Competing Mechanisms with Multi-Unit Consumer Demand

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Abstract

The competing mechanisms literature is extended to a market setting in which firms have fixed capacity, and there is a continuum of consumers who desire multiple units and can only purchase from one firm. Firms choose incentive compatible mechanisms in which consumers report their utility types; consumption of the good and payments of the numeraire are continuous functions of the reports. Uniform price auctions with reserve prices, reinterpreted as direct mechanisms, are not consistent with equilibrium. However, modified auctions without reserve prices but with type-specific entry fees do constitute an equilibrium of the competing mechanisms game under additional regularity assumptions. When all firms announce fixed prices at the perfectly competitive level, this profile also constitutes an equilibrium of the competing mechanism game.

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1. Introduction

This paper extends the competing mechanisms literature to a market setting in which consumers demand multiple units. Specifically, consider a market with a finite number of firms with fixed capacity. There is a continuum of consumers, each of whom demands multiple units and can only visit one firm during the market period.[1] There is no aggregate uncertainty about demand. In the Competing Mechanisms Game, Γ, firms simultaneously announce mechanisms, then consumers learn their utility type, choose a firm, and participate in that firm's mechanism. A feasible mechanism asks each of its consumers to report his type, and specifies the amount of the good received by each consumer and the amount of the numeraire each consumer pays to the firm, as a function of the measures of each type reported to the firm. Mechanisms are required to be incentive compatible and continuous. Some indirect mechanisms that can be reinterpreted as feasible mechanisms include (i) uniform price auctions, possibly with reserve prices or entry fees; (ii) fixed-price-per-unit mechanisms, in which the firm specifies a price and consumers are allocated their utility maximizing demands if resources permit, and a per capita rationing limit clears the market if demand exceeds capacity; and (iii) fixed-price-per-share mechansisms, in which all consumers visiting the firm make a pre-specified payment and receive an equal share of the firm's capacity.

Much of the literature involves competition in which sellers with one indivisible unit choose second-price auctions with a reserve price. Typically, equilibrium entails a positive reserve price which approaches zero as the number of sellers approaches infinity. The results here, however, are considerably different. Reserve-price competition does not survive if more general mechanisms are allowed. There are no symmetric equilibria of Γ, even in mixed strategies, in which firms choose uniform price auctions with a reserve price. If there were such an equilibrium, and if the highest reserve price chosen in equilibrium is above the competitive equilibrium price, p^c, then a firm choosing this reserve price would face excess capacity. There is a profitable deviation to another mechanism that maintains its set of customers but fully allocates capacity. On the other hand, if the highest reserve price chosen in equilibrium is at or below p^c, then the reserve prices are not binding. There is a profitable deviation to another mechanism that exploits residual demand while fully allocating capacity. Under reasonable assumptions,

[1] For example, suppose each of n firms owns a mountain for downhill skiing, with a chair lift that can accommodate a fixed number of ski runs during a particular day.

we show that Γ has a symmetric equilibrium in which each firm's pure strategy is a modified uniform price auction with type-specific entry fees and zero reserve prices. Market power by sellers is exploited through entry fees, which in the model represents a transfer from buyers to sellers rather than an efficiency loss.[2] The concluding remarks discuss the conjecture that these main results carry over when the model is extended to include aggregate demand uncertainty.

Surprisingly, given the negative result for reserve price mechanisms, we show that Γ has a symmetric equilibrium in which every firm chooses a fixed-price-per-unit mechanism at p^c. There is a difference between all firms setting the price p^c and all firms setting a reserve price equal to p^c. In both cases, a firm that raises either its price or its reserve price above p^c will induce a new consumer equilibrium that loses some of its customers, and this outflow will lower the utility received by consumers visiting other firms. With fixed-price mechanisms, there will be excess demand and rationing at the other firms. With reserve price mechanisms, the auction price at other firms will rise above the reserve price to clear the market.

This result, that the competitive equilibrium allocation is achieved with a finite number of firms, is *not* the familiar Bertrand result. Osborne and Pitchik (1986) consider a duopoly model of price competition with capacity constraints. For the version of their model in which capacities are fixed, they find that equilibrium is often in mixed strategies. Their model allows consumers to purchase from both firms, so the competitive result here depends on the subtleties of the rationing rule. The issue is discussed in Section 4. The concluding remarks discuss the conjecture that fixed-price-per-unit mechanisms will not survive the extension to include aggregate demand uncertainty.

Additional results are available for the case of one consumer type. If there is one consumer type, then there is an equilibrium of Γ in which all firms choose a fixed-price-per-share mechanism, but profits are higher than in the competitive equilibrium.[3] We also show that there is yet another equilibrium in which firms extract all consumer surplus.

There is a considerable literature on competing mechanisms. McAfee (1993) provides a model in which sellers with one unit of a good choose efficient auctions

[2] In a different context, Levin and Smith (1994) consider a single seller with one unit, and show that the seller will choose a zero reserve price when values are i.i.d. and bidders face an entry cost.

[3] For the skiing example, a fixed-price-per-share mechanism is simply a lift ticket. The customer would pay for the right to go on multiple runs, with the lift queue guaranteeing that all customers receive the same quantity, as determined by the lift capacity and the number of customers. See Barro and Romer (1987).

in equilibrium, out of a general class of mechanisms. The number of sellers is assumed to be large enough that they can ignore their effect on the broader market. Peters and Severinov (1997) restrict the space of mechanisms to second-price auctions with a reserve price, and show that reserve prices converge to zero as the number of sellers approaches infinity. Their solution concept specifies beliefs about the distribution of customers they will receive if they deviate from the common reservation price, and these beliefs are correct in the limit. Given these beliefs, firms maximize their expected profit. Burguet and Sákovics (1999) model a game with two buyers and two sellers who choose second-price auctions with a reserve price. Although they cannot fully characterize the symmetric equilibrium, they show that it involves mixed strategies, and the support of the equilibrium reserve price is bounded above zero. Pai (2014) models competition between two sellers, each with a single unit, who choose mechanisms from the space of "extended auctions" (which includes posted prices). It is shown that two forms of inefficiency exist in equilibrium: sellers sometimes withhold the good, and the good is sometimes allocated to an agent that does not have the highest valuation. Coles and Eeckhout (2003) consider a model with two sellers, each with one unit, and two identical buyers. When sellers can choose arbitrary anonymous mechanisms within this environment, it is shown that there is a continuum of equilibria, including price-posting and auctions with a reserve price. Virág (2007) generalizes Coles and Eeckhout (2003) by introducing multiple buyer types, so that price-posting is no longer efficient. Virág (2007) shows that, with two types, only ex-post efficient mechanisms such as auctions are consistent with equilibrium. Peters and Severinov (2006) consider a dynamic competing auctions setting, where buyers have multiple opportunities to place bids on any auction. A perfect Bayesian equilibrium is characterized in which buyers adopt the simple strategy of augmenting the lowest available standing bid by the minimum increment, as long as their bid does not exceed their value. It is shown that when the number of sellers is large, they choose a zero reserve price. All of the above-mentioned papers assume that sellers own one unit of the good and that buyers have unit demands.

A few papers study competing mechanisms in more general and abstract settings. Peters and Troncoso-Valverde (2013) prove a folk-theorem, showing that an allocation can be supported as an equilibrium outcome whenever it is incentive compatible and individually rational. The construction requires players to coordinate on very complicated messages involving encryption keys. See also Epstein and Peters (1999) and Peters and Szentes (2012).

The present paper is unique in that it imposes an economic structure in which sellers have multiple units and consumers have downward sloping demand curves. As a result, the set of available mechanisms is large. However, consumers report their demand types, rather than complex messages about the mechanisms chosen by other firms. There is a continuum of consumers of each type, and a single consumer's arrival choice or report has a negligible influence on any other agent in the economy. This structure provides a tractable way to study imperfect competition by the firms offering the mechanisms. The equilibrium mechanisms turn out to be very simple, and often involve no reports whatsoever. Multiplicity of equilibrium, a hallmark of the competing mechanisms literature, obtains here, even though we rule out collusive outcomes achieved by consumers reporting on whether competing firms have deviated. Intuitively, what the mechanism is prepared to offer consumers, off the equilibrium path, can serve to soften or strengthen competition. However, the only equilibrium mechanism we have identified, which is robust to multiple types and seems to be robust to demand uncertainty, is an auction with entry fees.

Section 2 sets up the economic environment and defines the competing mechanisms game. Section 3 contains results about auction mechanisms. Section 4 contains results about price-per-unit mechanisms. Section 5 contains additional results for the case of one consumer type. Section 6 contains some examples. Section 7 contains some brief concluding remarks. Proofs of all results are given in the Appendix.

2. The Competing Mechanisms Game

We consider a market with n firms selling a homogeneous good, and for simplicity, we assume that they all have the same capacity, normalized to 1, and no costs. There are I types of consumers, and a continuum of consumers of each type. Denote the measure of type i consumers as $r_i n$. Each consumer of type i has the quasilinear utility function $u_i(x_i) + M_i$, where x_i is the consumption of the (divisible) good and M_i is the consumption of the numeraire (or money). We assume that each consumer has a sufficiently large endowment of money to make any desired purchases, and that the utility function for each i satisfies $u_i'(x_i) > 0$ and $u_i''(x_i) < 0$ for all x_i.

Although we think of firms as being geographically separated, so that a consumer can visit at most one firm, the competitive-equilibrium benchmark will be useful. A consumer of type i facing price p will choose the quantity of the good satisfying

$$u_i'(x_i) = p,$$

whose solution we denote by the demand function, $d_i(p)$. Each firm inelastically supplies its capacity, so the competitive equilibrium price, denoted by p^c, is the unique solution to

$$\sum_{i=1}^{I} r_i n d_i(p^c) = n. \tag{2.1}$$

We assume that types can be ranked in terms of willingness to pay, so that $i < h$ implies $d_i(p) > d_h(p)$ for all p.

In the Competing Mechanisms Game, denoted by Γ, firms simultaneously select a mechanism from a class of mechanisms, M, defined below. We restrict attention to incentive-compatible direct-revelation mechanisms, where consumers report their utility type. Let ρ_i^f denote the measure of agents participating in firm f's mechanism and reporting type i, and define $\rho^f = (\rho_1^f, ..., \rho_I^f)$. A mechanism for firm f, denoted by m^f, consists of continuous functions $x_i^f(\rho^f)$ and $P_i^f(\rho^f)$, satisfying for all ρ^f the feasibility condition,

$$\sum_{i=1}^{I} \rho_i^f x_i^f(\rho^f) \leq 1.$$

Given the reports ρ^f, $x_i^f(\rho^f)$ is the consumption of the good received by a consumer reporting type i at firm f, and $P_i^f(\rho^f)$ is the money payment made by a consumer reporting type i at firm f. The profit or payoff to firm f is given by $\sum_{i=1}^{I} \rho_i^f P_i^f(\rho^f)$.

