2, Condition 4 states that either (Condition 4(i)) all consumers’ updated belief is that the high demand state is more likely or (Condition 4(ii)) all consumers’ updated belief is that the low demand state is “sufficiently” more likely. Condition 4 will be used to prove Lemma 3, which says that for a mechanism such that \( V_1, V_2, \) and \( V_{12} \) are all non-empty, these sets must be adjacent intervals.

**Condition 4** (agreement over which state is more likely) Either (i) \( 1 \geq \frac{\alpha_1}{\alpha_2} Z(v) \) holds for all \( v \in V \); or

(ii) \( \frac{\alpha_1}{\alpha_2} Z(v) > 1 - Z'(v) \frac{\alpha_1}{\alpha_2} (\bar{v} - v) \) holds for all \( v \in V \).

**Lemma 3(a):** Consider a mechanism such that \( V_1, V_2, \) and \( V_{12} \) are all non-empty. Given Conditions 1-3, Condition 4(i) implies that for any BIC and EIR payment and good allocation scheme, the following hold: \( v^*_1 < v^*_2 < v^*_{12}, \) and \( V_1 = [v^*_1, v^*_2], V_2 = [v^*_2, v^*_{12}], \) and \( V_{12} = [v^*_1, \bar{v}] \).

**Lemma 3(b):** Consider a mechanism such that \( V_1, V_2, \) and \( V_{12} \) are all non-empty. Given
Conditions 1-3, Condition 4(ii) implies that for any BIC and EIR payment and good allocation scheme, the following hold: \( v_2^* < v_1^* < v_{12}^* \), and \( V_2 = [v_2^*, v_1^*], \ V_1 = [v_1^*, v_{12}^*], \) and \( V_{12} = [v_{12}^*, \bar{v}] \).

The proofs for Lemma 3(a) and 3(b) are given in the Appendix.

Lemma 3(a) states that under Conditions 1-3, BIC, and EIR, if Condition 4(i) holds, then (using Fact 1 and Lemma 2) the firm’s profit in the \( V_1, V_2, \) and \( V_{12} \) all non-empty case is:

\[
\int_{v_1^*}^{v_2^*} v_1^* \alpha_1(-D_1'(v))dv + \int_{v_2^*}^{v_{12}^*} t_2(v_2^*)\alpha_2(-D_2'(v))dv \\
+ \int_{v_{12}^*}^{\bar{v}} [t_1(v)\alpha_1(-D_1'(v)) + t_2(v)\alpha_2(-D_2'(v))]dv.
\]

On the other hand, under Conditions 1-3, BIC and, EIR, if Condition 4(ii) holds, then (using Fact 1 and Lemma 2) the firm’s profit in the \( V_1, V_2, \) and \( V_{12} \) all non-empty case is:

\[
\int_{v_2^*}^{v_1^*} v_2^* \alpha_2(-D_2'(v))dv + \int_{v_2^*}^{v_{12}^*} t_1(v_1^*)\alpha_1(-D_1'(v))dv \\
+ \int_{v_{12}^*}^{\bar{v}} [t_1(v)\alpha_1(-D_1'(v)) + t_2(v)\alpha_2(-D_2'(v))]dv.
\]

3 The Optimal Mechanism

Now that we have conditions under which the sets \( V_1, V_2, \) and \( V_{12} \) are intervals (whenever non-empty), we are in a position to describe the monopoly firm’s optimal mechanism under these conditions. Proposition 1 describes sufficient conditions under which the firm’s optimal mechanism is to set the monopoly price in each demand state.

**Proposition 1:** For \( i = 1, 2 \), let \( p_i^{\text{m}} \) be the profit maximizing monopoly price in demand state \( i \). Suppose Conditions 1-3 hold. If Condition 4(ii) holds and \( \alpha_1 \) is sufficiently large, or, if condition 4(i) holds, \( \alpha_2 \) is sufficiently large, and

\[
\frac{\partial p_i^{\text{m}}}{\partial v} D_2(p_i^{\text{m}}) > \frac{\int_{v_i^*}^{\bar{v}} D_2(v)(-Z'(v))dv}{Z(p_i^{\text{m}})}
\]

holds, then, within the class of BIC and EIR mechanisms, the monopoly firm’s optimal mech-
anism is state-by-state monopoly pricing. The state-by-state monopoly pricing mechanism is as follows:

\[
\begin{cases}
t_1(v^*_1) = t_1(v^*_2) = v^*_1 = p^m_1, \\
t_2(v^*_12) = v^*_12 = p^m_2, \\
x_1(v) = 1 \forall v \geq p^m_1, & x_1(v) = 0 \forall v < p^m_1, \\
x_2(v) = 1 \forall v \geq p^m_2, & x_2(v) = 0 \forall v < p^m_2.
\end{cases}
\] (13)

**Proof Sketch.** The formal proof of Proposition 1 is given in the Appendix. The proof starts by showing that under sufficient conditions for the sets \( V_1, V_2 \) and \( V_{12} \) to be intervals (when all three sets are non-empty), it will not be optimal for the firm to choose a mechanism such that all three sets are non-empty. Next, using only the Conditions 1-3 and BIC and EIR constraints, we show that compared to all other mechanisms, except mechanisms where only \( V_2 \) and \( V_{12} \) are non-empty, the state-by-state monopoly pricing mechanism (henceforth, SBSMP mechanism) yields higher profit. To show that SBSMP yields higher profit than mechanisms where only \( V_2 \) and \( V_{12} \) are nonempty, we have to use that either condition 4(ii) holds and \( \alpha_1 \) is sufficiently large, or, condition 4(i) holds, \( \alpha_2 \) is sufficiently large, and (12) holds.

**Remark.** Proposition 1 makes strong assumptions, but the following structure satisfies all of our assumptions for \( \frac{\alpha_1}{\alpha_2} \) sufficiently small or sufficiently large:

\[
\begin{align*}
D_1(v) &= K - \frac{v^a}{K^{a-1}}, \\
D_2(v) &= K - \frac{v^b}{K^{b-1}}, \\
a &< b < a + 1, \\
K &> 2.
\end{align*}
\]
Furthermore, these sufficient conditions are far from necessary. We know of no counterexamples to SBSMP that satisfy our basic assumption about the information effect given in Condition 2.

### 3.1 A Counterexample to SBSMP

Here we provide an example in which SBSMP is not optimal for the monopolist. For simplicity, we consider an example with 3 valuation types, then we discuss how the example can be extended to a continuum of types without mass points.

There are two states, each occurring with prior probability one half. There are three types, where type 1 consumers have valuation 18, type 2 consumers have valuation 20, and type 3 consumers have valuation 40. In state 1, the measure of type 1 consumers is 18, the measure of type 2 consumers is 2, and the measure of type 3 consumers is 5. In state 2, the measure of type 1 consumers is 2, the measure of type 2 consumers is 18, and the measure of type 3 consumers is 5.

Conditional on being type 1, the probability of state 1 is 0.9; conditional on being type 2, the probability of state 1 is 0.1; and conditional on being type 3, the probability of state 1 is 0.5. Since, for the case of continuous demand, $\frac{\alpha_1}{\alpha_2} Z(v)$ is the ratio of the conditional belief of state 1 to the conditional belief of state 2, the analog of $Z(v)$ for this example would yield $Z(18) = 9$, $Z(20) = \frac{1}{9}$, and $Z(40) = 1$. Thus, condition 2 is violated, because the middle type assigns the highest probability to the high demand state.

It is easy to see that $p_1^m = 18$ holds, yielding monopoly profits of 450 in state 1, and $p_2^m = 20$ holds, yielding monopoly profits of 460 in state 2. Thus, SBSMP yields expected profits of 455.

Consider instead the following mechanism. Those reporting type 1 consume in state 1 and pay 18 (so $t_1(18) = 18$), and they consume nothing and pay nothing in state 2. Those reporting type 2 consume in state 2 and pay $\frac{178}{9}$ (so $t_2(20) = \frac{178}{9}$), and they consume nothing and pay nothing in state 1. Those reporting type 3 consume in both states and pay 29.
Clearly, the mechanism satisfies all EIR constraints and BIC constraints for type 1. To see that BIC constraints are satisfied for type 2, it suffices to show that they weakly prefer truthful reporting to reporting type 1,

\[
\frac{9}{10}(20 - \frac{178}{9}) \geq \frac{1}{10}(20 - 18),
\]

which holds with equality. To see that BIC constraints are satisfied for type 3, it suffices to show that they weakly prefer truthful reporting to reporting type 1,

\[
40 - 29 \geq \frac{1}{2}(40 - 18),
\]

which holds with equality. The firm’s profits are 485, which exceeds the SBSMP profit level.

If the firm is subject only to IIR constraints and BIC constraints, then full surplus can be extracted à la Crémer and McLean. Have all consumers pay their reported valuation and consume in all states. In addition, those reporting type 1 make the side bet in which they pay the firm 9000 in state 2 and receive 1000 from the firm in state 1; those reporting type 2 pay the firm 9000 in state 1 and receive 1000 in state 2; and those reporting type 3 do not have a side bet. It is easy to see that these side bets yield zero expected payments under truthful reporting, and that they yield large losses if a consumer misreports. IIR and BIC are satisfied, and the firm receives profits of 580.

**Extending the Example.** In the example, types 1 and 2 each consume with probability \( \frac{9}{10} \). If we were to modify the example to make the prior probability of state 1 slightly greater than \( \frac{1}{2} \), then the computations would be messier, but we would have the additional result that the conditional probability of consuming is not weakly monotonic in valuation, because type 1 consumers would be more likely to consume than type 2 consumers.

Finally, we could modify the example to a continuum of valuation types with no mass points. Spread the measure of type 1 consumers in each state to be uniformly distributed between valuations over the interval, \([18, 18 + \varepsilon]\), spread the measure of type 2 consumers in
each state to be uniformly distributed between valuations over $[20, 20 + \varepsilon]$, and spread the
measure of type 3 consumers in each state to be uniformly distributed between valuations
over $[40, 40 + \varepsilon]$. Then include additional consumers with all valuations in between (in both
states), where the total measure of additional consumers is $\varepsilon$. For small enough $\varepsilon$, it is clear
that we have $p_1^m = 18$ and $p_2^m = 20$, and that a mechanism very close to the one presented
above will yield higher profits than SBSMP.

4 Concluding Remarks

Our main result is that, under strong regularity conditions, SBSMP is optimal among all
anonymous, deterministic, ex post IR (EIR), and interim IC (BIC) mechanisms. The result is
far from obvious, as illustrated by our counterexample when the regularity conditions are not
satisfied. It would be nice to allow for randomized mechanisms and generalize beyond two
states, but much of the Myerson machinery is unavailable and very few results are available
in the literature when types are correlated. Our strong regularity conditions are needed to
allow us to evaluate the monopoly profit from different possible price discrimination schemes.
For tractability, it is important to show that the sets $V_1$, $V_2$, and $V_{12}$ are intervals. If $v_1^* < v_2^*$
holds, then Condition 2 (the single crossing property saying that higher types attach higher
probabilities to the higher demand state) ensures that, if type $v_2^*$ weakly prefers being in set
$V_2$ to set $V_1$, then higher types strictly prefer being in set $V_2$ to set $V_1$. However, Condition
2 works against incentive compatibility if $v_2^* < v_1^*$ holds. In that case, if $v_1^*$ weakly prefers
being in set $V_1$ to set $V_2$, then higher types may strictly prefer being in set $V_2$ to set $V_1$.
Thus, the information effect specified in Condition 2 cannot be too strong, so we impose
Condition 3. Providing conditions to rule out all the alternatives to SBSMP can become a
balancing act, even though we believe that counterexamples are unusual.

Our counterexample illustrates the differences between our setting and the multi-good
setting in which consumers purchase state-contingent consumption. Consumer 1’s vector
of willingness to pay for state-contingent consumption is (16.2, 1.8), consumer 2’s vector of willingness to pay for state-contingent consumption is (2, 18), and consumer 3’s vector of willingness to pay for state-contingent consumption is (20, 20). The multi-good analogue of the mechanism in our counterexample is to offer state-1 contingent consumption to type 1 consumers at the price 16.2, state-2 contingent consumption to type 2 consumers at the price 17.8, and both state-1 and state-2 contingent consumption to type 3 consumers at the price of 29 for the bundle. Both mechanisms yield the same expected profits from each consumer type, but there are differences regarding individual rationality and incentive compatibility. In the multi-good setting, interim individual rationality holds, but individual rationality does not hold ex post. Some type 1 consumers do not consume in state 2 and some type 2 consumers do not consume in state 1; if these consumers were to observe all of the reports, they would prefer not to participate.

Perhaps more significantly, incentive compatibility differs across the two mechanisms, due to the fact that beliefs differ across consumers. In our mechanism, type 2 and type 3 consumers are indifferent between truthfully reporting their type and instead reporting themselves as type 1. For example, a type 2 consumer pays less than her valuation in state 2 (she pays \( \frac{178}{9} \) and her valuation is 20), because she could receive positive surplus by claiming to be type 1 and paying only 18 in state 1. However, in the analogous mechanism in the multi-good setting, incentive compatibility for type 2 and type 3 consumers holds strictly. For example, a type 2 consumer would receive negative surplus by claiming to be type 1, because she would pay 16.2 for state-1 contingent consumption and only values that consumption at 2. Thus, aggregate uncertainty causes beliefs about the state to differ across consumer types, which breaks the reinterpretation of the monopolist’s problem as a standard bundling problem.\(^9\)

\(^9\)For the special case of multiplicative uncertainty, where there exists \( \alpha \) such that \( D_1(v) = \alpha D_2(v) \) for all \( v \), then all consumers have the same beliefs about the demand state, and the incentive compatibility issues discussed here do not arise.
5 Appendix

Proof of Lemma 1. Proof of (i). First, we will show that \( v_1 < v_{12} \) must hold. Suppose not; that is, let \( v_1 > v_{12} \) hold (note that we cannot have \( v_1 \) equal to \( v_{12} \) because a valuation cannot belong to both \( V_1 \) and \( V_{12} \)). Consider the BIC constraints of \( v_{12} \) with respect to \( v_1 \) and of \( v_1 \) with respect to \( v_{12} \):

\[
\begin{align*}
(BIC: v_{12}) (v_1 - t_1(v_{12})) & \leq (v_1 - t_2(v_{12})) + (v_{12} - t_2(v_{12})) \alpha_2(-D'_2(v_{12})) \\
& \quad + (v_{12} - t_1(v_1)) \alpha_1(-D'_1(v_{12}));
\end{align*}
\]

and

\[
\begin{align*}
(BIC: v_1) (v_{12} - t_1(v_{12})) & \geq (v_1 - t_2(v_{12})) \alpha_2(-D'_2(v_1));
\end{align*}
\]

These can be rewritten as

\[
\begin{align*}
(BIC: v_{12}) (v_{12} - t_2(v_{12})) & \frac{\alpha_2}{\alpha_1 Z(v_{12})} \geq (v_{12} - t_1(v_1)); \tag{14}
\end{align*}
\]

and

\[
\begin{align*}
(BIC: v_1) (v_{12} - t_1(v_1)) & \geq (v_1 - t_2(v_{12})) \frac{\alpha_2}{\alpha_1 Z(v_1)}. \tag{15}
\end{align*}
\]

Together, (14) and (15) imply:

\[
\begin{align*}
(v_{12} - t_2(v_{12})) \frac{\alpha_2}{\alpha_1 Z(v_{12})} & \geq (v_1 - t_2(v_{12})) \frac{\alpha_2}{\alpha_1 Z(v_1)}. \tag{16}
\end{align*}
\]

Next, note that \( v_1 > t_2(v_{12}) \) must hold. This is because, by EIR, \( v_{12} \geq t_2(v_{12}) \) must hold; further, \( v_1 > v_{12} \) holds by assumption. Thus, \( v_1 > v_{12} \geq t_2(v_{12}) \) holds. Note that \( v_1 > v_{12} \) implies \( (v_1 - t_2(v_{12})) > (v_{12} - t_2(v_{12})) \). Further, by Condition 2, \( \frac{1}{Z(v)} \) is increasing for all \( v \).
Thus, \( \frac{a_2}{\alpha_1 Z(v_1)} > \frac{a_2}{\alpha_1 Z(v_{12})} \) holds, which implies,

\[
(v_{12} - t_2(v_{12})) \frac{a_2}{\alpha_1 Z(v_{12})} < (v_1 - t_2(v_{12})) \frac{a_2}{\alpha_1 Z(v_1)},
\]

which is a contradiction of (16). Thus it must be the case that \( v_{12} > v_1 \) holds.

**Proof of (ii).** The aim is to show \( v_2 < v_{12} \). The proof is by contradiction, that is, suppose that \( v_{12} < v_2 \) holds. Consider the BIC constraints of \( v_{12} \) with respect to \( v_2 \) and of \( v_2 \) with respect to \( v_{12} \).

\[
(BIC: v_{12}) \quad (v_{12} - t_1(v_{12})) \alpha_1 (-D_1'(v_{12})) + (v_{12} - t_2(v_{12})) \alpha_2 (-D_2'(v_{12})) \geq (v_{12} - t_2(v_2)) \alpha_2 (-D_2'(v_2)),
\]

\[
(BIC: v_2) \quad (v_2 - t_2(v_2)) \alpha_2 (-D_2'(v_2)) \geq (v_2 - t_1(v_{12})) \alpha_1 (-D_1'(v_{12})) + (v_2 - t_2(v_{12})) \alpha_2 (-D_2'(v_{12})).
\]

These can be rewritten, respectively, as

\[
(BIC: v_{12}) \quad (v_{12} - t_1(v_{12})) \frac{\alpha_1 Z(v_{12})}{\alpha_2} \geq (t_2(v_{12}) - t_2(v_2)), \quad (17)
\]

and

\[
(BIC: v_2) \quad (t_2(v_{12}) - t_2(v_2)) \geq (v_2 - t_1(v_{12})) \frac{\alpha_1 Z(v_2)}{\alpha_2}. \quad (18)
\]

Together, (17) and (18) imply

\[
(v_{12} - t_1(v_{12})) \frac{\alpha_1 Z(v_{12})}{\alpha_2} \geq (v_2 - t_1(v_{12})) \frac{\alpha_1 Z(v_2)}{\alpha_2}.
\]

This can be rewritten as

\[
(v_{12} - t_1(v_{12})) Z(v_{12}) \geq (v_2 - t_1(v_{12})) Z(v_2). \quad (19)
\]
Note that \(-t_1(v_{12})Z(v_{12}) \leq -t_1(v_{12})Z(v_2)\) must hold, because \(Z(v_2) < Z(v_{12})\) holds, given that \(Z(v)\) is strictly decreasing (Condition 2) and due to the assumption that \(v_{12} < v_2\) holds. Further, \(v_{12}Z(v_{12}) < v_2Z(v_2)\) holds, because of the assumption that \(v_{12} < v_2\) holds and because Condition 3 stipulates that \(vZ(v)\) is strictly increasing in \(v\). Therefore \((v_{12} - t_1(v_{12}))Z(v_{12}) < (v_2 - t_1(v_{12}))Z(v_2)\) must hold, which contradicts (19).

**Proof of Lemmas 3(a) and 3(b).** For the purpose of both Lemma 3(a) and 3(b) recall that, as argued in the main text, the BIC conditions imply that there cannot be a valuation \(v\) such that \(v > v_i^*\) holds for some \(i \in \{1, 2, 12\}\) and \(v \in V_2\).

**Proof of Lemma 3(a).** We fix an arbitrary payment scheme, \(t_i : V \rightarrow [0, \overline{v}]\) for \(i \in \{1, 2\}\), and an arbitrary allocation scheme, \(x_i : V \rightarrow 0, 1\) for \(i \in \{1, 2\}\) satisfying BIC and EIR.

Let \(V_i \succeq^v V_j\) (respectively \(V_i \succeq^v V_j\)) denote the statement “type \(v\) prefers (strictly prefers) to be in the set \(V_i\) than in the set \(V_j\).” This notation is well defined because, by Fact 1, all types within \(V_1\) must be treated the same and all types within \(V_2\) must be treated the same.

Claims L3a1 and L3a2 (below), along with Lemma 1, complete the proof for Lemma 3(a).

**Claim L3a1:** Given Conditions 1-3 and the BIC and EIR conditions, Condition 4(i) implies \(v_1^* < v_2^*\).

**Proof of Claim L3a1.** The proof is by contradiction. Suppose \(v_2^* < v_1^*\) holds. Then Lemma 1 and Lemma 2 imply \(t_2(v) = v_2^* = v\) for all \(v \in V_2\). The IC of \(v_1^*\) with respect to \(V_2\) implies \(t_1(v_1^*) < v_1^*\), because otherwise \(v_1^*\) can report himself as \(v_2^*\) and earn a strictly positive surplus of \((v_1^* - v_2^*)\alpha_2(-D_2'(v_1^*))\). Further, \([v_2^*, v_1^*) \subset V_2\) holds, because \(v_2^*, v_1^*\), and \(v_{12}^*\) are infima of \(V_1\), \(V_2\), and \(V_{12}\), and because Lemma 1 implies \(\max\{v_1^*, v_2^*\} < v_{12}^*\). Thus, there exists a \(v \in (t_1(v_1^*), v_1^*)\) such that \(v \in V_2\). If we show that \(V_2 \succeq^v V_1\) implies that \(V_2 \succeq^v V_1\) must hold, then this will contradict the IC of \(v_1^*\) with respect to \(V_2\), and the proof shall be complete. So consider \(v \in (t_1(v_1^*), v_1^*)\) such that \(V_2 \succeq^v V_1\) holds. The statement \(V_2 \succeq^v V_1\) can be written as
\[(v - v_2^*)\alpha_2(-D'_2(v)) \geq (v - t_1(v_1^*))\alpha_1(-D'_1(v)), \text{ or,}\]
\[(v - v_2^*) \geq (v - t_1(v_1^*))\frac{\alpha_1\alpha_2}{\alpha_2}Z(v)\]  \(\text{(20)}\)

The derivative of the left side of (20) with respect to \(v\) is equal to 1. The derivative of the right side of (20) with respect to \(v\) is equal to \(\frac{\alpha_1\alpha_2}{\alpha_2}(v - t_1(v_1^*))\frac{\alpha_1\alpha_2}{\alpha_2}Z'(v)\). By Condition 4(i), \(1 \geq \frac{\alpha_1\alpha_2}{\alpha_2}Z(v)\) holds for all \(v \in V\), and, by Condition 2, \(Z'(v) < 0\) holds for all \(v \in V\). Thus, we must have
\[1 > \frac{\alpha_1\alpha_2}{\alpha_2}(v_2 - t_1(v_1^*))\frac{\alpha_1\alpha_2}{\alpha_2}Z'(v_2),\]
which implies \(V_2 \succ^v V_1\), contradicting the BIC of \(v_1^*\).

Claim L3a2: Given conditions 1-3 and the BIC and EIR conditions, \(v_1^* < v_2^*\) implies that there does not exist a \(v\) such that \(v > v_2^*\) and \(V_1 \succ^v V_2\) hold. So, \(V_1 = [v_1^*, v_2^*], V_2 = [v_2^*, v_{12}^*], V_{12} = [v_{12}^*, v]\) hold.

Proof of Claim L3a2. Suppose there exists a \(v\) such that \(v > v_2^*\) and \(V_1 \succ^v V_2\) hold. We will show that this implies that \(V_1 \succ^v V_2\) holds. Using Claim L3a1 and Lemma 2, we know that \(t_1(v_1^*) = v_1^*\) holds. So \(V_1 \succ^v V_2\) implies:
\[(v - v_1^*)\alpha_1(-D'_1(v)) \geq (v - t_2(v_2^*))\alpha_2(-D'_2(v)). \text{ Or,}\]
\[\frac{\alpha_1}{\alpha_2}Z(v) \geq \frac{(v - t_2(v_2^*))}{(v - v_1^*)}.\]  \(\text{(22)}\)

Since we have assumed \(v > v_2^*\) and since \(Z(v)\) is strictly decreasing (Condition 2), we have \(\frac{\alpha_1}{\alpha_2}Z(v_2^*) > \frac{\alpha_1}{\alpha_2}Z(v)\). Further, \(t_2(v_2^*) \geq v_1^*\) must hold, because otherwise type \(v_1^*\) would receive positive surplus in \(V_2\) and receive zero surplus in \(V_1\). Therefore, we have
\[\frac{(v - t_2(v_2^*))}{(v - v_1^*)} \geq \frac{(v_2^* - t_2(v_2^*))}{(v_2^* - v_1^*)}.\]  \(\text{(23)}\)
Thus, (22) and (23) imply

\[
\frac{\alpha_1}{\alpha_2}Z(v_2^*) > \frac{(v_2^* - t_2(v_2^*))}{(v_2^* - v_1^*)}. \tag{24}
\]

Substituting \(Z(v_2^*) = \frac{-D'_1(v_2^*)}{-D'_2(v_2^*)}\) into (24) and cross-multiplying yields

\[
(v_2^* - v_1^*)\alpha_1(-D'_1(v_2^*)) > (v_2^* - t_2(v_2^*))\alpha_2(-D'_2(v_2^*)).
\]

which implies \(V_1 \succ v_2^* V_2\), contradicting the IC of \(v_2^*\) with respect to \(V_1\). Thus, for all types \(v\) such that \(v > v_2^*\) holds, \(V_2 \succ v V_1\) must hold. Since \(v_2^*\), the infimum of \(V_2\), and \(v_{12}^*\), the infimum of \(V_{12}\) are both strictly greater than \(v_1^*\), it follows that \([v_1^*, v_2^*] \subset V_1\) holds. Since we have \(V_2 \succ v V_1\), for all \(v > v_2^*\), it then follows that \(V_1 = [v_1^*, v_2^*]\).