The timing of Γ is as follows. First, firms simultaneously choose a mechanism. Then consumers observe the profile of mechanisms selected by the firms, denoted by $m = (m^1, ..., m^n)$. Finally, consumers choose which firm to visit, report a type, and participate in that firm's mechanism.[4] Our solution concept is subgame perfect Nash equilibrium in which all consumers of the same type choose the same mixed strategy. That is, for any profile of mechanisms m, all consumers of the same type choose the same mixed strategy over arrivals. We denote an equilibrium by SPNE, and unless otherwise specified we will consider equilibria in which firms use pure strategies. Since we only consider type-symmetric equilibria

[4] We do not impose individual rationality restrictions. However, individual rationality holds on the equilibrium path for all of the mechanisms considered in this paper. Also, individual rationality on and off the equilibrium path would hold if we were to impose the Inada condition, $u(0) = -\infty$.

and the relevant subgame will always be clear from the context, we can denote the probability that a consumer of type i visits firm f as β_i^f.[5] Given a non-zero vector of arrival probabilities at firm f, $\beta^f = (\beta_1^f, ..., \beta_I^f)$, a mechanism is *incentive compatible* if reports satisfy the truth-telling condition,

$$\begin{aligned} u_i(x_i^f(\rho^f)) - P_i^f(\rho^f) &\geq u_i(x_h^f(\rho^f)) - P_h^f(\rho^f), \text{ for all } i, h, \text{ where} \\ \rho_i^f &= r_i n \beta_i^f \text{ holds.} \end{aligned} \tag{2.2}$$

We define M to be the set of continuous functions from reported types into a quantity consumed and payment by each type, satisfying (2.2) for all β^f.

Example: Fixed-Price-Per-Unit Mechanism.

If firm f chooses a fixed price per unit, p^f, then if there is no excess demand at firm f, each consumer receives his utility maximizing consumption and pays the per unit price p^f. If there is excess demand, then some consumers are rationed but each consumer continues to pay the per unit price p^f. We assume that there is a maximum quantity that any consumer can choose, $\overline{x}^f$, which clears the market as defined below. Consumers whose demand exceeds $\overline{x}^f$ consume at the maximum limit, and consumers whose demand is less than $\overline{x}^f$ consume their utility maximizing quantity.[6]

Here is the mechanism in which firm f chooses a fixed price per unit, p^f.

$$\begin{aligned} \text{For} \sum_{i=1}^{I} \rho_i^f d_i(p^f) &\leq 1, \\ x_i^f(\rho^f) &= d_i(p^f) \quad \text{and} \quad P_i^f(\rho^f) = p^f d_i(p^f). \\ \text{For} \sum_{i=1}^{I} \rho_i^f d_i(p^f) &> 1, \\ x_i^f(\rho^f) &= \min[d_i(p^f), \overline{x}^f(\rho^f)] \quad \text{and} \\ P_i^f(\rho^f) &= p^f \min[d_i(p^f), \overline{x}^f(\rho^f)], \end{aligned} \tag{2.3}$$

where $\overline{x}^f(\rho^f)$ is the solution to

[5] We assume that the conclusion of the law of large numbers holds, so that if all type i consumers use the mixing probability β_i^f, then the measure of type i consumers visiting firm f is $r_i n \beta_i^f$.

[6] This rationing rule would arise if we think of consumers as waiting in a queue, purchasing one unit at a time, then getting back into the queue if they desire to purchase more units.

$$\sum_{i=1}^{I} \rho_i^f \min[d_i(p^f), \overline{x}^f(\rho^f)] = 1. \tag{2.4}$$

From (2.4), it follows that whenever the consumption limit matters, $\sum_{i=1}^{I} \rho_i^f d_i(p^f) > 1$, then $\overline{x}^f(\rho^f)$ is uniquely defined. It is easy to see that this mechanism is continuous in ρ^f and is incentive compatible.

The class of allowable mechanisms, M, is fairly broad. However, M is far from completely general. Consumers are reporting their valuation-types, but not their full type which includes information about the other firms' mechanisms. See Epstein and Peters (1999) for an analysis of how to build a universal type space. It seems reasonable, when we are modeling competition by firms who set up their own markets to sell their capacity, to rule out incredibly complicated mechanisms requiring higher order reports about the mechanisms of other firms.[7] Whether or not a mechanism is incentive compatible can depend on β^f, and requiring that incentive compatibility holds for all β^f can be restrictive.[8] However, this restriction is only used to guarantee the existence of a Nash equilibrium of all subgames off the equilibrium path. The price-per-unit and price-per-share mechanisms have the nice property that no reports of any sort are required, and are obviously incentive compatible for any β^f.

Without the continuity assumption, we would typically have the problem that no SPNE exists. The reason is that, following a deviation in which firm f chooses a mechanism that is not continuous in ρ^f, the resulting consumer subgame often has no Nash equilibrium, due to the fact that we have a continuum of consumers.

For example, suppose there is only one type of consumer and all firms other than firm f choose a fixed price-per-unit of p^c. Suppose firm f chooses the following mechanism (for some positive ε), which is not continuous:

$$\begin{aligned} x_1^f(\rho^f) &= \frac{1}{\rho_1^f} \\ P_1^f(\rho^f) &= (p^c + \varepsilon) d_1(p^c) \quad \text{if} \quad \rho_1^f \geq r_1 \\ P_1^f(\rho^f) &= 0 \quad \text{if} \quad \rho_1^f < r_1. \end{aligned}$$

[7] Peters and Troncoso-Valverde (2013) develop the notion of sequential communication mechanism. The mechanisms allow players to report sequentially, first reports about types and then reports about the first-round reports of other players.

[8] See Peck (1997) for an example, in a different context, in which the revelation principle might fail when only valuation-types are reported.

For this profile of mechanisms, the consumer subgame has no Nash equilibrium. There cannot be a NE with $\beta_i^f \geq \frac{1}{n}$, because consumers are not rationed at the other firms, and receive their competitive equilibrium utility; at firm f, consumers pay more than what they would pay at other firms and receive less consumption, so consumers visiting firm f are not best responding. There cannot be a NE with $\beta_i^f < \frac{1}{n}$, because consumers visiting other firms are rationed and receive less than their competitive equilibrium utility; consumers at firm f receive well above their competitive equilibrium utility, so consumers visiting other firms are not best responding.

Fortunately, Lemma 1 shows that, for our class of allowable mechanisms, which requires continuity and incentive compatibility, the consumer subgame always has a Nash equilibrium.

Lemma 1: *For any profile of mechanisms, $m = (m^1, ..., m^n)$, where $m^f \in M$ for $f = 1, ..., n$, the resulting consumer subgame has a type-symmetric NE.*

3. Auction Mechanisms

3.1. Uniform Price Auctions with Reserve Prices

Many types of auctions are included within the set of allowable mechanisms, M. For example, if firm f holds a uniform price auction with reserve price R^f, its consumers submit demand functions specifying, for each price greater than or equal to R^f, the total bid at or above that price. The firm collects the submitted demand functions, and the auction price is the highest rejected bid. That is, the auction price is the price that clears the market if the total bid at R^f exceeds the firm's supply of 1, and the auction price is R^f otherwise. Since consumers are negligible and cannot affect the auction price, in equilibrium a type i consumer will bid his demand function, $d_i(p)$, for $p \geq R^f$.

Here is the mechanism corresponding to a uniform price auction with reserve price R^f. Given ρ^f, let p solve

$$\sum_{i=1}^{I} \rho_i^f d_i(p) = 1$$

and let $p^R \equiv \max[p, R^f]$. Then we have

$$\begin{aligned} x_i^f(\rho^f) &= d_i(p^R) \text{ and} \\ P_i^f(\rho^f) &= p^R d_i(p^R). \end{aligned} \tag{3.1}$$

Incentive compatibility and continuity are clearly satisfied.

In order to compare our setting to the competing auctions literature with single-unit demands, before considering whether profiles of auction mechanisms are equilibria to Γ, we consider the "reserve price game" Γ^R in which firms must choose a mechanism as specified in (3.1) for some R^f. In any equilibrium of the consumer subgame, the auction price at all firms attracting consumers must be the same. If not, a consumer could instead choose a firm with a lower auction price and receive his utility maximizing quantity at that price. For any firm f setting a reserve price that is non-binding in the ensuing consumer equilibrium, obviously it will sell its entire capacity. If there is a firm f attracting customers but whose reserve price, R^f, is binding in the ensuing consumer equilibrium, the auction price at all firms will be R^f, and it must be the case that $R^f > p^c$ holds.[9] Therefore, the firms setting lower (non-binding) reserve prices sell all their capacity, and firm f would not sell all of its capacity since there is excess capacity in the market at this price.[10]

How does the SPNE of Γ^R relate to the competing auctions literature? Burguet and Sákovics (1999) consider a duopoly reserve price game with two consumers and single-unit demand. They show that symmetric equilibrium involves mixed strategies by firms, and that the support of the equilibrium distribution of reserve prices is bounded above zero. In the present setting, however, Γ^R could have a SPNE in pure strategies in which all firms choose a zero reserve price. To see this, consider the profit function of firm f, if all other firms choose a zero reserve price. For $R^f > p^c$, firm f can only make positive profits if R^f is close enough to p^c so that overall market demand at the price R^f exceeds the capacity of the

[9] A firm setting $R^f \leq p^c$ will have a non-binding reserve price in the ensuing consumer equilibrium, even if all other firms set a reserve price of zero.

[10] If several firms set the same (binding) reserve price as firm f, then it would be impossible for all of these firms to sell all of their capacity. In the most natural consumer equilibrium, these firms would be treated identically and all of them would have excess capacity.

other firms, $n-1$. In this case, $\{\beta_i^f\}_{i=1}^I$ satisfies

$$\begin{aligned}\sum_{i=1}^{I} r_i n(1-\beta_i^f) d_i(R^f) &= n-1, \text{ or} \\ \sum_{i=1}^{I} r_i n \beta_i^f d_i(R^f) &= \sum_{i=1}^{I} r_i n d_i(R^f) - (n-1). \qquad (3.2)\end{aligned}$$

From (3.2), we can write the profits of firm f as

$$\begin{aligned}\pi^f(R^f) &= \sum_{i=1}^{I} r_i n \beta_i^f d_i(p^f) R^f \\ &= R^f[\sum_{i=1}^{I} r_i n d_i(R^f) - (n-1)]. \qquad (3.3)\end{aligned}$$

Differentiating (3.3) with respect to R^f and evaluating at $R^f = p^c$ yields

$$(\pi^f)'(p^c) = 1 + p^c \sum_{i=1}^{I} r_i n d_i'(p^c). \qquad (3.4)$$

Whenever (3.4) is negative, and the second order condition,

$$2\sum_{i=1}^{I} r_i n d_i'(R^f) + R^f \sum_{i=1}^{I} r_i n d_i''(R^f) < 0$$

holds for all $R^f \geq p^c$, then there is a SPNE where all firms set a zero reserve price.

To recap our analysis of the reserve price game, there is a SPNE in which all firms choose a zero reserve price when (3.4) is negative. It can be shown that (3.4) is negative when the market price elasticity of demand (in absolute value) is greater than $1/n$. Thus, unlike the literature that focuses on a finite number of consumers with unit demand, the competitive outcome of all reserve prices being zero can happen in equilibrium. This is likely to occur when there are many firms.

Now let us consider the competing mechanisms game, Γ. It turns out that there is no SPNE in which all firms choose a uniform price auction with any reserve price, even allowing for mixed strategies by firms. The idea is that, if firm f sets R^f at the upper support of equilibrium reserve prices, it knows that some of its capacity will be wasted. There is a profitable deviation to another mechanism that allocates all of its capacity, while continuing to attract the same set of customers as before.

Proposition 1: *There is no SPNE of Γ in which all firms choose reserve price mechanisms, even allowing for mixed strategies by firms.*

3.2. Uniform Price Auctions with Entry Fees

With additional structure on utility functions, the following proposition establishes that there is a SPNE to Γ in which firms choose a "modified auction with entry fees" ($\varepsilon - MAWEF$) mechanism. First, an "auction with entry fees" mechanism for firm f involves a zero reserve price and a set of entry fees, E_i^f, satisfying: (1) $x_i^f(\rho^f) = d_i(p(\rho^f))$, (2) $P_i^f(\rho^f) = p(\rho^f)d_i(p(\rho^f)) + E_i^f$, and (3) the auction price $p(\rho^f)$ is determined by the market-clearing condition,

$$\sum_{i=1}^{I} \rho_i^f d_i(p(\rho^f)) = 1.$$

If E_i^f varies across types, the mechanism might not be incentive compatible, for example, when the measure of arriving consumers is small enough that the market clearing price is near zero. Hence, we modify the concept to $\varepsilon - MAWEF$, in which the entry fees fully apply in an ε neighborhood of p^c, and linearly drop to zero as the price reaches $p^c - 2\varepsilon$ or $p^c + 2\varepsilon$. Thus, in an $\varepsilon - MAWEF$, the entry fee is given by

$$\begin{array}{rcl} E_i^f \quad \text{if} \quad p^c - \varepsilon & \leq & p(\rho^f) \leq p^c + \varepsilon, \\ \left(\dfrac{p(\rho^f) - p^c + 2\varepsilon}{\varepsilon}\right) E_i^f \quad \text{if} \quad p^c - 2\varepsilon & \leq & p(\rho^f) \leq p^c - \varepsilon, \\ \left(\dfrac{p^c + 2\varepsilon - p(\rho^f)}{\varepsilon}\right) E_i^f \quad \text{if} \quad p^c + \varepsilon & \leq & p(\rho^f) \leq p^c + 2\varepsilon, \\ & & \text{and } 0 \text{ otherwise.} \end{array}$$

Note that the profile of mechanisms given in Proposition 2 below remain consistent with equilibrium without this modification; the modification is only used to ensure that the mechanisms are within our allowable set, M.

Proposition 2: *Assume that for* $i = 1, ..., I$*, demand is of the form* $d_i(p) = a_i d(p)$*, where* a_i *is a positive parameter and* $d'(p) < 0$*. Then if* n *is sufficiently large and* ε *is sufficiently small, there is a SPNE of* Γ *in which, for* $f = 1, ..., n$*, firm* f *chooses the following* $\varepsilon - MAWEF$ *mechanism:*

$$E_i^* = -\frac{a_i [d(p^c)]^2}{(n-1) d'(p^c)} \quad \text{for } i = 1, ..., I.$$

Along the equilibrium path, we have $\beta_i^f = \frac{1}{n}$ *for all* i *and* f*, each firm's auction price is* p^c*, and each firm's profit is*

$$p^c - \frac{d(p^c)}{(n-1) d'(p^c)}.$$

Using entry fees rather than reserve prices guarantees that capacity is utilized efficiently by the firm and its customers. With entry fees as specified in Proposition 2, raising the entry fee sends some customers away and lowers the auction price, but it also increases entry fee revenue, and these effects offset.

4. Fixed-Price-Per-Unit Mechanisms

The following proposition establishes that there is a SPNE of Γ in which all firms choose the fixed-price-per-unit mechanism with the competitive equilibrium price.

Proposition 3: *For sufficiently large* n*, there is a SPNE of* Γ *in which, for* $f = 1, ..., n$*, firm* f *chooses the price-per-unit mechanism defined in (2.3) with* $p^f = p^c$*, and* $\beta_i^f = \frac{1}{n}$ *for all* i *and* f *along the equilibrium path.*

The proof of Proposition 3 solves an optimization problem for a potential deviator, firm f, which chooses its mechanism, $x_i^f(\rho^f)$ and $P_i^f(\rho^f)$, and chooses consumer behavior, β, to maximize profits subject to its resource constraint and the consumer indifference condition necessary for β to be an equilibrium to the consumer subgame. It is shown that no profitable deviation is possible.

It may seem surprising that firms are choosing the competitive equilibrium price in equilibrium, but here is the intuition for why the firm would not choose a different price. If firm f decided to raise its price slightly above p^c, the utility offered to its customers declines, necessitating a reduction in the measure of consumers visiting firm f and an increase in the measure of consumers visiting the other firms. Thus, type 1 consumers visiting other firms will be rationed.[11] The resulting equilibrium of the consumer subgame will involve type 1 consumers being indifferent between buying as many units as they want at firm f at the higher price, vs. paying the price p^c and being rationed at one of the other firms. For a small price increase above p^c by firm f, the rate at which utility at firm f is reduced is approximately $d_1(p^c)$. For type 1 consumers choosing some other firm, j, the rate at which utility is reduced is approximately

$$-\frac{\partial x_1^j}{\partial p}[u_1'(x_1^j) - p^c],$$

where x_1^j is less than $d_1(p^c)$ due to excess demand and rationing. However, when the price increase is small, the term in brackets in the above expression is approximately zero, due to the envelope theorem. For the indifference condition to be satisfied, the above expression must equal $d_1(p^c)$, so $\frac{\partial x_1^j}{\partial p}$ must equal negative infinity. That is, we must have an infinite rate of outflow of demand from firm f at the margin. In other words, for a small increase in the price chosen by firm f, the reduction in the quantity sold is an order of magnitude greater than the increase in the price. Locally, raising or lowering the price from p^c strictly lowers profits. The assumption that n is sufficiently large is only needed as a sufficient condition to establish the relevant second-order conditions. For the examples presented in the Appendix, the competitive pricing result holds for all $n > 1$.

Other SPNE of Γ exist yielding the same price and allocation, but a different pattern of consumer types across firms. Rather than all firms seeing the same distribution of consumer types, as in Proposition 3, any mixed strategy profile by consumers is consistent with equilibrium, as long as the total demand at each firm, at the price p^c, is exactly equal to the total capacity at each firm, 1.

A corollary of Proposition 3 is that competitive pricing is an equilibrium of the game, Γ^{PPU}, in which firms are restricted to choose a price per unit. This form of price competition, where the rationing rule requires consumers to purchase from only one firm, has not been discussed before in the IO literature. It is worth

[11] Usually type 1 consumers will be the only type rationed in the consumer subgame following the deviation, but the proof of Proposition 3 takes into account the possibility that several types could be rationed at other firms.

emphasizing that the reason for the perfectly competitive outcome is completely different from that of the Bertrand model without capacity constraints. True, when all firms choose a price per unit, p, where $p > p^c$ holds, then firms will not sell all of their capacity, and a firm that deviates to a slightly lower price can profitably sell more or all of its capacity. However, when all firms choose a price per unit where $p < p^c$ holds, then a firm that deviates to a higher or lower price *does not* face an infinite rate of outflow or inflow of customers, even at the margin; there would be a profitable deviation to a higher price. The actual case in which all firms set $p = p^c$ is a borderline case. All capacity is used, so a firm could never benefit from lowering its price. However, a firm does not want to raise its price either, due to the envelope theorem logic explained above. Osborne and Pitchik (1986) study a duopoly model with fixed capacities, which they later endogenize. Their model with fixed capacity is very similar to this one, but with a rationing rule that allows consumers to purchase from both firms. Competitive pricing does not necessarily obtain in their model when demand and capacity parameters correspond to what we assume here.[12]

5. Additional Results for $I = 1$

5.1. Fixed-Price-Per-Share Mechanisms

With a fixed-price-per-share mechanism, rather than setting a price for each unit and letting consumers decide how many units to buy, here a firm sets a price for the right to consume an equal share of the firm's capacity. Here is the mechanism in which firm f chooses a fixed price-per-share, P^f.

$$\begin{aligned} x_i^f(\rho^f) &= \frac{1}{\sum_{h=1}^{I} \rho_h^f} \\ P_i^f(\rho^f) &= P^f. \end{aligned} \tag{5.1}$$

In (5.1), capacity is divided evenly across all consumers at firm f, and each consumer makes a total payment equal to P^f, independent of how many consumers

[12] For the Osborne and Pitchik (1986) rationing rule which allows consumers to purchase from multiple firms, whether or not there is a pure strategy equilibrium with $p = p^c$ depends on the price elasticity of demand. If demand is sufficiently inelastic, then if all firms choose $p = p^c$, a local deviation to a higher price is profitable. Here, in contrast, a local deviation to a higher price is always unprofitable.

choose firm f. This mechanism implies infinite consumption when the measure of consumers is zero, so we will have to modify it slightly to avoid a discontinuity at zero. A modified fixed-price-per-share mechanism caps consumption at X, with the idea that X is so large that it cannot bind in equilibrium.

$$\begin{aligned} x_i^f(\rho^f) &= \min[\frac{1}{\sum_{h=1}^{I}\rho_h^f}, X] \\ P_i^f(\rho^f) &= P^f. \end{aligned} \tag{5.2}$$

This mechanism is continuous and incentive compatible. Proposition 4, below, shows that for the special case of one consumer type there is a SPNE of Γ in which all firms choose a modified fixed-price-per-share mechanism. In this equilibrium, consumers receive the same consumption of the good as in the competitive equilibrium, but their total payment is higher than in the competitive equilibrium. The cap, $X = \frac{n}{r_1}$, is only needed to ensure that the mechanism is continuous everywhere and within the allowable set, M; it is not needed for the result itself.

Proposition 4: *If we have $I = 1$, then for sufficiently large n, there is a SPNE of Γ in which all firms choose a modified fixed-price-per-share mechanism defined in (5.2). That is, each firm f chooses the share price*

$$P^f = P^* \equiv \frac{np^c}{(n-1)r_1}$$

and any (non-binding) cap, $X > \frac{n}{r_1}$. Consumers (on the equilibrium path) choose a mixed strategy that assigns probability $\frac{1}{n}$ to each firm.

What is the intuition for why firms raise its effective price[13] above p^c with price-per-share competition, but not with price-per-unit competition? The answer is that, in the two equilibria, there is a different effect of a price increase on the utility received in the consumer subgame. In both equilibria, a price increase by firm f sends consumers to the other firms, so that the quantity they consume at other firms in the consumer subgame falls. With price-per-unit competition, the payment consumers make at other firms also falls. However, with price-per-share

[13] The effective price of firm f with price-per-share competition (Proposition 4) is defined to be P^f/x^f.

competition, when firm f increases its price and consumers shift to the other firms, the payment consumers make at other firms does not fall. It turns out that this softening of competition provides an incentive to raise the share price above p^c. In Proposition 4, the assumption that n is "sufficiently large" is only used as a convenient way of demonstrating the second order conditions. Examples indicate that n can be as small as 2.