Last, note that \(V_2 = [v_2^*, v_{12}^*]\) holds because there cannot exist a \(v \in [v_2^*, v_{12}^*]\) such that \(v \in V_{12}\) holds as that would contradict the definition of \(v_{12}^*\) as the infimum of \(V_{12}\). Further, \(V_{12} = [v_{12}^*, \bar{v}]\) holds by Lemma 1. So we must have \(V_1 = [v_1^*, v_2^*], V_2 = [v_2^*, v_{12}^*],\) and \(V_{12} = [v_{12}^*, \bar{v}]\).

**Proof of Lemma 3(b).**

**Claim L3b1:** *Given Conditions 1-3 and the BIC and EIR conditions, Condition 4(ii) implies \(v_2^* < v_1^*\).*

**Proof of Claim L3b1.** The proof is by contradiction. Suppose \(v_1^* < v_2^*\) holds. Then Lemma 1 and Lemma 2 imply \(t_1(v_1^*) = v_1^* = t_1(v)\) for all \(v \in V_1\). The IC of \(v_2^*\) with respect to \(V_1\) implies \(t_2(v_2^*) < v_2^*\), because otherwise \(v_2^*\) can report himself as \(v_1^*\) and earn a strictly positive surplus of \((v_2^* - v_1^*)\alpha_1(-D'_1(v_2^*))\). Further, \([v_1^*, v_2^*] \subset V_1\) holds, because \(v_1^*, v_2^*,\) and \(v_{12}^*\) are the infima of \(V_1, V_2,\) and \(V_{12}\), and because Lemma 1 implies \(max\{v_1^*, v_2^*\} < v_{12}^*\). Consider a valuation \(v \in (t_2(v_2^*), v_2^*)\) such that \(v \in V_1\). As \(v \in V_1\), by BIC, \(V_1 \succeq v V_2\) must hold. If we show that \(V_1 \succeq v V_2\) implies \(V_1 \succ v_2^* V_2\), then this will contradict the BIC of \(v_2^*\) with respect to \(V_1\), and the proof of Claim L3b1 shall be complete. The statement \(V_1 \succeq v V_2\) can be written
as

\[(v - v^*_1)α_1(-D'_1(v)) ≥ (v - t_2(v^*_2))α_2(-D'_2(v)),\] or,

\[(v - v^*_1) \frac{α_1Z(v)}{α_2} ≥ (v - t_2(v^*_2))\]

The derivative of the right side of (25) with respect to \(v\) is equal to 1. The derivative of the left side of (25) with respect to \(v\) is equal to \(\frac{α_1Z(v)}{α_2} + (v - v^*_1)\frac{α_1Z'(v)}{α_2}\). By Condition 4(ii),

\[\frac{α_1Z(v)}{α_2} + (v - v^*_1)\frac{α_1Z'(v)}{α_2} > 1\]

holds. So the derivative of the left side of (25) is strictly greater than the derivative of the right side of (25). Thus, (25) holds with strict inequality for \(v^*_2\), which is strictly greater than \(v\) (due to the assumption: \(v^*_1 < v^*_2\)). That is,

\[(v^*_2 - v^*_1)\frac{α_1Z(v^*_2)}{α_2} > (v^*_2 - t_2(v^*_2))\]

holds. (26) can be rearranged to

\[(v^*_2 - v^*_1)α_1(-D'_1(v^*_2)) > (v^*_2 - t_2(v^*_2))α_2(-D'_2(v^*_2)),\]

which implies \(V_1 ≻ v^*_2 V_2\), which contradicts the definition of \(v^*_2\) as the valuation that finds it BIC to be the infimum of \(V_2\).

**Claim L3b2:** Given Conditions 1-3, 4(ii) and the BIC and EIR conditions, \(v^*_2 < v^*_1\) implies that there does not exist a \(v\) such that \(v > v^*_1\) and \(V_2 ≽^v V_1\) hold. So, \(V_2 = [v^*_2, v^*_1]\), \(V_1 = [v^*_1, v^*_12]\), and \(V_{12} = [v^*_12, \overline{v}]\) hold.

**Proof of Claim L3b2.** Suppose there exists \(v\) such that \(v > v^*_1\) and \(V_2 ≽^v V_1\) hold. Using Lemma 2, and \(v^*_2 < v^*_1\), \(t_2(v^*_2) = v^*_2\) holds. The statement \(V_2 ≽^v V_1\) can be written as:
\[(v - v^*_2)\alpha_2(-D'_2(v)) \geq (v - t_1(v^*_1))\alpha_1(-D'_1(v)).\]

Which simplifies to,
\[
(v - v^*_2) \geq (v - t_1(v^*_1))\frac{\alpha_1}{\alpha_2}Z(v).
\] (27)

The derivative of the left side of (27) with respect to \(v\) is 1. The derivative of the right side of (27) with respect to \(v\) is
\[
\frac{\alpha_1}{\alpha_2}Z(v) + Z'(v)\frac{\alpha_1}{\alpha_2}(v - t_1(v^*_1)).
\]

Condition 4(ii) states that, we have, for all \(v\),
\[
\frac{\alpha_1}{\alpha_2}Z(v) > 1 - Z'(v)\frac{\alpha_1}{\alpha_2}(\bar{v} - v),
\]
which implies
\[
1 < \frac{\alpha_1}{\alpha_2}Z(v) + Z'(v)\frac{\alpha_1}{\alpha_2}(v - t_1(v^*_1)).
\]

Thus, the derivative of the left side of (27) is strictly less than the derivative of the right side of (27). Then since \(v^*_1 < v\) holds, (27) must hold strictly for \(v^*_1\), yielding
\[
(v^*_1 - v^*_2) > (v^*_1 - t_1(v^*_1))\frac{\alpha_1}{\alpha_2}Z(v^*_1).
\] (28)

Using \(Z(v^*_1) = -\frac{D'_1(v^*_1)}{D'_2(v^*_1)}\), inequality (28) becomes
\[
(v^*_1 - v^*_2)\alpha_2(-D'_2(v^*_1)) > (v^*_1 - t_1(v^*_1))\alpha_1(-D'_1(v^*_1)),
\]
which is equivalent to \(V_2 \succ^v V_1\). This contradicts the BIC of \(v^*_1\) with respect to \(V_2\). Thus, there cannot exist a type \(v\) such that \(v > v^*_1\) and \(V_2 \succeq^v V_1\) hold. In other words, all types \(v > v^*_1\) strictly prefer \(V_1\) to \(V_2\), which implies \(V_2 = [v^*_2, v^*_1]\). Finally, for \(v \in [v^*_1, v^*_2]\), type \(v\) cannot prefer \(V_{12}\) to \(V_1\), because that would imply \(v \in V_{12}\), contradicting the definition
of \( v_{12}^* \) as the infimum of \( V_{12} \). Thus, we have \([v_1^*, v_{12}^*] \subset V_1 \). By Lemma 1, there does not exist a valuation \( v \) greater than \( v_{12}^* \) such that \( v \in V_1 \). Thus, we have \( V_1 = [v_1^*, v_{12}^*] \), and \( V_{12} = [v_{12}^*, \bar{v}] \).

\[ \]

5.1 Proof of Proposition 1

5.1.1 Ruling out \( V_1, V_2, \) and \( V_{12} \) all non-empty.

**Claim:** Under Conditions 1-4, in the monopoly firm’s optimal mechanism within the class of BIC and EIR mechanisms, the sets \( V_1, V_2, \) and \( V_{12} \) cannot all be non-empty.

**Proof.** The proof of this claim relies on Lemmas 4-7 detailed below. The following is the road-map for the proof. Lemma 4 will characterize the firm’s optimal payment scheme for the case where Condition 4(i) holds. The profit expression for this case is given by (10). In Lemma 5 we will show that compared to the profit from setting \( V_1 = [v_1^*, v_2^*] \), \( V_2 = [v_2^*, v_{12}^*] \), and \( V_{12} = [v_{12}^*, \bar{v}] \), and setting the payment scheme as the firm’s optimal corresponding payment scheme, the firm can strictly increase profits by shrinking the interval \( V_2 \). Lemma 6 and Lemma 7 will deal with the case where Condition 4(ii) holds. The profit expression for this case is given by (11). The firm’s optimal payment scheme for this case is characterized in Lemma 6. Lemma 7 will then show that we arrive at a contradiction if \( V_2 = [v_2^*, v_1^*] \), \( V_1 = [v_1^*, v_{12}^*] \), and \( V_{12} = [v_{12}^*, \bar{v}] \) hold, and the firm charges the optimal payment scheme for these sets, and we assume that all these sets are non-empty in the firm’s optimal mechanism.

**Lemma 4:** Consider the class of mechanisms where \( V_1, V_2, \) and \( V_{12} \) are all non-empty. Suppose Conditions 1-3 and 4(i) hold at the optimal mechanism within the class of BIC and EIR mechanisms, then for all \( v_1 \in V_1, v_2 \in V_2, \) and \( v_{12} \in V_{12} \), the optimal payments are characterized by
\[ t_1(v_1) = v_1^*, \]
\[ t_2(v_2) = v_2^* - (v_2^* - v_1^*) \frac{\alpha_1}{\alpha_2} Z(v_2^*), \]
\[ t_1(v_12) = v_1^*, \]
\[ t_2(v_12) = v_2^* - (v_2^* - v_1^*) \frac{\alpha_1}{\alpha_2} Z(v_2^*) + (v_{12}^* - v_1^*) \frac{\alpha_1}{\alpha_2} Z(v_{12}^*). \]

**Proof of Lemma 4.** Since Conditions 1-3, 4(i), BIC and EIR hold, Lemma 3(a) applies. So \( v_1^* < v_2^* < v_{12}^* \) holds and \( V_1 = [v_1^*, v_2^*], V_2 = [v_2^*, v_{12}^*], \) and \( V_{12} = [v_{12}^*, \bar{v}] \) hold. By Lemma 2, \( t_1(v_1) = v_1^* \) must hold for all \( v_1 \in V_1 \). Recall that by Fact 1, \( t_2(v_2) \) is the same for all \( v_2 \in V_2 \). Further, \( v_2^* \) is the infimum of the set \( V_2 \). So for all \( v_2 \in V_2 \), \( t_2(v_2) \) must be such that type \( v_2^* \) is indifferent between reporting \( v_1^* \) and reporting \( v_2^* \). If instead type \( v_2^* \) strictly prefers reporting \( v_2^* \) over reporting \( v_1^* \), then by continuity, for a valuation \( v_1 \in V_1 \) less than \( v_2^* \), but close enough to \( v_2^* \), we will have that \( v_1 \) also strictly prefers reporting \( v_2^* \) rather than \( v_1^* \), which contradicts either the BIC of \( v_1 \) or the definition of \( v_2^* \) as the infimum of \( V_2 \). Thus, the following holds:

\[(v_2^* - t_2(v_2^*))\frac{\alpha_1}{\alpha_2}(-D'_2(v_2^*)) = (v_2^* - v_1^*)\frac{\alpha_1}{\alpha_2}(-D'_1(v_2^*)),\] or

\[ t_2(v_2^*) = v_2^* - (v_2^* - v_1^*)\frac{\alpha_1}{\alpha_2} Z(v_2^*). \] (29)

Now consider the payment scheme over the set \( V_{12} \). The IC for type \( v_1^* \) requires \( t_1(v_{12}^*) \geq t_1(v_1^*) = v_1^* \), because otherwise type \( v_1^* \) could report \( v_{12}^* \), receiving a lower price in state 1 and refusing to purchase in state 2. Similarly, the IC for type \( v_2^* \) requires \( t_2(v_{12}^*) \geq t_2(v_2^*) \), because otherwise a type \( v_2^* \) could report \( v_{12}^* \), receiving a lower price in state 2 and refusing to purchase in state 1. Also, EIR implies \( t_i(v_{12}^*) \leq v_{12}^* \) for \( i \in \{1, 2\} \). Claim L4 demonstrates that the payment scheme in Lemma 4 maximizes the firm’s expected profit from \( V_{12} \) subject to a subset BIC and EIR constraints. Subsequently, we will argue that the payment scheme in Lemma 4 satisfies all the BIC and EIR constraints, which will complete the proof of Lemma 4.
**Claim L4:** Given \( t_1(v_1^*) = v_1^* \) and \( t_2(v_2^*) \) according to (29), the payment scheme

\[
t_1(v_{12}) = v_1^* \quad \forall v_{12} \in V_{12},
\]

\[
t_2(v_{12}) = t_2(v_2^*) + (v_{12}^* - v_1^*) \frac{\alpha_1}{\alpha_2} Z(v_{12}^*) \quad \forall v_{12} \in V_{12},
\]

maximizes profits from \( V_{12} \), subject to: (i) the BIC constraint of \( v_{12}^* \) with respect to \( v_2^* \), (ii) the BIC constraint of types \( v \in [V_{12} - \{v_{12}^*\}] \) with respect to \( v_{12}^* \), (iii) the EIR constraints of \( v_{12}^* \), (iv) the BIC constraint of \( v_1^* \) with respect to \( v_{12}^* \), i.e. \( t_1(v_{12}^*) \geq v_1^* \), and (v) the IC constraint of \( v_2^* \) with respect to \( v_{12}^* \), i.e. \( t_2(v_{12}^*) \geq t_2(v_2^*) \).