When $I > 1$ holds, there is no hope for a SPNE in which all firms choose the same fixed-price-per-share mechanism, because that would entail the same consumption by all consumers. Barro and Romer (1987) assume that firms are perfectly competitive, and argue that, with heterogenous consumers, there will be an equilibrium in which firms specialize in serving one consumer type. Translated into the current notation, a firm serving type i will choose a share price (they call it a lift-ticket price) equal to $p^c d_i(p^c)$. The number of firms serving type i, denoted by n_i, would be determined by the condition that per capita consumption is $d_i(p^c)$, so $n_i = r_i n d_i(p^c)$ would hold. Is there a similar result for Γ with imperfect competition? The answer, generically, is no, due to integer constraints.[14] To prevent a local deviation targeting the same consumer type, the consumption of type i consumers will generally have to differ from $d_i(p^c)$. As a result, marginal rates of substitution will not be equated across consumer types, creating the incentive for a firm to adopt a more complicated mechanism that attracts multiple consumer types.[15]

5.2. Full Surplus Extraction with $I = 1$

When there is only one consumer type, $I = 1$, there are other equilibria to Γ in which firms receive even higher profits than the equilibria of Propositions 3 and 4, based on mechanisms that become very generous to consumers when fewer than expected arrive at the firm. Proposition 5 shows that full surplus extraction is possible in equilibrium.

Proposition 5: *Assume that there is only one consumer type, $I = 1$, and that $u_1(0)$ is finite. Then for a sufficiently large parameter (of the mechanism), A,*

[14] Barro and Romer (1987) ignore integer constraints, presumably because they imagine that n is large.

[15] Details were in an earlier version of the paper, and are available upon request.

there is a SPNE of Γ in which, for $f = 1, ..., n$, firm f chooses the following mechanism:

$$
\begin{aligned}
x_1^f(\rho^f) &= \frac{1}{\rho_1^f} \\
P_1^f(\rho^f) &= u_1(\frac{1}{\rho_1^f}) - u_1(0) \quad \text{if } \rho_1^f \geq r_1 \\
P_1^f(\rho^f) &= A(\rho_1^f - r_1) + \ u_1(\frac{1}{r_1}) - u_1(0) \ \text{ if } \ \rho_1^f < r_1.
\end{aligned}
$$

On the equilibrium path, we have $\beta_1^f = \frac{1}{n}$ for all f, and firms extract full surplus.

The intuition for full surplus extraction in Proposition 5 is that firms drastically reduce the payment consumers make when they receive fewer customers than "expected," even allowing the payment to be negative. The mechanism is continuous, but no firm will want to steal any consumers at all from the other firms. However, if other firms receive at least as many customers as expected, they leave consumers with zero surplus. Therefore, the best response is to split the market but extract all surplus. With more than one type of consumer, a consumer could pretend to be a lower-demand type, so incentive compatibility precludes full surplus extraction.

6. Examples

In this section, examples with $I = 1$ and $I = 2$ are computed, to illustrate the results and to demonstrate that the number of firms need not be very large. Consider the class of examples in which utility is of the form, $u_i(x_i) = -\frac{a_i}{x_i}$, which implies $d_i(p) = (\frac{a_i}{p})^{1/2}$ and $d_i'(p) = -\frac{1}{2}(a_i)^{1/2}p^{-3/2}$.

6.1. Example 1: $I = 1, a_1 = 1, r_1 = 1$.

Since there is only one type, we omit the subscript denoting type. The competitive equilibrium price satisfies the market clearing condition,

$$(\frac{1}{p})^{1/2} = 1,$$

so $p^c = 1$. To see that the fixed-price-per-unit mechanism equivalent to setting $p^f = 1$ is part of an equilibrium of Γ, consider the optimization problem of firm f

within the space of all mechanisms that fully allocate capacity (i.e., $x^f = 1/(n\beta^f)$, given that others are setting the competitive price. Clearly the firm will not set a price below 1, so we have

$$\begin{aligned} & \max_{\beta^f \leq \frac{1}{n}, P^f} n\beta^f P^f \\ & \text{subject to} \\ -n\beta^f - P^f \quad = & \quad -(\frac{n(1-\beta^f)}{n-1}) - \frac{n-1}{n(1-\beta^f)}. \end{aligned}$$

The constraint characterizes the equilibrium of all subgames following a unilateral deviation. Substituting the constraint into the objective, we have the unconstrained problem to maximize

$$\beta^f \left[(\frac{n(1-\beta^f)}{n-1}) + \frac{n-1}{n(1-\beta^f)} - n\beta^f \right].$$

Setting the derivative with respect to β^f equal to zero and solving the cubic equation (messy details omitted), there are three roots, but only the root $\beta^f = \frac{1}{n}$ lies between 0 and $\frac{1}{n}$. The second-order conditions are satisfied for any $n > 1$, because the second derivative is increasing in β^f and takes the value

$$-\frac{2(n-2)n^2}{(n-1)^2}$$

at $\beta^f = \frac{1}{n}$. Therefore, the best response for firm f is to set $\beta^f = \frac{1}{n}$, yielding the same profits as it receives by setting the constant price, $p^f = 1$.

Now consider the equilibrium of Γ in which all firms choose the fixed-price-per-share mechanism,

$$P^* = \frac{n}{n-1}.$$

The best response for firm f is the solution to

$$\begin{aligned} & \max_{\beta^f \leq 1, P^f} n\beta^f P^f \\ & \text{subject to} \\ -n\beta^f - P^f \quad = & \quad -(\frac{n(1-\beta^f)}{n-1}) - \frac{n}{n-1}. \end{aligned}$$

Substituting the constraint into the objective, we have the unconstrained problem to maximize

$$\beta^f \left[(\frac{n(1-\beta^f)}{n-1}) + \frac{n}{n-1} - n\beta^f \right].$$

Taking the derivative with respect to β^f yields the expression,

$$-\frac{2n(\beta^f n - 1)}{n-1},$$

so the appropriate solution to the first-order condition is $\beta^f = \frac{1}{n}$, yielding the same profits as it receives by choosing the fixed-price-per-share mechanism, $P^f = \frac{n}{n-1}$. The second-order conditions are satisfied, because the second derivative of the objective is negative for $n > 1$.

Finally, consider the equilibrium of Γ in which for all f, firm f chooses the $\varepsilon - MAWEF$ mechanism given by

$$E^* = -\frac{d_i(p^c)}{(n-1)\sum_{h=1}^{I} r_h \frac{\partial d_h(p^c)}{\partial p}} = \frac{2}{(n-1)}.$$

It follows that, for a non-deviating firm f', we have $\beta^{f'} = \frac{1-\beta^f}{n-1}$, so the market clearing price at firm f' is $[\frac{n(1-\beta^f)}{n-1}]^2$. Therefore, consumption offered by firm f' is $\frac{n-1}{n(1-\beta^f)}$, so utility offered by firm f' is $-(\frac{2n(1-\beta^f)}{n-1}) - \frac{2}{n-1}$. The best response for firm f is the solution to

$$\max_{\beta^f \leq 1, P^f} n\beta^f P^f$$

subject to

$$-n\beta^f - P^f = -(\frac{2n(1-\beta^f)}{n-1}) - \frac{2}{n-1}.$$

Substituting the constraint into the objective, we have the unconstrained problem to maximize

$$\beta^f \left[\frac{2n(1-\beta^f)}{n-1} + \frac{2}{n-1} - n\beta^f \right].$$

Taking the derivative with respect to β^f yields the expression,

$$-\frac{2(\beta^f n - 1)(n+1)}{n-1},$$

so the appropriate solution to the first-order condition is $\beta^f = \frac{1}{n}$, yielding the same profits as it receives by choosing the $\varepsilon - MAWEF$ mechanism. The second-order conditions are satisfied, because the second derivative of the objective is negative for $n > 1$.

It is interesting to compare outcomes in these three equilibria. In all three equilibria, per capita consumption of the good is 1 unit, but the total payment by consumers and firm profits differ. With fixed-price mechanisms, the price per unit is 1, and profits are at the competitive level, equal to 1. With fixed-price-per-share mechanisms, the effective price per unit and profit are equal to $\frac{n}{n-1}$. With $\varepsilon - MAWEF$ mechanisms, the effective price per unit and profit are equal to $1 + \frac{2}{n-1}$. Thus, the $\varepsilon - MAWEF$ mechanisms yield the highest profit, followed by the fixed-price-per-share mechanisms, with fixed-price mechanisms exhibiting no market power and yielding the lowest profit.

6.2. Example 2: $I = 2, a_1 = 4, a_2 = 1, r_1 = r_2 = \frac{1}{3}$.

The competitive equilibrium price satisfies the market clearing condition,

$$\frac{1}{3}(\frac{4}{p})^{1/2} + \frac{1}{3}(\frac{1}{p})^{1/2} = 1,$$

so $p^c = 1$. To see that the fixed-price-per-unit mechanism equivalent to setting $p^f = 1$ is part of an equilibrium of Γ, see the proof of Proposition 3, which shows that the best deviation for a firm is to attract only type 1 consumers. We can write (8.33), the derivative of profits, as

$$\begin{aligned} & -\frac{4n\beta_1^f}{3} + \frac{4n(1-\beta_1^f)}{3(\frac{2n}{3}-1)} + \frac{3(\frac{2n}{3}-1)}{n(1-\beta_1^f)} \\ & +\beta_1^f \left[-\frac{4n}{3} - \frac{4n}{3(\frac{2n}{3}-1)} + \frac{3(\frac{2n}{3}-1)}{n(1-\beta_1^f)^2} \right]. \end{aligned} \tag{6.1}$$

Setting expression (6.1) equal to zero and solving the cubic equation yields only one sensible root, $\beta_1^f = \frac{3}{2n}$, which yields firm f the same profit as it earns by setting the fixed price equal to the competitive price, 1. Second order conditions can be shown to be satisfied whenever $n > 2$ holds. It can also be shown that when $n = 2$ holds, profits are maximized at $\beta_1^f = \frac{3}{2n}$, even though profits are not globally concave in β_1^f.

Now consider the equilibrium of Γ in which for all f, firm f chooses the $\varepsilon - MAWEF$ mechanism given by

$$E_1^* = \frac{4}{(n-1)} \quad \text{and} \quad E_2^* = \frac{2}{(n-1)}.$$

The best response for firm f is the solution to

$$\begin{aligned} & \max_{\beta^f, P_1^f, P_2^f, x_1^f, x_2^f} \beta_1^f P_1^f + \beta_2^f P_2^f \\ & \text{subject to} \\ \frac{1}{3} n \beta_1^f x_1^f + \frac{1}{3} n \beta_2^f x_2^f &= 1, \\ \frac{4}{x_1^f} + P_1^f &= 4\sqrt{\widetilde{p}} + \frac{4}{(n-1)}, \\ \frac{1}{x_2^f} + P_2^f &= 2\sqrt{\widetilde{p}} + \frac{2}{(n-1)}, \\ \beta^f &\geq 0, \end{aligned}$$

where $\widetilde{p}$ satisfies market clearing condition,

$$1 = \frac{\frac{1}{3}n(1-\beta_1^f)}{n-1} \frac{2}{\sqrt{\widetilde{p}}} + \frac{\frac{1}{3}n(1-\beta_2^f)}{n-1} \frac{1}{\sqrt{\widetilde{p}}},$$

solved to be

$$\widetilde{p} = \left[\frac{(3 - 2\beta_1^f - \beta_2^f)n}{3(n-1)} \right]^2.$$

Imposing the necessary condition that marginal utilities are equated yields $x_1^f = 2x_2^f \equiv x$. Then, defining $B^f \equiv 2\beta_1^f + \beta_2^f$, we can rewrite the constraints as

$$\begin{aligned} \frac{6}{nB^f} &= x, \\ P_1^f &= 4 \left[\frac{(3 - B^f)n}{3(n-1)} \right] + \frac{4}{(n-1)} - \frac{4}{x}, \\ P_2^f &= 2 \left[\frac{(3 - B^f)n}{3(n-1)} \right] + \frac{2}{(n-1)} - \frac{2}{x}. \end{aligned}$$

From the two indifference constraints, we see that $P_1^f = 2P_2^f \equiv P^f$ must hold, so we can simplify the optimization problem to the unconstrained problem,

$$\max_{B^f} B^f \left[\left[\frac{(3-B^f)n}{3(n-1)} \right] + \frac{1}{(n-1)} - \frac{nB^f}{6} \right].$$

Setting the derivative of this objective equal to zero and solving, we have $B^f = \frac{3}{n}$. The second derivative of this objective is

$$-\frac{n(n+1)}{3(n-1)},$$

so the second order conditions are satisfied. Firm f has no incentive to deviate, since following the equilibrium satisfies the resource and indifference constraints and achieves $\beta_1^f = \beta_2^f = \frac{1}{n}$ and $B^f = \frac{3}{n}$.

7. Concluding Remarks

This paper develops a framework for studying competing mechanisms in an economic environment where firms sell and consumers demand multiple units. There is a literature in which firms selling a single unit compete by choosing auctions with a reserve price. In the present setting, where consumers are negligible and there is no aggregate uncertainty, we find that these reserve-price mechanisms are not used in equilibrium. Under certain assumptions, equilibrium exists in which firms choose auctions with type-specific entry fees but no reserve price. This is the most compelling equilibrium, because if we are to augment the model with aggregate demand uncertainty, an $\varepsilon - MAWEF$ mechanism is flexible enough to allocate the good efficiently across consumers, no matter how many consumers show up. I conjecture that the competiting mechanisms game with demand uncertainty will have an equilibrium in which all firms choose an $\varepsilon - MAWEF$ mechanism for suitably chosen entry fees.

We also show that Γ has an equilibrium in which all firms choose a fixed-price-per-unit mechanism with the price equal to the competitive equilibrium price. While this result is a contribution to the literature on price competition with capacity constraints, I can confidently conjecture that fixed-price-per-unit mechanisms will not be consistent with equilibrium of the competiting mechanisms game when there is aggregate demand uncertainty. The reason is that, if the price at firm f clears the market in one demand state, then there will be excess demand

and rationing in higher demand states, and wasteful excess supply in lower demand states. Holding fixed the probabilities with which consumers choose firm f, the firm could deviate to a mechanism that maintains the expected utility of each consumer type at firm f, while efficiently allocating the good and generating higher total surplus. Such a deviation would have to be profitable. Introducing demand uncertainty is a subject for future work.

Much of the tractability of this model stems from the fact that individual consumers are unable to affect anyone's allocation other than their own. Besides enhancing tractability, the assumption of negligible consumers may be a desirable description of certain markets. Although the model is quite tractable, some of the proofs are difficult, owing to the requirement that mechanisms be continuous and incentive compatible off the equilibrium path. For example, if all firms chose auctions with entry fees as specified in Proposition 2, but where the entry fee did not depend on the reports of other consumers, this profile would be a Nash equilibrium of Γ. However, this mechanism might not be incentive compatible if the measure of arriving consumers is very small.

Throughout the paper, we assume that consumers must commit to a single mechanism. This rules out more complicated environments in which a consumer could contact a firm, attempt to arrange a transaction, and contact a different firm if a favorable transaction could not be completed. See Peters and Severinov (2006) and Peters (2015) for important steps in this direction. It might be both useful and tractable to combine the possibility of consumers contacting multiple firms with the framework of a continuum of consumers who cannot individually affect the market.

8. Appendix: Proofs

Proof of Lemma 1. Fix a profile of mechanisms, m. For $f = 1, ..., n$, let $U_i^f(\beta)$ denote the utility received by type i consumers who choose firm f when consumers mix across firms according to β. Note that $U_i^f(\beta)$ depends on β only through β^f. Given that all mechanisms are incentive compatible for any β, it follows that $U_i^f(\beta)$ is continuous in β. Consider the mapping $g : \triangle^{nI} \to \triangle^{nI}$, where

$$g_i^f(\beta) = \frac{\beta_i^f + \max[0, U_i^f(\beta) - \sum_{j=1}^n \beta_i^j U_i^j(\beta)]}{1 + \sum_{j'=1}^n \max[0, U_i^{j'}(\beta) - \sum_{j=1}^n \beta_i^j U_i^j(\beta)]}.$$

Because each $U_i^f(\beta)$ is continuous, it follows that g is a continuous function. The simplex is a compact, convex set. Applying Brouwer's fixed point theorem, we have at the fixed point,

$$\begin{aligned} & \beta_i^f \sum_{j'=1}^{n} \max[0, U_i^{j'}(\beta) - \sum_{j=1}^{n} \beta_i^j U_i^j(\beta)] \\ = \quad & \max[0, U_i^f(\beta) - \sum_{j=1}^{n} \beta_i^j U_i^j(\beta)] \text{ for all } i, f. \end{aligned} \tag{8.1}$$

Consider the possible cases. From (8.1), if

$$U_i^f(\beta) < \sum_{j=1}^{n} \beta_i^j U_i^j(\beta) \tag{8.2}$$

holds, then the right side of (8.1) is zero. Then either $\beta_i^f = 0$ holds, or we have $U_i^{j'}(\beta) \leq \sum_{j=1}^{n} \beta_i^j U_i^j(\beta)$ for all j'. But the latter cannot occur, because it would imply $U_i^{j'}(\beta) = U_i^j(\beta)$ for all j, j', in contradiction to (8.2). Thus, for all f, i, we have either $\beta_i^f = 0$ or

$$U_i^f(\beta) \geq \sum_{j:\beta_i^j > 0} \beta_i^j U_i^j(\beta). \tag{8.3}$$

Applying (8.3) to all j' such that $\beta_i^{j'} > 0$, it follows that $U_i^f(\beta) = U_i^{j'}(\beta)$ for all i and for all f, j' such that $\beta_i^f > 0$ and $\beta_i^{j'} > 0$. Therefore, all consumer choices are best responses, and the fixed point is a Nash equilibrium of the subgame. ■

Proof of Proposition 1. Suppose that there is a SPNE of Γ in which all firms choose reserve price mechanisms, possibly in mixed strategies. Let $\overline{R}$ denote the supremum of reserve prices in the support of the equilibrium profile of mechanisms.

Case 1. We have $\overline{R} > p^c$.

First, it cannot be the case that two or more firms have a mass point at $\overline{R}$. If so, there would be a positive probability that all of the firms with a mass point at $\overline{R}$ choose that reserve price. Total sales by firms setting the reserve price $\overline{R}$ must be positive in this situation, or else $\overline{R}$ would always yield zero revenue. Let firm f be a firm such that, of all the firms setting the reserve price $\overline{R}$ in this situation, firm f is selling the least of its capacity. Firm f must be selling strictly less than all of its capacity, since $\overline{R} > p^c$ holds. However, firm f could slightly reduce its reserve price, selling strictly more of its capacity, thereby increasing its profits.

Second, suppose that $\overline{R}$ is in the support of reserve prices chosen by firm f.[16] Since no firm, other than possibly firm f, can have a mass point at $\overline{R}$, when firm f chooses the reserve price $\overline{R}$, it knows that it is the only firm with a reserve price that high. If firm f receives zero revenue in any consumer equilibrium, this contradicts the fact that $\overline{R}$ is a best response to the mixed strategies of the other firms. Therefore, in the ensuing consumer equilibrium, the auction price at all firms is $\overline{R}$ and all firms other than firm f sell all their capacity. It follows that, no matter what reserve prices the other firms choose, the consumer equilibrium β satisfies the market clearing condition

$$\sum_{i=1}^{I} r_i n(1 - \beta_i^f) d_i(\overline{R}) = n - 1, \tag{8.4}$$

and the profits of firm f are

$$\sum_{i=1}^{I} r_i n \beta_i^f d_i(\overline{R}) \overline{R}.$$

Substituting (8.4) into the profit expression yields profits of

$$\left[\sum_{i=1}^{I} r_i n d_i(\overline{R}) - (n-1) \right] \overline{R}. \tag{8.5}$$

Furthermore, any consumer mixed strategy profile β satisfying (8.4) is an equilibrium of the consumer subgame, yielding the profits given in (8.5). Let β^* be a consumer equilibrium satisfying $\beta_1^{*f} > 0$.

We will construct a profitable deviation for firm f. Intuitively, the new mechanism is constructed so that utility is unaffected if consumers continue to mix according to β^*, so this remains an equilibrium of the consumer subgame. If reports are consistent with β^*, then some of the capacity that was not utilized under the mechanism $\overline{R}$ is allocated to type 1 consumers in exchange for an additional payment. Then the mechanism is extended to other reports to maintain continuity, incentive compatibility, and profitability.

[16] For the third situation, where $\overline{R}$ is not in the support of reserve prices chosen by any firm, then for R^f sufficiently close to $\overline{R}$, firm f knows that its reserve price is highest with probability arbitrarily close to one, and the argument mirrors the second situation of $\overline{R}$ in the support.

Here is the deviation mechanism for Case 1, denoted by m^f.

If there is no excess demand at the price $\overline{R}$ based on the reports, so $\sum_{i=1}^{I} \rho_i^f d_i(\overline{R}) \leq 1$ holds, then we have

$$\begin{aligned}
x_i^f(\rho^f) &= d_i(\overline{R}) \quad \text{for } i > 1 \\
P_i^f(\rho^f) &= \overline{R} d_i(\overline{R}) \quad \text{for } i > 1 \\
x_1^f(\rho^f) &= d_1(\overline{R}) + \frac{1 - \sum_{i=1}^{I} \rho_i^f d_i(\overline{R})}{A + \rho_1^f} \\
P_1^f(\rho^f) &= \overline{R} d_1(\overline{R}) + u_1(x_1^f(\rho^f)) - u_1(d_1(\overline{R})) \\
&\quad + \varepsilon[1 - \sum_{i=1}^{I} \rho_i^f d_i(\overline{R})] \max[0, r_1 n \beta_1^{*f} - \rho_1^f].
\end{aligned} \tag{8.6}$$

The positive parameters A and ε are chosen as part of the mechanism. The purpose of A is to guarantee that type 1 consumption is well defined even if $\rho_1^f = 0$ holds. Larger values of A mean that less of the excess supply is allocated to type 1 consumers. The term involving ε is used below to ensure that there cannot be an equilibrium to the consumer subgame in which too few consumers report type 1.