**Proof of Claim L4.** The following is the statement of maximization problem described above.

\[
\max_{t_1(v), t_2(v)} \int_{v_{12}^*}^{v_0} [t_1(v)\alpha_1(-D'_1(v)) + t_2(v)\alpha_2(-D'_2(v))] dv.
\]

Subject to:

The BIC constraints of \( v \in [V_{12} - \{v_{12}^*\}] \) with respect to \( v_{12}^* \):

\[
(v - t_1(v))\alpha_1(-D'_1(v)) + (v - t_2(v))\alpha_2(-D'_2(v)) \geq 0 \tag{30}
\]

The BIC constraint of \( v_{12}^* \) with respect to \( v_2^* \):

\[
(v_{12}^* - t_1(v_{12}^*))\alpha_1(-D'_1(v_{12}^*)) + (v_{12}^* - t_2(v_{12}^*))\alpha_2(-D'_2(v_{12}^*)) \geq 0 \tag{31}
\]

The EIR constraints of \( v_{12}^* \) and the BIC constraints of \( v_1^* \) and \( v_2^* \) with respect to \( v_{12}^* \):

\[
t_1(v_{12}^*) \leq v_{12}^*; \quad t_2(v_{12}^*) \leq v_{12}^*; \quad t_1(v_{12}^*) \geq v_1^*; \quad t_2(v_{12}^*) \geq t_2(v_2^*). \tag{32}
\]
Simplifying (30) yields

\[ t_1(v)\alpha_1(-D'_1(v)) + t_2(v)\alpha_2(-D'_2(v)) \leq \]
\[ t_1(v'_{12})\alpha_1(-D'_1(v)) + t_2(v'_{12})\alpha_2(-D'_2(v)). \]  \hspace{1cm} (33)

Note that the term on the left side of (33) is exactly the expression in the maximand, which is less than the right side of (33). Thus, by offering all types \( v \in V_2 \) the payment scheme \( t_1(v'_{12}) \), and \( t_2(v'_{12}) \), the BIC constraints (30) are satisfied and the maximand is weakly higher than under any other BIC scheme. It follows that the maximization problem can be restated as follows.

\[
\max_{t_1(v'_{12}), t_2(v'_{12})} t_1(v'_{12})\alpha_1 \int_{v'_{12}}^6 (-D'_1(v)) dv + t_2(v'_{12})\alpha_2 \int_{v'_{12}}^6 (-D'_2(v)) dv. \]  \hspace{1cm} (34)

Subject to (31) and (32). Rearranging (31) yields:

\[
t_1(v'_{12})\alpha_1(-D'_1(v'_{12})) + t_2(v'_{12})\alpha_2(-D'_2(v'_{12})) \leq \]
\[ v'_{12}\alpha_1(-D'_1(v'_{12})) + t_2(v'_{2})\alpha_2(-D'_2(v'_{12})). \]  \hspace{1cm} (35)

At the optimum, (35) will bind. Further, Condition 1(iii) and Condition 2 imply

\[
\frac{\int_{v'_{12}}^6 (-D'_2(v)) dv}{(-D'_2(v'_{12}))} > \frac{\int_{v'_{12}}^6 (-D'_1(v)) dv}{(-D'_1(v'_{12}))}. \]  \hspace{1cm} (36)

We will argue later that (36) is a sufficient condition for the payment scheme stated in Claim L4 to be optimal for the set \( V_1 \). To see why Condition 1(iii) and Condition 2 imply (36), first rewrite (36) as:

\[
\frac{\int_{v'_{12}}^6 (-D'_2(v)) dv}{\int_{v'_{12}}^6 (-D'_1(v)) dv} > \frac{(-D'_2(v'_{12}))}{(-D'_1(v'_{12}))}. \]  \hspace{1cm} (37)
Note that the left side of (37) is equal to
\[ \frac{\int_{v_{12}}^{v} \frac{(-D'(v))}{Z(v)} dv}{\int_{v_{12}}^{v} (-D'_1(v)) dv}. \]

\( Z(v) \) and \((-D'_1(v))\) are non-negative and strictly decreasing due to Condition 2 and Condition 1(iii), respectively. Thus, it follows from Wang (1993, Lemma 2)\(^{10}\) that we have
\[ \frac{\int_{v_{12}}^{v} \frac{(-D'(v))}{Z(v)} dv}{\int_{v_{12}}^{v} (-D'_1(v)) dv} > \frac{1}{Z(v_{12})}. \]

Because \( Z(v) \) is strictly decreasing and we have \( v \geq v_{12}^* \) for all \( v \in V_{12} \), it follows that the right side of (38) exceeds \( \frac{1}{Z(v_{12})} \). Therefore, we have
\[ \frac{\int_{v_{12}}^{v} \frac{(-D'(v))}{Z(v)} dv}{\int_{v_{12}}^{v} (-D'_1(v)) dv} > \frac{1}{Z(v_{12})}, \]
which implies (37), and its equivalent, (36).

Due to the linearity of the maximand, (34), and the constraint, (35), in \( t_1(v_{12}^*) \) and \( t_2(v_{12}^*) \), it follows from (36) that the solution is to set \( t_1(v_{12}^*) \) as low as possible and \( t_2(v_{12}^*) \) as high as possible, subject to (35), \( t_1(v_{12}^*) \geq v_1^* \) and \( t_2(v_{12}^*) \leq v_{12}^* \). We claim that, \( t_1(v) = v_1^* \) and \( t_2(v) = t_2(v_2^*) + (v_{12}^* - v_1^*) \frac{\alpha_1}{\alpha_2} Z(v_{12}^*) \) for all \( v \in V_{12} \) is optimal (derived using \( t_1(v_{12}^*) = v_1^* \) in (35)).

To verify this claim, we show that setting \( t_2(v_{12}^*) = t_2(v_2^*) + (v_{12}^* - v_1^*) \frac{\alpha_1}{\alpha_2} Z(v_{12}^*) \) satisfies \( t_2(v_{12}^*) \leq v_{12}^* \). Using (29), we can express \( (v_{12}^* - t_2(v_{12}^*)) \) as
\[ (v_{12}^* - v_2^*) + (v_2^* - v_1^*) \frac{\alpha_1}{\alpha_2} Z(v_{12}^*) - (v_{12}^* - v_1^*) \frac{\alpha_1}{\alpha_2} Z(v_{12}^*). \]

Evaluated at \( v_{12}^* = v_2^* \), expression (39) is zero, so we will be done with this claim if we show that the expression is non-decreasing in \( v_{12}^* \). Differentiating with respect to \( v_{12}^* \) yields
\[ 10^{\text{In Wang’s notation, } x(\phi) = 1, y(\phi) = (-D'_1(v)), \text{ and } z(\phi) = \frac{1}{Z(v)}}. \]
\[1 - (v_{12}^* - v_i^*) \frac{\alpha_1}{\alpha_2} Z'(v_{12}^*) - \frac{\alpha_1}{\alpha_2} Z(v_{12}^*). \]  

(40)

Because \(Z'(v_{12}^*)\) is negative and \(1 - \frac{\alpha_1}{\alpha_2} Z(v_{12}^*)\) is non-negative by Condition 4(i), the expression (40) is non-decreasing.

To check that all EIR conditions are satisfied, note that \(t_i(v) \leq v\) holds for all \(v \in V\) and for \(i \in \{1, 2\}\). We now show that all BIC constraints, including those omitted from the simplified problem, are satisfied when payments are as specified in Lemma 4 and the sets, \(V_1, V_2, \) and \(V_{12}\), are the stated intervals. For each of these sets, Lemma 4 specifies that the payment depends only on the set and not the valuation type within that set; thus, for each set, the BIC with respect to other valuations within the same set are satisfied. For types \(v < v_i^*\), BIC is clearly satisfied, since the required payment in any state exceeds their value.

**BIC for \(V_1\) with respect to \(V_2\).** For \(v \in [v_1^*, v_2^*]\), we first check the BIC with respect to the set \(V_2\). If \(v \in [v_1^*, t_2(v_2^*)]\), then the BIC of \(v\) with respect to \(V_2\) clearly holds since the required payment in state 2 is more than \(v\). To see that the BIC of \(v \in (t_2(v_2^*), v_2^*)\) also holds, consider the BIC of \(v_2^*\). By the construction of \(t_2(v_2^*)\), \((v_2^* - t_2(v_2^*))\alpha_2(-D_2'(v_2^*)) = (v_2^* - v_1^*)\alpha_1(-D_1'(v_2^*))\) holds. Rewriting yields,

\[
\frac{(v_2^* - t_2(v_2^*))}{(v_2^* - v_1^*)} = \frac{\alpha_1}{\alpha_2} Z(v_2^*). 
\]  

(41)

Note that for the payment scheme in Lemma 4, \(t_2(v_2^*) \geq v_1^*\) holds. Thus, substituting \(v \in (t_2(v_2^*), v_2^*)\) instead of \(v_2^*\) into this equality yields

\[
\frac{(v - t_2(v_2^*))}{(v - v_1^*)} \leq \frac{(v_2^* - t_2(v_2^*))}{(v_2^* - v_1^*)} = \frac{\alpha_1}{\alpha_2} Z(v_2^*) < \frac{\alpha_1}{\alpha_2} Z(v),
\]
where the last inequality follows because $Z(v)$ is strictly decreasing in $v$ and $v < v^*_2$ holds. So \( \frac{(v - t_2(v^*_2))}{(v - v^*_1)} < \frac{\alpha_1}{\alpha_2} Z(v) \) holds $\forall v \in (t_2(v^*_2), v^*_2)$. Cross-multiplying yields,

\[
(v - t_2(v^*_2)) \alpha_2 (-D'_2(v)) < (v - v^*_1) \alpha_1 (-D'_1(v)) \quad \forall v \in (t_2(v^*_2), v^*_2).
\]

Thus, for all the valuations in $V_1$, the BIC with respect to $V_2$ is satisfied.

**BIC for $V_1$ with respect to $V_{12}$.** Next, we check the BIC for $v \in [v^*_1, v^*_2)$ with respect to the set $V_{12}$. To show that this BIC is satisfied we show that $t_2(v^*_12) > v^*_2$ holds. To see this, first note that $v^*_12 \alpha_1 \alpha_2 Z(v^*_12) > v^*_2 \alpha_1 \alpha_2 Z(v^*_2)$ holds by Condition 3, given $v^*_12 > v^*_2$; second, $v^*_1 \alpha_1 \alpha_2 Z(v^*_1) > v^*_12 \alpha_1 \alpha_2 Z(v^*_12)$ holds given that $Z(v)$ is strictly decreasing in $v$ (Condition 2). Putting these two facts together, we have

\[
(v^*_12 - v^*_1) \frac{\alpha_1}{\alpha_2} Z(v^*_12) > (v^*_2 - v^*_1) \frac{\alpha_1}{\alpha_2} Z(v^*_2).
\]

Therefore, we have

\[
t_2(v^*_12) = v^*_2 - (v^*_2 - v^*_1) \frac{\alpha_1}{\alpha_2} Z(v^*_2) + (v^*_12 - v^*_1) \frac{\alpha_1}{\alpha_2} Z(v^*_12) > v^*_2.
\]

Thus, according to the payment scheme in Lemma 4, if one reports a valuation in $V_{12}$, the required payment in state 1 is $v^*_1$ and the required payment in state 2, $t_2(v^*_12)$, satisfies $t_2(v^*_12) > v^*_2$. Therefore the best a valuation $v$ in $[v^*_1, v^*_2)$ can do by reporting a valuation in $V_{12}$ is to buy in state 1 at price $v^*_1$ and refuse to buy in state 2. But this is exactly the scheme available upon truthful reporting. So the BIC for $v \in [v^*_1, v^*_2)$ with respect to the set $V_{12}$ is satisfied.
**BIC for \( V_2 \) with respect to \( V_1 \).** Next we will show that for all types \( v \) such that \( v > v_2^* \) holds, \( v \) strictly prefers the set \( V_2 \) over the set \( V_1 \). Using (41) and \( v > v_2^* \) yields,

\[
\frac{(v - t_2(v_2^*))}{(v - v_1^*)} \geq \frac{(v_2^* - t_2(v_2^*))}{(v_2^* - v_1^*)} = \frac{\alpha_1}{\alpha_2} Z(v_2^*) > \frac{\alpha_1}{\alpha_2} Z(v),
\]

where the first inequality follows because \( t_2(v_2^*) \geq v_1^* \) holds, and the last inequality follows because \( Z(v) \) is strictly decreasing and \( v > v_2^* \) holds by assumption. Thus,

\[
\frac{(v - t_2(v_2^*))}{(v - v_1^*)} > \frac{\alpha_1}{\alpha_2} Z(v) \text{ holds.}
\]

Cross-multiplying yields,

\[
(v - t_2(v_2^*))\alpha_2(-D_2'(v)) > (v - v_1^*)\alpha_1(-D_1'(v)) \forall v > v_2^*.
\]

Thus, for any type \( v \) such that \( v > v_2^* \) holds, \( v \) strictly prefers the set \( V_2 \) over the set \( V_1 \). So the BICs of the types \( v \in V_2 \) with respect to \( V_1 \) are satisfied.