If there is excess demand at the price $\overline{R}$ based on the reports, so $\sum_{i=1}^{I} \rho_i^f d_i(\overline{R}) > 1$ holds, then we implement a uniform price auction. Defining p^f as the solution to$\sum_{i=1}^{I} \rho_i^f d_i(p^f) = 1$, we have

$$\begin{aligned}
x_i^f(\rho^f) &= d_i(p^f) \\
P_i^f(\rho^f) &= p^f d_i(p^f).
\end{aligned}$$

This completes the definition of m^f.

m^f is feasible for large A and small ε, satisfying continuity and IC:

It is immediate that m^f is continuous in ρ^f. Incentive compatibility holds for sufficiently large A and sufficiently small ε, which follows from the fact that, when there is excess supply at the price $\overline{R}$, a type 1 consumer receives utility close to $u_1(d_1(\overline{R})) - \overline{R} d_1(\overline{R})$, which is higher than utility from reporting any other type. Also, if a type $i > 1$ were to report type 1, his net increase in utility is given by

$$\begin{aligned}
& u_i(x_1^f(\rho^f)) - P_1^f(\rho^f) - u_i(d_i(\overline{R})) + \overline{R} d_i(\overline{R}) \\
= \; & u_i(x_1^f(\rho^f)) - \overline{R} d_1(\overline{R}) - u_1(x_1^f(\rho^f)) + u_1(d_1(\overline{R})) \\
& - u_i(d_i(\overline{R})) + \overline{R} d_i(\overline{R}) - \varepsilon \max[0, r_1 n \beta_1^{*f} - \rho_1^f].
\end{aligned} \tag{8.7}$$

In the limit, as $A \to \infty$ and $\varepsilon \to 0$, (8.7) approaches

$$\left[u_i(d_1(\overline{R})) - \overline{R}d_1(\overline{R})\right] - \left[u_i(d_i(\overline{R})) - \overline{R}d_i(\overline{R})\right] < 0.$$

The mechanism m^f yields higher profits than the mechanism $\overline{R}$:
First, if the consumer equilibrium following the deviation to m^f satisfies

$$\sum_{i=1}^{I} r_i n \beta_i^{*f} d_i(\overline{R}) \leq \sum_{i=1}^{I} \rho_i^f d_i(\overline{R}) \leq 1,$$

then we must have $\rho_1^f \geq r_1 n \beta_1^{*f} > 0$, because otherwise the auction price at firms other than f is at most $\overline{R}$ and type 1 consumers would be receiving strictly lower utility from firm f, due to the ε term in (8.6). Therefore, the profits are greater than the profits of the mechanism $\overline{R}$ by at least

$$\left[u_1(x_1^f(\rho^f)) - u_1(d_1(\overline{R}))\right] \rho_1^f,$$

which is strictly positive.

Second, if the consumer equilibrium following the deviation to m^f satisfies

$$\sum_{i=1}^{I} \rho_i^f d_i(\overline{R}) < \sum_{i=1}^{I} r_i n \beta_i^{*f} d_i(\overline{R}),$$

then the auction price at firms other than f is strictly greater than $\overline{R}$, so consumers of type $i > 1$ are strictly better off at firm f, contradicting consumer equilibrium.[17]

Third, if the consumer equilibrium following the deviation to m^f satisfies

$$\sum_{i=1}^{I} \rho_i^f d_i(\overline{R}) > 1,$$

then firm f sells all of its capacity at a price greater than $\overline{R}$, so again the mechanism m^f yields higher profits than the mechanism $\overline{R}$.

Case 2. We have $\overline{R} \leq p^c$.

[17] If there is only one type, $I = 1$, then set $\varepsilon = 0$ and the argument goes through since incentive compatibility is not an issue.

In this case, consumers distribute themselves across firms so that none of the reserve prices is binding, and all of the auction prices are equal to p^c. We now show that there is a profitable deviation to a mechanism that, roughly speaking, raises the auction price at the other firms to some price $\widetilde{p}$, allocates $d_i(\widetilde{p})$ to its type i customers at price $\widetilde{p}$, and allocates its remaining capacity as extra consumption to type 1 customers. Then the mechanism is extended continuously to satisfy feasibility and incentive compatibility off the equilibrium path.

Here is the deviation mechanism for Case 2, denoted by $\widetilde{m}^f$.

Below, we treat A, ε, and $\widetilde{p}$ as parameters of the mechanism. First, let $C(\rho^f)$ be defined by

$$C(\rho^f) = \sum_{i=1}^{I} \rho_i^f d_i(\widetilde{p}) - \left[\sum_{i=1}^{I} r_i n d_i(\widetilde{p}) - (n-1) \right].$$

The economic interpretation of $C(\rho^f)$ is the amount by which the actual demand at firm f at price $\widetilde{p}$, based on reported types, exceeds the anticipated demand, based on the residual market demand faced by firm f. We will suppress the dependence on ρ^f and refer to $C(\rho^f)$ as C.

For $C \leq 0$, $\widetilde{m}^f$ is given by[18]

$$\begin{aligned}
x_i^f(\rho^f) &= d_i(\widetilde{p}) \quad \text{for } i > 1 \\
P_i^f(\rho^f) &= \widetilde{p} d_i(\widetilde{p}) \quad \text{for } i > 1 \\
x_1^f(\rho^f) &= d_1(\widetilde{p}) + \frac{n - \sum_{i=1}^{I} r_i n d_i(\widetilde{p})}{A + \rho_1^f} \\
P_1^f(\rho^f) &= \widetilde{p} d_1(\widetilde{p}) + u_1(x_1^f(\rho^f)) - u_1(d_1(\widetilde{p})) - \varepsilon \max[0, r_1 - \rho_1^f].
\end{aligned}$$

For $C > 0$, $\widetilde{m}^f$ is given by

[18]If we have only one type, $I = 1$, then $\widetilde{m}^f$ must be modified, but a simpler proof along the same lines is available, which eliminates the term, $\varepsilon \max[0, r_1 - \rho_1^f]$. We omit the details to save space.

$$
\begin{aligned}
x_i^f(\rho^f) &= d_i(\widehat{p}) \quad \text{for } i > 1 \\
P_i^f(\rho^f) &= \widehat{p} d_i(\widehat{p}) \quad \text{for } i > 1 \\
x_1^f(\rho^f) &= d_1(\widehat{p}) + \frac{n - \sum_{i=1}^{I} r_i n d_i(\widetilde{p})}{A + \rho_1^f} \\
P_1^f(\rho^f) &= \widehat{p} d_1(\widehat{p}) + u_1(x_1^f(\rho^f)) - u_1(d_1(\widehat{p})) - \varepsilon \max[0, r_1 - \rho_1^f].
\end{aligned}
$$

where $\widehat{p}$ is the unique solution to

$$
\sum_{i=1}^{I} \rho_i^f d_i(\widehat{p}) = \sum_{i=1}^{I} r_i n d_i(\widetilde{p}) - (n-1).
$$

Note that $C > 0$ implies $\widehat{p} > \widetilde{p}$.

$\widetilde{m}^f$ is feasible for small ε, satisfying continuity and IC:

Because we have $A > 0$, and as C approaches 0 from above $\widehat{p}$ approaches $\widetilde{p}$, it is immediate that $\widetilde{m}^f$ is continuous. For small enough ε, the consumption of each type is arbitrarily close to the utility maximizing consumption at price $\widetilde{p}$ or $\widehat{p}$, depending on C. Thus, there is no incentive for a consumer to report a different type.

The mechanism $\widetilde{m}^f$ yields higher profits than the mechanism $\overline{R}$:

We claim that for $\widetilde{p}$ close enough to p^c, there is a consumer equilibrium, β, satisfying $\beta_1^f = \frac{1}{n}$, in which the auction price at other firms will be $\widetilde{p}$. The price at other firms is $\widetilde{p}$ if and only if we have

$$
\begin{aligned}
\sum_{i=1}^{I} r_i n (1 - \beta_i^f) d_i(\widetilde{p}) &= (n-1), \quad \text{or} \\
\sum_{i=1}^{I} r_i n \beta_i^f d_i(\widetilde{p}) &= \sum_{i=1}^{I} r_i n d_i(\widetilde{p}) - (n-1). \qquad (8.8)
\end{aligned}
$$

For $\widetilde{p}$ close enough to p^c, the right side of (8.8) is positive. Then any β that satisfies (8.8) gives rise to $C = 0$ under truthful reporting. The mechanism $\widetilde{m}^f$ then delivers the same utility to each type as they would receive at other firms if $\beta_1^f = \frac{1}{n}$ and therefore $r_1 = \rho_1^f$ holds, ensuring that β is an equilibrium of the consumer subgame.[19]

The profits for firm f are then given by

$$\sum_{i=1}^{I} r_i n \beta_i^f d_i(\widetilde{p})\widetilde{p} + r_1[u_1(d_1(\widetilde{p}) + \frac{n - \sum_{i=1}^{I} r_i n d_i(\widetilde{p})}{A + r_1}) - u_1(d_1(\widetilde{p}))], \tag{8.9}$$

which, from (8.8), can be written as

$$\begin{aligned} \pi^f(\widetilde{p}) &= \widetilde{p}\left[\sum_{i=1}^{I} r_i n d_i(\widetilde{p}) - (n-1)\right] \\ &\quad + r_1[u_1(d_1(\widetilde{p}) + \frac{n - \sum_{i=1}^{I} r_i n d_i(\widetilde{p})}{A + r_1}) - u_1(d_1(\widetilde{p}))]. \end{aligned} \tag{8.10}$$

Differentiating (8.10) yields

$$\begin{aligned} (\pi^f)'(\widetilde{p}) &= \sum_{i=1}^{I} r_i n d_i(\widetilde{p}) - (n-1) + \widetilde{p}\sum_{i=1}^{I} r_i n d_i'(\widetilde{p}) \\ &\quad + r_1 u_1'(x_1^f(\rho^f))\left[d_1'(\widetilde{p}) - \frac{\sum_{i=1}^{I} r_i n d_i'(\widetilde{p})}{A + r_1}\right] \\ &\quad - r_1 u_1'(d_1(\widetilde{p}))d_1'(\widetilde{p}). \end{aligned} \tag{8.11}$$

Evaluating (8.11) at $\widetilde{p} = p^c$, which implies $\sum_{i=1}^{I} r_i n d_i(\widetilde{p}) = n$ and $u_1'(x_1^f(\rho^f)) = u_1'(d_1(\widetilde{p})) = p^c$, yields

$$(\pi^f)'(p^c) = 1 + \left(1 - \frac{r_1}{A + r_1}\right) p^c \sum_{i=1}^{I} r_i n d_i'(p^c).$$

For A sufficiently close to zero, $(\pi^f)'(p^c)$ is positive, so for $\widetilde{p}$ slightly greater than p^c.

[19] If we have $I = 1$, then there is a unique consumer equilibrium, but β_1^f will be slightly less than $\frac{1}{n}$ for $\widetilde{p}$ close enough to p^c. With $I > 1$, there will be a consumer equilibrium with $\beta_1^f = \frac{1}{n}$ for $\widetilde{p}$ close enough to p^c, because types $i > 1$ can go to other firms.