**BIC for \( V_2 \) with respect to \( V_{12} \).** Now we show that for all types \( v \) such that \( v_2^* \leq v < v_{12}^* \) holds, we must have that \( v \) prefers the set \( V_2 \) over the set \( V_{12} \). First, consider valuations in \([v_2^*, t_2(v_{12}^*)]\). If such a valuation reports \( v_{12}^* \) then, given the payment scheme in Lemma 4, the best possible outcome for them is: Buy in state 1 at \( v_1^* \) and refuse to buy in state 2. But this is exactly the scheme from reporting \( v_1^* \). We have already shown that that is worse than truthful reporting. Now consider valuations in \((t_2(v_{12}^*), v_{12}^*)\). By the construction in Lemma 4, \( t_1(v_{12}^*) = v_1^* \) and \( t_2(v_{12}^*) \) is such that (31) binds. The latter implies

\[
(v_{12}^* - v_1^*)\alpha_1(-D_1'(v_{12}^*)) + (v_{12}^* - t_2(v_{12}^*))\alpha_2(-D_2'(v_{12}^*)) = (v_{12}^* - t_2(v_2^*))\alpha_2(-D_2'(v_{12}^*)).\]
Dividing through by \( \alpha_2(-D'_2(v^*_1)) \) and rearranging yields

\[
(v^*_1 - v^*_1) \frac{\alpha_1}{\alpha_2} Z(v^*_1) + t_2(v^*_1) - t_2(v^*_1) = 0. \tag{42}
\]

For \( v \) such that \( v_2^* \leq v < v^*_1 \) holds, \( (v - v^*_1) \frac{\alpha_1}{\alpha_2} Z(v) < (v^*_1 - v^*_1) \frac{\alpha_1}{\alpha_2} Z(v^*_1) \) holds because, by Condition 3, \( v^*_1 \frac{\alpha_1}{\alpha_2} Z(v) < v^*_1 \frac{\alpha_1}{\alpha_2} Z(v^*_1) \) holds, and, by Condition 2, \( v^*_1 \frac{\alpha_1}{\alpha_2} Z(v) > v^*_1 \frac{\alpha_1}{\alpha_2} Z(v^*_1) \) holds. So we have

\[
(v - v^*_1) \frac{\alpha_1}{\alpha_2} Z(v) + t_2(v^*_2) - t_2(v^*_1) < 0.
\]

After adding \( v \) to both sides of the inequality, this can be rearranged into

\[
(v - v^*_1)\alpha_1(-D'_1(v)) + (v - t_2(v^*_1))\alpha_2(-D'_2(v)) < (v - t_2(v^*_1))\alpha_2(-D'_2(v)). \tag{43}
\]

Inequality (43) implies that for all types \( v \in [v^*_2, v^*_1] \), type \( v \) prefers truthful reporting to reporting a valuation in \( V_{12} \).

**BIC for \( V_{12} \) with respect to \( V_2 \) and \( V_1 \).** Now we show that for all types \( v \) such that \( v > v^*_1 \) holds, we must have that \( v \) prefers the set \( V_{12} \) over the set \( V_2 \). By the construction in Lemma 4, \( t_1(v^*_1) = v^*_1 \) and \( t_2(v^*_1) \) is such that (31) binds. The latter implies (42). Conditions 2 and 3 imply that \( (v - v^*_1)\frac{\alpha_1}{\alpha_2} Z(v) > (v^*_1 - v^*_1)\frac{\alpha_1}{\alpha_2} Z(v^*_1) \) holds for \( v > v^*_1 \), which along with (42) implies

\[
(v - v^*_1)\frac{\alpha_1}{\alpha_2} Z(v) + t_2(v^*_2) - t_2(v^*_1) > 0. \tag{44}
\]

Inequality (44) can be rearranged to

\[
(v - v^*_1)\alpha_1(-D'_1(v)) + (v - t_2(v^*_1))\alpha_2(-D'_2(v)) > (v - t_2(v^*_2))\alpha_2(-D'_2(v)). \tag{45}
\]
Inequality (45) implies that, for all types $v$ such that $v > v_{12}^*$ holds, $v$ strictly prefers the set $V_{12}$ over the set $V_2$. In demonstrating the BIC of $V_2$ with respect to $V_1$, we have already shown that for all $v$ such that $v > v_2^*$ holds, $v$ strictly prefers the set $V_2$ over the set $V_1$. Thus it must be the case that for all types $v$ such that $v > v_{12}^*$ holds, $v$ strictly prefers the set $V_{12}$ over the set $V_1$. ■

Lemma 5: If Conditions 1-3 and 4(i) hold, then at the firm’s optimal mechanism within the class of BIC and EIR mechanisms, it cannot be the case that $v_1^* < v_2^* < v_{12}^*$ holds and the sets $V_1$, $V_2$, and $V_{12}$ are all non-empty.

Proof of Lemma 5. The BIC and EIR conditions and Conditions 1, 2, 3 and 4(i) imply that $v_1^* < v_2^* < v_{12}^*$ holds and $V_1 = [v_1^*, v_2^*)$, $V_2 = [v_2^*, v_{12}^*)$, and $V_{12} = [v_{12}^*, v]$ hold. To prove Lemma 5, we will show that for any mechanism such that $v_2^* < v_{12}^*$ holds, the firm can do strictly better by reducing the gap between $v_2^*$ and $v_{12}^*$, thereby making $V_2 = [v_2^*, v_{12}^*)$ smaller. Using Conditions 1-3, 4(i) and the optimal payment scheme in Lemma 4, we can rewrite the profit expression, (10), as

$$
\pi(v_1^*, v_2^*, v_{12}^*) = v_1^* \alpha_1 D_1(v_1^*) + t_2(v_{12}^*) \alpha_2 D_2(v_{12}^*) + v_1^* \alpha_1 [D_1(v_1^*) - D_1(v_2^*)] + t_2(v_2^*) \alpha_2 [D_2(v_2^*) - D_2(v_{12}^*)].
$$

Substituting $t_2(v_2^*)$ and $t_2(v_{12}^*)$ from Lemma 4, we have the following profit expression:

$$
\pi(v_1^*, v_2^*, v_{12}^*) = \alpha_1 v_1^* [D_1(v_1^*) - D_1(v_2^*) + D_1(v_{12}^*)] + \alpha_2 [(v_{12}^* - v_1^*) \alpha_2 Z(v_{12}^*)] D_2(v_{12}^*) + \alpha_2 D_2(v_2^*) (v_2^* - (v_2^* - v_1^*) \alpha_1 \alpha_2 Z(v_2^*)].
$$

It will be convenient to write the last term in this expression, i.e. $\alpha_2 D_2(v_2^*) (v_2^* - (v_2^* - v_1^*) \alpha_1 \alpha_2 Z(v_2^*))$, as

$$
\alpha_2 v_2^* D_2(v_2^*) (1 - \frac{\alpha_1}{\alpha_2} Z(v_2^*)) + v_1^* \alpha_1 Z(v_2^*) D_2(v_2^*).
$$
Thus, we have

\[
\pi(v_1^*, v_2^*, v_{12}^*) = \alpha_1 v_1^* [D_1(v_1^*) - D_1(v_2^*) + D_1(v_{12}^*)] + \alpha_2 [(v_{12}^* - v_1^*) \frac{\alpha_1}{\alpha_2} Z(v_{12}^*)] D_2(v_{12}^*)
\]

\[
+ \alpha_2 v_2^* D_2(v_2^*) (1 - \frac{\alpha_1}{\alpha_2} Z(v_2^*)) + v_1^* \alpha_1 Z(v_2^*) D_2(v_2^*).
\]

Taking the derivative of the profit expression in (46) with respect to \(v_2^*\) yields

\[
\frac{\partial \pi}{\partial v_2^*} = \alpha_1 v_1^* D'_1(v_2^*) + \alpha_2 \frac{\partial (v_2^* D_2(v_2^*))}{\partial v_2^*} (1 - \frac{\alpha_1}{\alpha_2} Z(v_2^*)) - \alpha_1 (v_2^* - v_1^*) D_2(v_2^*) Z'(v_2^*) - \alpha_1 v_1^* Z(v_2^*) D'_2(v_2^*).
\]

Now consider the case where \(v_2^*\) is strictly less than the monopoly price in state 2 (henceforth \(p_2^m\)), i.e. \(v_2^* < p_2^m\) holds. By simplifying the derivative, for the case where \(v_2^* < p_2^m\) holds, we can see that

\[
\frac{\partial \pi}{\partial v_2^*} = \alpha_2 \frac{\partial (v_2^* D_2(v_2^*))}{\partial v_2^*} (1 - \frac{\alpha_1}{\alpha_2} Z(v_2^*)) - \alpha_1 (v_2^* - v_1^*) D_2(v_2^*) Z'(v_2^*) > 0
\]

holds. Where the last inequality follows because \(\frac{\partial (v_2^* D_2(v_2^*))}{\partial v_2^*} > 0\) holds because we are in the case where \(v_2^*\) is strictly less than \(p_2^m\); further, \(\frac{\alpha_1}{\alpha_2} Z(v_2^*) \leq 1\) holds by Condition 4(i), \(Z'(v_2^*) < 0\) holds by Condition 2, and \(v_2^* > v_1^*\) holds by Lemma 3(a). The implication of \(\frac{\partial \pi}{\partial v_2^*} > 0\) is that whenever \(v_2^* < v_{12}^*\) holds, and \(v_2^* < p_2^m\) holds, the firm can strictly increase profits by increasing \(v_2^*\) towards \(v_{12}^*\) so that \(V_2 = [v_2^*, v_{12}^*]\) shrinks.

Next, consider the case where \(p_2^m \leq v_2^*\) holds. By Lemma 3(a), \(p_2^m \leq v_2^* < v_{12}^*\) must hold. Consider the derivative of profit (46) with respect to \(v_{12}^*\),

\[
\frac{\partial \pi}{\partial v_{12}^*} = \alpha_1 v_1^* D'_1(v_{12}^*) + \alpha_1 \frac{\partial (v_{12}^* D_2(v_{12}^*))}{\partial v_{12}^*} Z(v_{12}^*)
\]

\[
+ \alpha_1 (v_{12}^* - v_1^*) Z'(v_{12}^*) D_2(v_{12}^*) - \alpha_1 v_1^* Z(v_{12}^*) D'_2(v_{12}^*)
\]

\[
= \alpha_1 \frac{\partial (v_{12}^* D_2(v_{12}^*))}{\partial v_{12}^*} Z(v_{12}^*) + \alpha_1 (v_{12}^* - v_1^*) Z'(v_{12}^*) D_2(v_{12}^*) < 0.
\]

Where the last inequality follows because \(\frac{\partial (v_{12}^* D_2(v_{12}^*))}{\partial v_{12}^*} < 0\) holds, since we are in the case where \(v_{12}^* > p_2^m\) holds. Further, \(Z'(v_{12}^*) < 0\) holds by Condition 2. The implication of
\(\frac{\partial m}{\partial v_{12}} < 0\) is that, whenever \(v_2^* < v_{12}^*\) holds, and \(v_2^* \geq p_{m2}^*\) holds, the firm can strictly increase profits by decreasing \(v_{12}\) toward \(v_2^*\), so that \(V_2 = [v_2^*, v_{12}^*]\) becomes smaller.

To summarize, in either case, \(v_2^* < p_m^2\) or \(v_2^* > p_m^2\), the firm strictly increases profit by making the interval \(V_2 = [v_2^*, v_{12}^*]\) smaller by reducing \((v_{12}^* - v_2^*)\). Therefore it cannot be the case that at the firm’s optimum \((v_{12}^* - v_2^*) > 0\) holds. ■

Lemma 5 establishes that, under Conditions 1-3 and 4(i), it cannot be the case that \(V_1\), \(V_2\), and \(V_{12}\) are all non-empty at the firm’s optimal mechanism within the class of BIC and EIR mechanisms. We now establish the same conclusion under Conditions 1-3 and 4(ii).

Under these conditions, by Lemma 3(b), \(V_2 = [v_2^*, v_1^*]\), \(V_1 = [v_1^*, v_{12}^*]\), and \(V_{12} = [v_{12}^*, \bar{v}]\) hold.

**Lemma 6:** Suppose Conditions 1-3, and 4(ii), hold at the optimal mechanism within the class of BIC and EIR mechanisms. Then for all \(v_1 \in V_1\), \(v_2 \in V_2\), and \(v_{12} \in V_{12}\), and payments are characterized by

\[
\begin{align*}
t_2(v_2) &= v_2^*, \\
t_1(v_1) &= v_1^* - (v_1^* - v_2^*) \frac{\alpha_2}{\alpha_1 Z(v_1^*)}, \\
t_1(v_{12}) &= v_1^* - (v_1^* - v_2^*) \frac{\alpha_2}{\alpha_1 Z(v_1^*)}, \\
t_2(v_{12}) &= v_{12}^*.
\end{align*}
\]

**Proof of Lemma 6.** By Lemma 3(b), we must have \(V_2 = [v_2^*, v_1^*], V_1 = [v_1^*, v_{12}^*],\) and \(V_{12} = [v_{12}^*, \bar{v}]\). By Lemma 2, \(t_2(v_2) = v_2^*\) must hold for all \(v_2 \in V_2\). Recall that by Fact 1, \(t_1(v_1)\) is the same for all \(v_1 \in V_1\). Further, \(v_1^*\) is the infimum of the set \(V_1\). So for all \(v_1 \in V_1\), \(t_1(v_1)\) must be such that type \(v_1^*\) is indifferent between reporting \(v_2^*\) and reporting \(v_1^*\). If instead type \(v_1^*\) strictly prefers reporting \(v_1^*\) over reporting \(v_2^*\), then by continuity, for a valuation \(v_2\) less than \(v_1^*\), but close enough to \(v_1^*\), we will have that \(v_2\) also strictly prefers reporting \(v_1^*\) rather than \(v_2^*\), which contradicts either the BIC of \(v_2\) or the definition of \(v_1^*\) as
the infimum of \( V_1 \). Thus, we have

\[
(v_1^* - t_1(v_1^*))\alpha_1(-D_1'(v_1^*)) = (v_1^* - v_2^*)\alpha_2(-D_2'(v_1^*)), \quad \text{or}
\]
\[
t_1(v_1^*) = v_1^* - (v_1^* - v_2^*)\frac{\alpha_2}{\alpha_1 Z(v_1^*)}.
\]

(47)

Now we move on to characterizing the payment scheme over the set \( V_{12} \). The BIC for type \( v_1^* \) requires \( t_1(v_{12}^*) \geq t_1(v_1^*) \), because otherwise a type \( v_1^* \) could report \( v_{12}^* \), receiving a lower price in state 1 and refusing to purchase in state 2. Similarly, the BIC for type \( v_2^* \) requires \( t_2(v_{12}^*) \geq v_2^* \), because otherwise a type \( v_2^* \) could report \( v_{12}^* \), receiving a lower price in state 2 and refusing to purchase in state 1. Also, type \( v_{12}^* \) would refuse to purchase in state \( i \) unless \( t_i(v_{12}^*) \leq v_{12}^* \) holds for \( i \in \{1, 2\} \). Claim L6 demonstrates that the payment scheme in Lemma 6 maximizes firm’s expected profit from \( V_{12} \) subject to a subset of BIC and EIR constraints. Subsequently, we argue that the payment scheme in Lemma 6 satisfies all the EIR and BIC conditions, which completes the proof of Lemma 6.

**Claim L6:** Given \( t_2(v_2^*) = v_2^* \), the payment scheme

\[
t_1(v_{12}) = t_1(v_1^*) = v_1^* - (v_1^* - v_2^*)\frac{\alpha_2}{\alpha_1 Z(v_1^*)} \quad \forall v_{12} \in V_{12},
\]

\[
t_2(v_{12}) = v_{12}^* \quad \forall v_{12} \in V_{12},
\]

maximizes profits from \( V_{12} \), subject to: (i) the BIC constraint of \( v_{12}^* \) with respect to \( v_1^* \), (ii) the BIC constraint of types \( v \in \left(V_{12} - \{v_{12}^*\}\right) \) with respect to \( v_{12}^* \), (iii) the EIR constraint of \( v_{12}^* \), (iv) the BIC constraint of \( v_2^* \) with respect to \( v_{12}^* \), i.e. \( t_2(v_{12}^*) \geq v_2^* \), and (v) the BIC constraint of \( v_1^* \) with respect to \( v_{12}^* \), i.e. \( t_1(v_{12}^*) \geq t_1(v_1^*) \).

**Proof of Claim L6.** The following is the statement of maximization problem described above.

\[
\max_{t_1(v), t_2(v)} \int_{v_{12}^*}^\theta [t_1(v)\alpha_1(-D_1'(v)) + t_2(v)\alpha_2(-D_2'(v))]dv.
\]

Subject to:
The BIC constraints of \( v \in [V_{12} - \{v_{12}^*\}] \) with respect to \( v_{12}^* \):

\[
(v - t_1(v))\alpha_1(-D_1'(v)) + (v - t_2(v))\alpha_2(-D_2'(v)) \geq 0.
\] (48)

The BIC constraint of \( v_{12}^* \) with respect to \( v_1^* \):

\[
(v_{12}^* - t_1(v_{12}^*))\alpha_1(-D_1'(v_{12}^*)) + (v_{12}^* - t_2(v_{12}^*))\alpha_2(-D_2'(v_{12}^*)) \geq 0.
\] (49)

The EIR constraints of \( v_{12}^* \) and the BIC constraints of \( v_1^* \) and \( v_2^* \) with respect to \( v_{12}^* \):

\[
t_1(v_{12}^*) \leq v_{12}^*; \quad t_2(v_{12}^*) \leq v_{12}^*; \quad t_1(v_{12}^*) \geq t_1(v_1^*); \quad t_2(v_{12}^*) \geq v_2^*.
\] (50)

Simplifying/rearranging (48) yields

\[
t_1(v)\alpha_1(-D_1'(v)) + t_2(v)\alpha_2(-D_2'(v)) \leq t_1(v_{12}^*)\alpha_1(-D_1'(v_{12}^*)) + t_2(v_{12}^*)\alpha_2(-D_2'(v_{12}^*)).
\] (51)

Note that the term on the left side of (51) is exactly the expression for \( v \) in the maximand, which is less than the right side of (51). Thus, by offering all types \( v \in V_{12} \) the payments \( t_1(v_{12}^*) \) and \( t_2(v_{12}^*) \), the BIC constraints (48) are satisfied and the maximand is weakly higher than under any other scheme. It follows that the maximization problem can be restated as follows.

\[
\max_{t_1(v_{12}^*), t_2(v_{12}^*)} \quad t_1(v_{12}^*)\alpha_1\int_{v_{12}^*}^v (-D_1'(v))dv + t_2(v_{12}^*)\alpha_2\int_{v_{12}^*}^v (-D_2'(v))dv.
\] (52)

Subject to (49) and (50). Rearranging (49) yields:

\[
t_1(v_{12}^*)\alpha_1(-D_1'(v_{12}^*)) + t_2(v_{12}^*)\alpha_2(-D_2'(v_{12}^*)) \leq t_1(v_1^*)\alpha_1(-D_1'(v_{12}^*)) + v_{12}^*\alpha_2(-D_2'(v_{12}^*)).
\] (53)
At the optimum, (53) will bind. The arguments in Claim L4 in the proof of Lemma 4 can be repeated to show that Condition 1(iii), Condition 2, and Lemma 2 of Wang (1993) imply:

\[
\frac{\int_{v_{i_2}}^{v} (-D'_2(v))dv}{(-D'_2(v_{i_2}^*))} > \frac{\int_{v_{i_2}}^{v} (-D'_1(v))dv}{(-D'_1(v_{i_2}^*))}.
\] (54)

Due to the linearity of the maximand, (52), and the constraint, (53), in the choice variable \(t_1(v_{i_2}^*)\) and \(t_2(v_{i_2}^*)\), it follows from (54) that the solution is to set \(t_1(v_{i_2}^*)\) as low as possible and \(t_2(v_{i_2}^*)\) as high as possible, subject to (53), \(t_1(v_{i_2}^*) \geq t_1(v_1^*)\) and \(t_2(v_{i_2}^*) \leq v_{i_2}^*\). Thus, \(t_1(v) = t_1(v_1^*)\) and \(t_2(v) = v_{i_2}^*\) for all \(v \in V_{i_2}\) is optimal.

It is straightforward to check that all EIR constraints are satisfied. We now show that BIC constraints are satisfied when payments are as specified in Lemma 6 and the sets, \(V_1\), \(V_2\), and \(V_{i_2}\), are the stated intervals. For each of these sets, Lemma 6 specifies that the payment scheme within the set is the same; thus, for each set, the BIC with respect to other valuations within the same set are satisfied. For types \(v < v_{i_2}^*\), BIC is clearly satisfied, since the required payment in any state exceeds their value. For \(v \in [v_{i_2}^*, v_1^*)\), we know that reporting \(v_1^*\) yields weakly higher utility than reporting \(v_{i_2}^*\), because \(t_1(v_{i_2}^*) = t_1(v_1^*)\) and \(t_2(v_{i_2}^*) = v_{i_2}^* > v\) hold. However, reporting \(v_1^*\) cannot yield a strictly higher utility than truthful reporting. This is clearly true for valuations in \([v_{i_2}^*, t_1(v_1^*)]\), as the required payment from reporting one’s type as \(v_1^*\) is greater than one’s valuation. Further, for valuations \(v \in (t_1(v_1^*), v_1^*)\), given Condition 4(ii), reporting \(v_1^*\) yields a strictly lower utility than truthful reporting. To see this, the net benefit of reporting \(v_1^*\), versus reporting, \(v_2^*\), for type \(v\) is

\[
(v - t_1(v_1^*))\alpha_1(-D'_1(v)) - (v - v_2^*)\alpha_2(-D'_2(v)).
\]

Therefore type \(v\) prefers to report \(v_1^*\) rather than \(v_2^*\) if and only if we have

\[
\frac{\alpha_1}{\alpha_2}(v - t_1(v_1^*))Z(v_1^*) - (v - v_2^*) \geq 0.
\] (55)
Differentiating (55) with respect to \( v \) yields

\[
\frac{\alpha_1}{\alpha_2}(v - t_1(v^*_1))Z'(v^*_1) + \frac{\alpha_1}{\alpha_2}Z(v) - 1,
\]

which is strictly positive by Condition 4(ii). Therefore, the left side of (55) is strictly increasing in \( v \), and equals zero at \( v = v^*_1 \), by construction. Therefore types \( v \in (t_1(v^*_1), v_1) \) strictly prefer truthful reporting to reporting \( v^*_1 \), so BIC for \( V_2 \) with respect to both \( V_1 \) and \( V_{12} \) are satisfied. It also follows that (55) holds for \( v > v^*_1 \), so BIC for \( V_1 \) with respect to \( V_2 \) is satisfied.

Further, for the types in \( V_1 \), BIC is satisfied with respect to \( V_{12} \), because \( t_1(v^*_1) = t_1(v^*_1) \) and \( t_2(v^*_1) = v^*_2 \) hold. (The payment in state 1 is the same, and the payment in state 2 is greater than \( v \).) Last, for types \( v \in [v^*_1, v^*_2] \), truthful reporting yields higher utility than reporting \( v^*_1 \), because truthful reporting entails the same payment in state 1 but positive surplus in state 2. The BIC of types in \( V_{12} \) with respect to \( V_2 \) is satisfied, because types in \( V_{12} \) prefer truthful reporting over reporting \( v^*_1 \), and, since \( v > v^*_1 \) holds, types in \( V_{12} \) prefer reporting \( v^*_1 \) over reporting \( v^*_2 \). ■

**Lemma 7:** If Conditions 1-3 and 4(ii) hold, then at the firm’s optimal mechanism within the class of BIC and EIR mechanisms, it cannot be the case that \( v^*_2 < v^*_1 < v^*_1 \) holds and the sets \( V_2, V_1, \) and \( V_{12} \) are all non-empty.