What if a different consumer equilibrium is selected in response to this deviation? If we have $C = 0$ and $\rho_1^f > r_1$, then profits are given by

$$\sum_{i=1}^{I} r_i n \beta_i^f d_i(\widetilde{p})\widetilde{p} + \rho_1^f[u_1(d_1(\widetilde{p}) + \frac{n - \sum_{i=1}^{I} r_i n d_i(\widetilde{p})}{A + r_1}) - u_1(d_1(\widetilde{p}))],$$

which is greater than the expression in (8.9), so once again the deviation is profitable.

If we have $C = 0$ and $\rho_1^f < r_1$, then type 1 consumers receive higher utility at firm f than at the other firms, which is inconsistent with equilibrium of the consumer subgame. If we have $C < 0$, then the auction price at the other firms will be greater than $\widetilde{p}$, so all consumers receive higher utility at firm f than at the other firms, which is inconsistent with equilibrium of the consumer subgame.

Now consider the possibility of $C > 0$. If somehow this is consistent with equilibrium of the consumer subgame, this would imply $\widehat{p} > \widetilde{p}$, and also that the auction price at other firms is less than $\widetilde{p}$. However, a consumer of type $i > 1$ is worse off at firm f, which is inconsistent with equilibrium of the consumer subgame. ■

Proof of Proposition 2. First, notice that the $\varepsilon - MAWEF$ mechanism in the statement of Proposition 2 is within the class of available mechanisms. Continuity follows from the facts that the market clearing price is continuous in β^f and competitive equilibrium consumption is continuous in price. When the auction price is outside the 2ε neighborhood of p^c, entry fees are zero, and incentive compatibility follows immediately. When the auction price is inside the 2ε neighborhood of p^c, incentive compatibility follows from the fact that when n is sufficiently large, entry fees are close to zero, and any difference in entry fees across types is swamped by the loss of utility associated with misreporting and receiving consumption that does not maximize utility given the equilibrium price.

For any profile of mechanisms, the ensuing consumer subgame has a type-symmetric Nash equilibrium, which follows from Lemma 1. Select an arbitrary type-symmetric Nash equilibrium following a deviation by two or more firms.

On the equilibrium path, since all firms are choosing the same mechanism, it is clear that $\beta_i^f = \frac{1}{n}$ for all i, f forms an equilibrium of the subgame. The auction price at each firm is therefore p^c, so the profits of each firm are given by

$$\sum_{i=1}^{I} r_i a_i d(p^c) p^c - \sum_{i=1}^{I} \frac{r_i a_i d(p^c)^2}{(n-1)d'(p^c)}.$$

Because the competitive equilibrium price satisfies $\sum_{i=1}^{I} r_i a_i d(p^c) = 1$, the profit expression simplifies to

$$p^c - \frac{d(p^c)}{(n-1)d'(p^c)}.$$

Now consider a potential deviation by a single firm, f. An upper bound to the profits available is the solution to the following optimization problem, in which firm f chooses its mechanism and its arrival vector, β^f, subject to its capacity constraint and the constraint that consumers are indifferent between firm f and the other firms[20]:

$$\begin{aligned}
&\max \sum_{i=1}^{I} r_i n \beta_i^f P_i^f \\
&\text{subject to}
\end{aligned}$$

$$\begin{aligned}
\sum_{i=1}^{I} r_i n \beta_i^f x_i^f &= 1 \qquad (8.12)\\
u_i(x_i^f) - P_i^f &= u_i(a_i d(\widetilde{p}(\beta^f))) - \widetilde{p}(\beta^f) a_i d(\widetilde{p}(\beta^f)) - E_i^*, \quad \text{for } i = 1, ..., I \\
1 &\geq \beta_i^f \geq 0 \;\; \text{for } i = 1, ..., I.
\end{aligned}$$

In (8.12), $\widetilde{p}(\beta^f)$ is defined to be the auction price at other firms when the arrival vector at firm f is β^f. Then, suppressing the dependence on β^f, $\widetilde{p}$ solves

$$1 = \sum_{h=1}^{I} \frac{r_h n (1 - \beta_h^f) a_h d(\widetilde{p})}{n-1}. \qquad (8.13)$$

Letting λ denote the Lagrange multiplier on the capacity constraint and λ_i denote the multiplier on the indifference constraint for type i, some of the *necessary* first-order conditions are, for $i = 1, ..., I$, the equality constraints in (8.12) and

$$\begin{aligned}
\lambda_i u_i'(x_i^f) &= \lambda r_i n \beta_i^f \qquad (8.14)\\
r_i n \beta_i^f &= \lambda_i \qquad (8.15)
\end{aligned}$$

[20] It is without loss of generality to impose the indifference constraint for each type in (**??**), because if some type i strictly prefers to visit other firms and $\beta_i^f = 0$ holds for some type at the solution, there is another solution in which type i consumers are indifferent and $\beta_i^f = 0$ holds.

In particular, we have $u_i'(x_i^f) = \lambda \equiv p^f$ for all i such that $1 > \beta_i^f > 0$. Equivalently, we have $x_i^f = a_i d(p^f)$ for all i, which implies

$$\sum_{i=1}^{I} r_i n \beta_i^f a_i = \frac{1}{d(p^f)}. \tag{8.16}$$

Based on the above necessary conditions, it follows that for any solution to (8.12), β^f must solve (suppressing the dependence of p^f and $\widetilde{p}$ on β^f through (8.13) and (8.16))

$$\max_{0 \le \beta_i^f \le 1, i=1,...,I} \sum_{i=1}^{I} r_i n \beta_i^f [u_i(a_i d(p^f)) - u_i(a_i d(\widetilde{p})) + \widetilde{p} a_i d(\widetilde{p}) + E_i^*]. \tag{8.17}$$

Because a continuous function on a compact set has a maximum, we know that there is a solution to (8.17). An interior solution must satisfy the first order conditions, simplified by the condition that $u_i'(a_i d(p)) = p$ holds and the compact notation $P_i^f = u_i(a_i d(p^f)) - u_i(a_i d(\widetilde{p})) + \widetilde{p} a_i d(\widetilde{p}) + E_i^*$, given by

$$r_i n P_i^f + (\sum_{h=1}^{I} r_h n \beta_h^f a_h) d(\widetilde{p}) \frac{\partial \widetilde{p}}{\partial \beta_i^f} + (\sum_{h=1}^{I} r_h n \beta_h^f a_h) p^f d'(p^f) \frac{\partial p^f}{\partial \beta_i^f} = 0 \tag{8.18}$$

for $i = 1, ..., I$.

The first order conditions (8.18) can be simplified further. Using (8.16), we have

$$r_i n P_i^f + \frac{d(\widetilde{p})}{d(p^f)} \frac{\partial \widetilde{p}}{\partial \beta_i^f} + \frac{p^f d'(p^f)}{d(p^f)} \frac{\partial p^f}{\partial \beta_i^f} = 0. \tag{8.19}$$

Differentiating (8.16), we derive

$$\frac{\partial p^f}{\partial \beta_i^f} = -\frac{d(p^f)^2 r_i n a_i}{d'(p^f)}. \tag{8.20}$$

Also, we can differentiate (8.13) to derive

$$\frac{\partial \widetilde{p}}{\partial \beta_i^f} = \frac{r_i n a_i d(\widetilde{p})^2}{(n-1) d'(\widetilde{p})}. \tag{8.21}$$

Substituting (8.20) and (8.21) into (8.19) yields

$$r_i n \left[P_i^f + \frac{a_i d(\widetilde{p})^3}{(n-1) d(p^f) d'(\widetilde{p})} - a_i p^f d(p^f) \right] = 0. \tag{8.22}$$

We now argue that adopting the same $\varepsilon - MAWEF$ mechanism adopted by the other firms is a solution to (8.12). This mechanism corresponds to $\beta_i^f = \frac{1}{n}$ for all i, which implies $p^f = \widetilde{p} = p^c$. Since the corresponding value of P_i^f is

$$a_i d(p^c)p^c - \frac{a_i d(p^c)^2}{(n-1)d'(p^c)},$$

it follows that (8.22) is satisfied.

This establishes that the $\varepsilon - MAWEF$ mechanism corresponds to β^f that solves (8.22) for each i, the necessary first order conditions to (8.12). To complete the proof, we first show that ignoring corner solutions is without loss of generality. Then we show any interior solution to (8.12) requires $p^f = \widetilde{p} = p^c$, which pins down the profits of firm f and establishes the $\varepsilon - MAWEF$ mechanism as a best response.

Claim: For any corner solution to (8.17), β^{*f}, there is an interior solution yielding the same payoff.

Proof of Claim: Any β^f satisfying

$$\sum_{i=1}^{I} r_i n \beta_i^f a_i = \sum_{i=1}^{I} r_i n \beta_i^{*f} a_i \tag{8.23}$$

yields the same payoff as β^{*f}. Since $\beta^{*f} = 0$ cannot be optimal, we must have $\beta_i^{*f} > 0$ for some i. For any $\beta_h^{*f} = 0$, we can reduce β_i^{*f} and increase β_h^{*f}, such that (8.23) holds. Similarly, if we have $\beta_i^{*f} < 1$ for some i, then for any $\beta_h^{*f} = 1$, we can increase β_i^{*f} and reduce β_h^{*f}, such that (8.23) holds.

The only remaining case is $\beta_i^{*f} = 1$ for all i. But this implies $\widetilde{p} = 0$. One can show that profits are strictly less than the profits from $\beta_i^f = \frac{1}{n}$ for all i, so this case is impossible, thereby proving the claim.

Claim: Any interior solution to (8.17) must yield $p^f = \widetilde{p} = p^c$, and all such solutions yield the same profits.

Proof of Claim: Substituting

$$P_i^f = u_i(a_i d(p^f)) - u_i(a_i d(\widetilde{p})) + \widetilde{p} a_i d(\widetilde{p}) + E_i^*$$

into (8.22) implies the necessary condition,

$$\begin{aligned} 0 &= u_i(a_i d(p^f)) - u_i(a_i d(\widetilde{p})) + \widetilde{p} a_i d(\widetilde{p}) + E_i^* \\ &\quad + \frac{a_i d(\widetilde{p})^3}{(n-1)d(p^f)d'(\widetilde{p})} - a_i p^f d(p^f). \end{aligned} \tag{8.24}$$

Using (8.13) and (8.16), we can express $\widetilde{p}$ in terms of p^f, given by

$$d(\widetilde{p}) = \frac{(n-1)d(p^f)}{d(p^f)\sum_{h=1}^{I} r_h n a_h - 1}, \tag{8.25}$$

so the right side of (8.24) depends only on p^f. Differentiating (8.24) with respect to p^f, and simplifying using the condition, $u_i'(a_i d(p)) = p$ yields the expression,

$$a_i d(\widetilde{p})\frac{\partial \widetilde{p}}{\partial p^f} - a_i d(p^f) + \frac{\partial}{\partial p^f}[\frac{a_i d(\widetilde{p})^3}{(n-1)d(p^f)d'(\widetilde{p})}]. \tag{8.26}$$

Also, differentiating (8.25) yields

$$\frac{\partial \widetilde{p}}{\partial p^f} = -\frac{(n-1)d'(p^f)}{[d(p^f)\sum_{h=1}^{I} r_h n a_h - 1]^2}. \tag{8.27}$$

For sufficiently large n, the last term in (8.26) is negligible, and it is clear from (8.27) that the first term in (8.26) is negligible. Therefore, the entire expression is arbitrarily close to $-a_i d(p^f)$, which is strictly negative. Thus, the second-order conditions with respect to p^f are satisfied, so the only value of p^f that satisfies (8.24) is p^c. ■

Proof of Proposition 3. First, for any profile of prices, the ensuing consumer subgame has a type-symmetric equilibrium, by Lemma 1. Select an arbitrary type-symmetric equilibrium following a deviation by two or more firms.