**Proof of Lemma 7.** Given BIC, EIR, Conditions 1-3 and 4(ii), by Lemma 3(b), if the firm sets \( V_1, V_2, \) and \( V_{12} \) non-empty, then \( V_2 = [v^*_2, v^*_1], V_1 = [v^*_1, v^*_1], \) and \( V_{12} = [v^*_2, v^*_1] \) hold. So, the optimal payment scheme given in Lemma 6 implies that we can express the firm’s profit (11) as

\[
\pi(v^*_2, v^*_1, v^*_1) = \alpha_1 D_1(v^*_1) t_1(v^*_1) + \alpha_2[D_2(v^*_2) - D_2(v^*_1)]v^*_2 + \alpha_2 D_2(v^*_1) v^*_1
\]

\[
= \alpha_1 D_1(v^*_1)[v^*_1 - (v^*_1 - v^*_1)\frac{\alpha_2}{\alpha_1 Z(v^*_1)}] + \alpha_2[D_2(v^*_2) - D_2(v^*_1)]v^*_2 + \alpha_2 D_2(v^*_1) v^*_1. \quad (56)
\]
The necessary first order condition, \( \frac{\partial \pi(v_1^*, v_2^*, v_{12}^*)}{\partial v_1} = 0 \), implies

\[
0 = \alpha_1 D_1(v_1^*)[1 - \frac{\alpha_2}{\alpha_1}(\frac{Z(v_1^*) - (v_1^* - v_2^*)Z'(v_1^*)}{Z(v_1^*)^2})] + \alpha_1 D_1'(v_1^*)v_1^* - \alpha_2 D_2'(v_2^*)v_2^*.
\]

(57)

Substituting \( \frac{D'(v_1^*)}{Z(v_1^*)} = D_2'(v_1^*) \) into (57) and simplifying yields

\[
0 = \alpha_1 D_1(v_1^*)[1 - \frac{\alpha_2}{\alpha_1}(\frac{Z(v_1^*) - (v_1^* - v_2^*)Z'(v_1^*)}{Z(v_1^*)^2})] + \alpha_1 D_1'(v_1^*)v_1^* - \alpha_2 D_2'(v_2^*)v_2^*.
\]

Rearranging terms, we have

\[
0 = \alpha_1[D_1(v_1^*) + D_1'(v_1^*)v_1^*] - \alpha_2[D_2'(v_1^*)v_1^* + \frac{D_1(v_1^*)}{Z(v_1^*)}] + \alpha_2 D_1'(v_1^*)v_1^* - \frac{\alpha_2 D_1(v_1^*)}{Z(v_1^*)}Z'(v_1^*).
\]

(58)

Substituting \( \frac{D'(v_1^*)}{Z(v_1^*)} = D_2'(v_1^*) \) into (58) and combining/rearranging terms, we have

\[
0 = \alpha_1[D_1(v_1^*) + D_1'(v_1^*)v_1^*][1 - \frac{\alpha_2}{\alpha_1}Z'(v_1^*)] + \frac{\alpha_2 D_1(v_1^*)}{Z(v_1^*)} - \frac{\alpha_2 D_1(v_1^*)}{Z(v_1^*)}Z'(v_1^*).
\]

(59)

Since \( v_1^* > v_2^* \) holds, the last term in (59) is negative. By the BIC of \( v_2^* \) with respect to \( v_1^* \), \([1 - \frac{\alpha_2}{\alpha_1}Z'(v_1^*)] > 0 \) holds. Thus, (59) yields the necessary condition, \([D_1(v_1^*) + D_1'(v_1^*)v_1^*] > 0 \), or in other words, \( v_1^* \) must be strictly less than the monopoly price in state 1, \( p_1^m \). Now let us compute

\[
\frac{\partial \pi(v_2^*, v_1^*, v_{12}^*)}{\partial v_2} = \frac{D_1(v_1^*)}{Z(v_1^*)} - D_2(v_1^*) + D_2(v_2^*) + v_2^* D_2'(v_2^*).
\]

(60)

From Condition 1, marginal revenue in state 2 is decreasing in \( v \), so from (60), \( \frac{\partial \pi(v_2^*, v_1^*, v_{12}^*)}{\partial v_2} \) is decreasing in \( v_2^* \). The necessary first order condition requires \( \frac{\partial \pi(v_1^*, v_2^*, v_{12}^*)}{\partial v_2} = 0 \). Since \( v_1^* > v_2^* \) holds, \( \frac{\partial \pi(v_2^*, v_1^*, v_{12}^*)}{\partial v_2} \) evaluated at \( v_2^* = v_1^* \), must be negative. That is, we have

\[
\frac{D_1(v_1^*)}{Z(v_1^*)} - D_2(v_1^*) + D_2(v_2^*) + v_1^* D_2'(v_1^*) < 0.
\]

(61)
Substituting $Z(v^*_1) = \frac{D'_1(v^*_1)}{D'_2(v^*_1)}$ into (61) and simplifying, we have

$$D'_2(v^*_1)[\frac{D_1(v^*_1)}{D'_1(v^*_1)} + v^*_1] < 0.$$ 

Therefore, since $D'_1(v^*_1) < 0$ and $D'_2(v^*_1) < 0$ hold, we have $D_1(v^*_1) + D'_1(v^*_1)v^*_1 < 0$, or in other words, $v^*_1$ must be strictly greater than the monopoly price in state 1, $p^*_1$, a contradiction.

Thus, Lemmas 4-7 demonstrate that under the sufficient conditions for $V_1$, $V_2$ and $V_{12}$ to be intervals, the firm will not choose a mechanism where $V_1$, $V_2$ and $V_{12}$ are all non-empty. Clearly, the firm will not choose a mechanism such that all sets are empty. Therefore, the following possibilities remain to be considered.

5.1.2 $V_1$ and $V_{12}$ non-empty: State-by-state monopoly pricing.

The first possibility is $V_1 - V_{12} - nonempty$. That is, the firm chooses a mechanism such that $V_1 = [v^*_1, v^*_{12}]$, and $V_{12} = [v^*_{12}, \bar{v}]$ are non-empty, and the set $V_2$ is empty. In this case, the profit is given by

$$\pi(V_1, V_{12}) = \alpha_1 t_1(v^*_{12})D_1(v^*_{12}) + \alpha_2 t_2(v^*_{12})D_2(v^*_{12}) + \alpha_1 v^*_1(D_1(v^*_1) - D_1(v^*_{12})). \quad (62)$$

This is obtained by two modifications in (8): First, set $t_1(v) = v^*_1$ for all $v \in V_1$ (using Lemma 2). Second, by the proofs of Lemma 4 and Lemma 6, satisfying the BIC condition for $v^*_{12}$ with respect to $V_1$ implies that the same is satisfied for all $v$ greater than $v^*_{12}$. As the set $V_2$ is empty, other than the EIR conditions, the firm needs to ensure $t_i(v^*_{12}) \geq v^*_1$ for $i = 1, 2$ (due to the BIC of $v^*_1$ with respect to $V_{12}$), and it needs to satisfy the BIC constraint of $v^*_{12}$ with respect to $V_1$. That is,

$$(v^*_{12} - t_1(v^*_{12}))\alpha_1(-D'_1(v^*_{12})) + (v^*_{12} - t_2(v^*_{12}))\alpha_2(-D'_2(v^*_{12})) \geq (v^*_2 - v^*_1)\alpha_1(-D'_1(v^*_{12})).$$
must be satisfied. This can be rewritten as

\[ v_1^* \alpha_1(-D_1'(v_1^*)) + v_{12}^* \alpha_2(-D_2'(v_{12}^*)) \geq t_1(v_{12}) \alpha_1(-D_1'(v_{12}^*)) + t_2(v_{12}) \alpha_2(-D_2'(v_{12}^*)) \]. \tag{63} \]

At optimum, (63) will bind. The arguments in Claim L4 can be repeated to show that Condition 1(iii), Condition 2, and Lemma 2 of Wang (1993) imply that

\[ Z(v) > D_1(v) D_2(v) \]

holds for all \( v < \bar{v} \). Thus, as in Lemma 5 and Lemma 7, it will be optimal for the firm to put the maximum possible weight on \( t_2(v_{12}^*) \) (subject to the EIR constraints and the BIC constraint of \( v_1^* \) with respect to \( V_{12} \)). This means that in the case of \( V_1 \) and \( V_{12} \) non-empty, the firm will optimally set:

\[
\begin{align*}
    t_2(v_{12}^*) &= v_{12}^*, \\
t_1(v_{12}^*) &= v_1^*.
\end{align*}
\]

So we can rewrite the firm’s profit in (62) as:

\[ \pi = \alpha_1 v_1^* D_1(v_1^*) + \alpha_2 v_{12}^* D_2(v_{12}^*). \tag{64} \]

Let the monopoly price in state \( i \) be \( p_{im} \), where \( i = 1, 2 \). That is, \( p_{im} \) satisfies

\[ D_i(p_{im}) + p_{im} D_i'(p_{im}) = 0 \]

or \( p_{im} \) solves \( p = -\frac{D_i(p)}{D_i'(p)} \). Note that as \( Z(v) > \frac{D_1(v)}{D_2(v)} \) holds for all \( v < \bar{v} \), \( p_{i1} < p_{i2} \) holds. It is straightforward to see that to maximize (64), the firm chooses the mechanism described in (13). The first order conditions of profit (64) with respect \( v_1^* \) and \( v_{12}^* \) yield unique stationary points: \( v_1^* = p_{11}^m \) and \( v_{12}^* = p_{12}^m \). By Condition 1(iii), the second order conditions are satisfied at these points. We will call the mechanism in (13) as state-by-state monopoly pricing (henceforth, SBSMP). It is straightforward to verify that SBSMP satisfies BIC and EIR for all valuations.
5.1.3 Ruling out only one of \( V_1, V_2, \) or \( V_{12} \) non-empty.

A second possibility is that the firm chooses a mechanism such that only set \( V_i \) is non-empty, where \( i = 1, 2 \). By Lemma 2, the firm must charge \( v_i^* \) to all valuations in \( V_i \). Thus, the firm’s highest possible expected profit for this case is \( \alpha_i \pi_i^m \), where \( \pi_i^m \) denotes the profit from charging the state-\( i \) monopoly price, \( p_i^m \), to all valuations in \( [p_i^m, \overline{v}] \). This is clearly sub-optimal for the firm compared to the SBSMP mechanism which yields expected profit equal to \( \alpha_1 \pi_1^m + \alpha_2 \pi_2^m \).

A third possibility is that the firm chooses a mechanism such that only the set \( V_{12} = [v_{12}^*, \overline{v}] \) is non-empty. The EIR constraint of \( v_{12}^* \) and the BIC constraint of valuations less than \( v_{12}^* \) imply \( t_1(v_{12}^*) = t_2(v_{12}^*) = v_{12}^* \). The BIC constraint of valuations greater than \( v_{12}^* \) with respect to \( v_{12}^* \) imply that charging \( t_1(v) = t_2(v) = v_{12}^* \) achieves the upper bound of the expected payment from such valuations. To see this, consider the BIC constraint of \( v \) with respect to \( v_{12}^* \), where \( v > v_{12}^* \) holds:

\[
(v - t_1(v))\alpha_1(-D'_1(v)) + (v - t_2(v))\alpha_2(-D'_2(v)) \leq (v - t_1(v_{12}^*))\alpha_1(-D'_1(v)) + (v - t_2(v_{12}^*))\alpha_2(-D'_2(v)).
\]

Simplifying yields,

\[
t_1(v)\alpha_1(-D'_1(v)) + t_2(v)\alpha_2(-D'_2(v)) \leq t_1(v_{12}^*)\alpha_1(-D'_1(v)) + t_2(v_{12}^*)\alpha_2(-D'_2(v)),
\]

where the term on the left side is the expected payment of valuation \( v \in V_{12} \) to the firm, and it is bounded above by charging \( v \) the same payment scheme as the scheme offered to \( v_{12}^* \). So, the profit for the case where only the set \( V_{12} = [v_{12}^*, \overline{v}] \) is non-empty is \( \alpha_1 v_{12}^* D_1(v_{12}^*) + \alpha_2 v_{12}^* D_2(v_{12}^*) \), which is strictly less than the SBSMP profit for all possible choices of \( v_{12}^* \) because \( p_1^m \neq p_2^m \).
5.1.4 Ruling out $V_1$ and $V_2$ non-empty.

A fourth possibility is that the firm chooses a mechanism such that only the sets $V_1$ and $V_2$ are non-empty, and $V_{12}$ is empty. Suppose $v_i^* < v_j^*$. Then, $V_i \subset [v_i^*, \overline{v}]$ and $V_j \subset [v_j^*, \overline{v}]$. By Lemma 2, $t_i(v) = v_i^*$ holds for all $v \in V_i$ and by Fact 1, $t_j(v) = t_j(v_j^*)$ holds for all $v \in V_j$. So the firm’s profit in this case is bounded above by

$$\alpha_i v_i^* D_i(v_i^*) + \alpha_j t_j(v_j^*) D_j(v_j^*),$$

where $t_j(v_j^*) = v_j^* - \text{discount}$. Due to the BIC of $v_j^*$ with respect to $v_i^*$, the discount is strictly greater than 0. So, checking term-wise, the profit in this case is strictly less than the SBSMP profit of

$$\alpha_i p_i^m D_i(p_i^m) + \alpha_j p_j^m D_j(p_j^m).$$

5.1.5 Ruling out $V_2$ and $V_{12}$ non-empty.

The last possibility, other than SBSMP, is $V_2 - V_{12} - \text{nonempty}$. That is, the firm chooses a mechanism such that $V_2 = [v_2^*, v_{12}^*]$, and $V_{12} = [v_{12}^*, \overline{v}]$ are non-empty and the set $V_1$ is empty. By Lemma 2, $t_2(v_2^*) = v_2^* = t_2(v)$ holds for all $v \in V_2$. The proof of Lemma 4 and Lemma 6 imply that at the optimal payment scheme for this case, $t_i(v) = t_i(v_{12}^*)$ holds for all $v \in V_{12}$ and $i = 1, 2$.

The arguments in Claim L4 can be repeated to show that Condition 1(iii), Condition 2, and Lemma 2 of Wang (1993) imply that $Z(v) > \frac{D_1(v)}{D_2(v)}$ holds for all $v < \pi$. Thus, as in Lemma 5 and Lemma 7, to optimize, the firm will set $t_2(v_{12}^*)$ as high as possible, subject to the BIC of $v_2^*$ with respect to $V_{12}$ (that is, $t_i(v_{12}^*) \geq v_2^* \text{ for } i = 1, 2$), the EIR of $v_{12}^*$ (that is, $t_i(v_{12}^*) \leq v_{12}^* \text{ for } i = 1, 2$), and the BIC of $v_{12}^*$ with respect to $V_2$. The latter, after rearranging/canceling terms, is given by

$$v_2^* \alpha_2(-D_2'(v_{12}^*)) + v_{12}^* \alpha_2(-D_2'(v_{12}^*)) \geq t_1(v_{12}^*) \alpha_1(-D_1'(v_{12}^*)) + t_2(v_{12}^*) \alpha_2(-D_2'(v_{12}^*)). \quad (65)$$
At optimum, (65) binds. There are two sub-cases for optimal \( \{t_1(v_{12}^*), t_2(v_{12}^*)\} \) under Conditions 4(i) and 4(ii). Sub-case (a): Under Condition 4(ii), \( t_2(v_{12}^*) = v_{12}^* \) and \( t_1(v_{12}^*) \), given by binding (65), equals \( [v_{12}^* - \frac{(v_{12}^* - v_2^*)\alpha_2}{\alpha_1 Z(v_{12}^*)}] \), which is greater than \( v_2^* \) under Condition 4(ii).

Sub-case (b): Under Condition 4(i), \( [v_{12}^* - \frac{(v_{12}^* - v_2^*)\alpha_2}{\alpha_1 Z(v_{12}^*)}] \) is strictly less than \( v_2^* \). Thus, under Condition 4(i), the BIC of \( v_2^* \) with respect to \( V_{12} \) will bind, so the firm will optimally set \( t_1(v_{12}^*) = v_2^* \) and \( t_2(v_{12}^*) = v_2^* + \frac{(v_{12}^* - v_2^*)\alpha_1 Z(v_{12}^*)}{\alpha_2} \).

Consider sub-case (a) (Condition 4(ii)) of the \( V_2 - V_{12} - nonempty \) case. We now show that in maximizing its profit in this sub-case (a), the firm will choose a mechanism such that \( v_2^* < v_{12}^* < p_2^m \) holds. Due to Lemma 1, showing \( v_{12}^* < p_2^m \) will suffice. The profit in sub-case (a) is given by

\[
\pi_a(v_2^*, v_{12}^*) = \alpha_1 [v_{12}^* - \frac{(v_{12}^* - v_2^*)\alpha_2}{\alpha_1 Z(v_{12}^*)}] D_1(v_{12}^*) + \alpha_2 v_2^* D_2(v_{12}^*) + \alpha_2 v_2^* (D_2(v_{12}^*) - D_2(v_{12}^*)). \tag{66}
\]

Taking a derivative of the profit expression in (66) with respect to \( v_{12}^* \), and rearranging/canceling terms using \( Z(v_{12}^*) = \frac{D_1(v_{12}^*)}{D_2(v_{12}^*)} \) yields:

\[
\frac{\partial \pi_a(v_2^*, v_{12}^*)}{\partial v_{12}^*} = \alpha_1 \frac{\partial D_1(v_{12}^*)}{\partial v_{12}^*} \left[ 1 - \frac{\alpha_2}{\alpha_1 Z(v_{12}^*)} \right] + \frac{\alpha_2 (v_{12}^* - v_2^*) Z'(v_{12}^*) D_1(v_{12}^*)}{Z^2(v_{12}^*)} + \alpha_2 \frac{\partial D_2(v_{12}^*)}{\partial v_{12}^*}. \tag{67}
\]

Setting the derivative equal to 0 for the first order condition with respect to \( v_{12}^* \) tells us that \( v_{12}^* < p_2^m \) must hold. \( v_{12}^* < p_2^m \) can be shown by contradiction. Suppose \( v_{12}^* \geq p_2^m > p_1^m \) holds. Then the first term on the right side of (67) is negative (given Condition 4(ii)), and the second term is negative because \( v_{12}^* > v_2^* \) holds and because \( Z'(v_{12}^*) < 0 \) holds, and the third term is weakly negative. So it cannot be the case that the first order condition is satisfied for \( v_{12}^* \geq p_2^m \). Thus, \( v_{12}^* < p_2^m \) must hold.

Next, consider sub-case (b) (Condition 4(i)) of the \( V_2 - V_{12} - nonempty \) case. We now show that in maximizing its profit in this sub-case (b), like in sub-case (a), the firm will choose a mechanism such that \( v_2^* < v_{12}^* < p_2^m \) holds. Due to Lemma 1, showing \( v_{12}^* < p_2^m \) will
suffice. The profit in sub-case (b) is given by

\[
\pi(v^*_2, v^*_1) = \alpha_1 v^*_2 D_1(v^*_1) + \alpha_2 [v^*_2 + \frac{(v^*_1 - v^*_2)\alpha_1 Z(v^*_1)}{\alpha_2}] D_2(v^*_1) + \alpha_2 v^*_2 (D_2(v^*_2) - D_2(v^*_1))
\]

\[
= \alpha_1 v^*_2 D_1(v^*_1) + \alpha_1 (v^*_1 - v^*_2) Z(v^*_1) D_2(v^*_1) + \alpha_2 v^*_2 D_2(v^*_2).
\] (68)

Taking a derivative of the profit expression in (68) with respect to \(v^*_1\), and rearranging/canceling terms using \(Z(v^*_1) = D'_1(v^*_1) D'_2(v^*_1)\) yields:

\[
\frac{1}{\alpha_1} \frac{\partial \pi(v^*_1, v^*_1)}{\partial v^*_1} = \frac{\partial(v^*_1 D_2(v^*_1))}{\partial v^*_1} Z(v^*_1) + (v^*_1 - v^*_2) Z'(v^*_1) D_2(v^*_1).
\] (69)

Setting the derivative in (69) equal to 0 for the first order condition with respect to \(v^*_1\) implies \(v^*_1 < p^m_2\). To see this, note that otherwise \(v^*_1 \geq p^m_2\) holds, which implies that the right side of (69) is negative. This is because the first term on the right side of (69) is weakly negative if \(v^*_1 \geq p^m_2\) holds, and the second term is negative because \(v^*_1 > v^*_2\) and \(Z'(v^*_1) < 0\) hold. So it cannot be the case that the first order condition is satisfied for \(v^*_1 \geq p^m_2\). Thus, \(v^*_1 < p^m_2\) must hold.

The analysis for \(V_2 - V_{12} - nonempty\) mechanisms yields: (A) At the optimal mechanism within such mechanisms, \(v^*_2 < v^*_1 < p^m_2\) holds. (B) Under Condition 4(ii), the profit expression is given by (66). So the difference between the profit in the SBSMP mechanism and the profit in the \(V_2 - V_{12} - nonempty\) mechanism is

\[
\pi(SBSMP) - \pi_a(V_2, V_{12}) = \alpha_1 (\pi^m_1 - v^*_1 D_1(v^*_1)) + \alpha_2 (\pi^m_2 - v^*_2 D_2(v^*_2)) - \alpha_2 (v^*_1 - v^*_2) (D_2(v^*_2) - D_2(v^*_1)) - \frac{D_1(v^*_1)}{Z(v^*_1)}.
\] (70)

(C) Under Condition 4(i), the profit expression is given by (68). So the difference between the profit in the SBSMP mechanism and the profit in the \(V_2 - V_{12} - nonempty\) mechanism
under Condition 4(i) is

\[
\pi(SBSMP) - \pi_b(V_2, V_{12}) = \alpha_1 (\pi_1^m - v_{12}^* D_1(v_{12}^*)) + \alpha_2 (\pi_2^m - v_2^* D_2(v_2^*)) - \alpha_1 (v_{12}^* - v_2^*) D_2(v_{12}^*) Z(v_{12}^*).
\]  

(71)

Now we will show that under the conditions of Proposition 1, (70 and 71) are positive. As \((-D_1'(v)) = Z(v)(-D_2'(v))\) holds by definition, integration by parts yields

\[
D_1(v) = Z(v)D_2(v) + \int_v^\pi D_2(v)Z'(v)dv.
\]  

(72)

Substituting for \(\frac{D_1(v_{12}^*)}{Z(v_{12}^*)}\) using (72) into (70) yields

\[
\pi(SBSMP) - \pi_a(V_2, V_{12}) = \alpha_1 (\pi_1^m - v_{12}^* D_1(v_{12}^*)) + \alpha_2 (\pi_2^m - v_2^* D_2(v_2^*)) + \alpha_2 (\frac{v_{12}^* - v_2^*}{Z(v_{12}^*)}) \int_{v_{12}^*}^\pi D_2(v)Z'(v)dv.
\]  

(73)

If we show that that the expression on the right side of (73) is positive then we would have shown that, under Conditions 1-3 and 4(ii), SBSMP is better than setting only \(V_2\) and \(V_{12}\) nonempty, and we will be done for this case. Notice that the proposition allows us to consider \(\alpha_1\) “sufficiently large” with condition 4(ii). So, consider \(\alpha_1\) arbitrarily close to 1. Then \(v_{12}^*\) has to be close to \(p_{11}^m\) or else the first term on the right side of (73) is strictly positive and it dominates the other two terms and we are done. If, with \(\alpha_1\) sufficiently close to 1, the derivative of the sum of terms 2 and 3, with respect to \(v_2^*\), is negative when evaluated at \(v_2^* = p_{11}^m\), then we are done, because the expression will be greater than the expression evaluated at \(v_2^* = v_{12}^*\), which is positive. Differentiating the last two terms with respect to \(v_2^*\) yields

\[
-\frac{\partial p_{11}^m D_2(p_{11}^m)}{\partial v_2^*} - \frac{\int_{p_{11}^m}^\pi D_2(v)Z'(v)dv}{Z(p_{11}^m)}
\]

so, for \(\alpha_1\) “sufficiently large,” the sufficient condition for the expression in (73) to be positive
is
\[
\frac{\partial p_{m}^{n} D_{2}(p_{m}^{n})}{\partial v_{2}^{*}} > \frac{\int_{p_{m}^{n}}^{\pi} D_{2}(v)(-Z'(v))dv}{Z(p_{m}^{n})},
\]
which holds due to (12) as given in the Proposition.

Now consider the difference between the SBSMP profit and the profit from the \(V_{2} - V_{12}-\text{nonempty}\) mechanism under Condition 4(i) (sub-case (b)). Substituting for \(D_{2}(v_{12}^{*})Z(v_{12}^{*})\) using (72) in (71) yields.

\[
\pi(SBSMP) - \pi_{b}(V_{2}, V_{12}) = \alpha_{1}[\pi_{1}^{m} - v_{12}^{*}D_{1}(v_{12}^{*}) + (v_{12}^{*} - v_{2}^{*}) \int_{v_{12}^{*}}^{\pi} D_{2}(v)Z'(v)dv]
+ \alpha_{2}(\pi_{2}^{m} - v_{2}^{*}D_{2}(v_{2}^{*})).
\] (74)

The conditions in the Proposition allow us to consider \(\alpha_{2}\) sufficiently “large” when Conditions 1-3 and 4(i) hold. So consider \(\alpha_{2}\) sufficiently close to 1. The last term on the right side of (74) is positive and dominates the expression unless we have \(v_{2}^{*} = p_{2}^{m}\). As \(v_{2}^{*}\) converges to \(p_{2}^{m}\), the last term is still positive, and because \(v_{2}^{*} < v_{12}^{*} < p_{2}^{m}\) holds, the term in square brackets converges to \(\pi_{1}^{m} - p_{2}^{m}D_{1}(p_{2}^{m})\), which is strictly positive since the two states have different monopoly prices.

So under Conditions 1-3, if Condition 4(ii) holds and \(\alpha_{1}\) is sufficiently large, or, if condition 4(i) holds, \(\alpha_{2}\) is sufficiently large, and (12) holds, then, SBSMP yields greater profit than \(V_{2} - V_{12}-\text{nonempty}\) mechanisms. We have already shown that SBSMP is strictly better than all other alternative mechanism for the monopoly firm. Thus, under the conditions of Proposition 1, state-by-state monopoly pricing is optimal for the monopoly firm. ■

References


discriminating monopolist when demands are interdependent. *Econometrica* 53(2), 345-361.