Now consider a potential deviation by a single firm, f. To show that the deviation is not profitable, we will show that there is no profitable deviation, even if firm f could choose any equilibrium of the subgame. Since it is without loss of generality to restrict attention to mechanisms that fully allocate capacity, thereby allowing a higher total payment, an upper bound to the profits available is the solution to the following optimization problem, in which firm f chooses its mechanism and its arrival vector, β^f, subject to its capacity constraint and the constraint that consumers are indifferent between firm f and the other firms:[21]

[21] It is an upper bound because we do not impose incentive compatibility, but as it turns out, incentive compatibility is not binding. It is without loss of generality to impose the consumer indifference condition. Also, $x_i(\beta^f)$ is the consumption level at other firms (utility maximizing demand or rationing level, whichever is smaller), which depends on firm f's choice of β^f.

$$\max \sum_{i=1}^{I} r_i n \beta_i^f P_i^f$$
$$\text{subject to}$$
$$\begin{aligned} \sum_{i=1}^{I} r_i n \beta_i^f x_i^f &= 1 \\ u_i(x_i^f) - P_i^f &= u_i(x_i(\beta^f)) - p^c x_i(\beta^f), \quad \text{for } i = 1, ..., I \\ \beta_i^f &\geq 0 \ \text{ for } i = 1, ..., I. \end{aligned} \tag{8.28}$$

Letting λ denote the Lagrange multiplier on the capacity constraint and λ_i denote the multiplier on the indifference constraint, the necessary first-order conditions with respect to x_i^f and P_i^f are

$$\begin{aligned} \lambda_i u_i'(x_i^f) &= \lambda r_i n \beta_i^f \ \text{ and} \\ r_i n \beta_i^f &= \lambda_i, \end{aligned}$$

implying $u_i'(x_i^f) = \lambda$ for all i. Therefore, we have $x_i^f = d_i(\lambda)$ for all i.

We now show that the solution to (8.28) involves $\beta_i^f = 0$ for all $i > 1$. Suppose instead that $\beta_i^f > 0$ for some $i > 1$. Firm f could increase profits by altering β^f in such a way as to increase the total capacity offered to type 1 consumers and decrease the total capacity offered to type i consumers, both by $\varepsilon > 0$. To see this, the measure of type 1 consumers visiting other firms goes down by $\frac{\varepsilon}{d_1(\lambda)}$, and the measure of type i consumers visiting other firms goes up by $\frac{\varepsilon}{d_i(\lambda)}$. Notice that $d_1(\lambda) > d_i(\lambda)$ holds, and both type 1 and type i consumers must be rationed at the other firms and purchasing the same amount, up to the rationed limit. Therefore, the net effect of this ε switch is to increase the measure of consumers purchasing up to the rationed limit at other firms, which lowers the utility offered at other firms and allows firm f to extract higher payments, a contradiction. Therefore, the best deviation for firm f is to attract only type 1 consumers. We know that for sufficiently large n, β_1^f must be small, and certainly less than 1, or else firm f would be offering type 1 consumers lower utility than the other firms, even if its price is zero.[22]

[22] Moreover, even if n is not large, there is still no profitable deviation in the neighborhood of $p^f = p^c$. Then the firm would be forced to try to attract all of type 1 consumers and some lower type consumers if it attempts to deviate, which would be even less profitable than if the firm had plenty of type 1 consumers to attract.

Thus, we can simplify the optimization problem of firm f as follows.

$$\begin{aligned}
& \max_{\beta_1^f, P_1^f} r_1 n \beta_1^f P_1^f \\
& \text{subject to} \qquad\qquad\qquad\qquad (8.29) \\
u_1\left(\frac{1}{r_1 n \beta_1^f}\right) - P_1^f \;=\; & u_1(\overline{x}) - p^c \overline{x},
\end{aligned}$$

where $\overline{x}$ is the rationing level offered by other firms, which depends on β_1^f. Therefore, $\overline{x}$ must be at least as high as what would obtain if type 1 consumers visit each of the other firms with probability $(1-\beta_1^f)/(n-1)$, other consumers visit each of the other firms with probability $1/(n-1)$, and only type 1 consumers are rationed. In this case, a lower bound for $\overline{x}$ (which represents an upper bound to firm f's profit) is the solution to

$$\frac{(1-\beta_1^f) r_1 n \overline{x}}{n-1} + \frac{\sum_{i=2}^I r_i n d_i(p^c)}{n-1} = 1. \tag{8.30}$$

From (2.1), we have

$$\sum_{i=2}^I r_i n d_i(p^c) = n - r_1 n d_1(p^c). \tag{8.31}$$

Combining (8.30) and (8.31), we have a lower bound for $\overline{x}$ given by

$$\overline{x} = \frac{r_1 n d_1(p^c) - 1}{(1-\beta_1^f) r_1 n}.$$

Since the solution to (8.29) will have the constraint hold with equality, we can substitute the constraint into the objective, so an upper bound to the firm's profit is the solution to

$$\max_{\beta_1^f} r_1 n \beta_1^f \left[u_1\left(\frac{1}{r_1 n \beta_1^f}\right) - u_1\left(\frac{r_1 n d_1(p^c) - 1}{(1-\beta_1^f) r_1 n}\right) + p^c \frac{r_1 n d_1(p^c) - 1}{(1-\beta_1^f) r_1 n} \right]. \tag{8.32}$$

Differentiating the profit expression in (8.32) with respect to β_1^f yields

$$\begin{aligned}
& r_1 n \left[u_1\left(\frac{1}{r_1 n \beta_1^f}\right) - u_1\left(\frac{r_1 n d_1(p^c) - 1}{(1-\beta_1^f) r_1 n}\right) + p^c \frac{r_1 n d_1(p^c) - 1}{(1-\beta_1^f) r_1 n} \right] \qquad (8.33) \\
& + \beta_1^f \left[u_1'\left(\frac{1}{r_1 n \beta_1^f}\right)\left(-\frac{1}{(\beta_1^f)^2}\right) - u_1'\left(\frac{r_1 n d_1(p^c) - 1}{(1-\beta_1^f) r_1 n}\right)\left(\frac{r_1 n d_1(p^c) - 1}{(1-\beta_1^f)^2}\right) + p^c \frac{r_1 n d_1(p^c) - 1}{(1-\beta_1^f)^2} \right].
\end{aligned}$$

Evaluating the first order condition at $\beta_1^f = \frac{1}{r_1 n d_1(p^c)}$, which implies $x_1^f = \overline{x} = d_1(p^c)$, (8.33) becomes

$$r_1 n p^c d_1(p^c) + [-u_1'(d_1(p^c))(r_1 n d_1(p^c)) + (p^c - u_1'(d_1(p^c)))\frac{1}{r_1 n d_1(p^c)}\frac{r_1 n d_1(p^c) - 1}{(1 - \frac{1}{r_1 n d_1(p^c)})^2}],$$

which is zero, due to the fact that $u_1'(d_1(p^c)) = p^c$. Thus, as long as the second order condition is satisfied, firm f can do no better than to offer the price p^c.

The derivative of (8.33) with respect to β_1^f is

$$\begin{aligned}
&2\left[u_1'(\frac{1}{r_1 n\beta_1^f})(-\frac{1}{(\beta_1^f)^2}) - u_1'(\frac{r_1 n d_1(p^c) - 1}{(1-\beta_1^f)r_1 n})(\frac{r_1 n d_1(p^c) - 1}{(1-\beta_1^f)^2}) + p^c\frac{r_1 n d_1(p^c) - 1}{(1-\beta_1^f)^2}\right] \\
&+\beta_1^f\left[u_1''(\frac{1}{r_1 n\beta_1^f})(\frac{1}{r_1 n(\beta_1^f)^4}) + u_1'(\frac{1}{r_1 n\beta_1^f})(\frac{2}{(\beta_1^f)^3})\right] \\
&-\beta_1^f\left[u_1''(\frac{r_1 n d_1(p^c) - 1}{(1-\beta_1^f)r_1 n})(\frac{r_1 n d_1(p^c) - 1}{r_1 n})(\frac{r_1 n d_1(p^c) - 1}{(1-\beta_1^f)^4})\right] \\
&-\beta_1^f\left[2u_1'(\frac{r_1 n d_1(p^c) - 1}{(1-\beta_1^f)r_1 n})(\frac{r_1 n d_1(p^c) - 1}{(1-\beta_1^f)^3})\right] + \beta_1^f\left[p^c\frac{r_1 n d_1(p^c) - 1}{(1-\beta_1^f)^3}\right],
\end{aligned}$$

which, after simplifying and substituting $\overline{x}$ for $\frac{r_1 n d_1(p^c) - 1}{(1-\beta_1^f)r_1 n}$, becomes

$$r_1 n\left[\frac{2\overline{x}}{(1-\beta_1^f)^3}([p^c - u_1'(\overline{x})] + u_1''(\frac{1}{r_1 n\beta_1^f})(\frac{1}{(r_1 n)^2(\beta_1^f)^3}) - u_1''(\overline{x})\frac{\beta_1^f\overline{x}^2}{(1-\beta_1^f)^4}\right]. \tag{8.34}$$

Because type 1 consumers are rationed at the other firms, we have $p^c < u_1'(\overline{x})$. Because n is large, firm f must choose β_1^f that is small enough that $n\beta_1^f$ is bounded from above, or else it would be impossible to satisfy the constraint in (8.29). Therefore, the first term in (8.34) is negative, the second term becomes unboundedly negative as n gets large, and the third term is positive but becomes negligible as n gets large. We conclude that the second order conditions are satisfied. ■

Proof of Proposition 4. First, for any profile of share prices, the ensuing consumer subgame has a type-symmetric equilibrium, which follows from Lemma 1. Select an arbitrary type-symmetric equilibrium following a deviation by two or more firms. On the equilibrium path, it is obvious that consumers are best responding to each other in the ensuing consumer subgame, by choosing each firm with probability $\frac{1}{n}$.

Now suppose that all firms, except possibly firm f, choose the share price $P^* = \frac{np^c}{(n-1)r_1}$. Consider a potential deviation by a single firm, f. To show that a deviation is not profitable, we will show that firm f cannot increase its profits, even if it could choose the equilibrium of the consumer subgame following its deviation. To show that there is no profitable deviation, we can restrict attention to mechanisms that fully allocate capacity, thereby allowing a higher total payment. Thus, the optimal deviation can be seen as choosing P_1^f and β_1^f to maximize profits, subject to the constraint of making type 1 consumers indifferent between firm f and the other firms (i.e., that consumers adjust their arrival probabilities to form a Nash equilibrium of the subgame):[23]

$$\begin{aligned} & \max r_1 n \beta_1^f P_1^f \\ & \text{subject to} \\ u_1(\frac{1}{r_1 n \beta_1^f}) - P_1^f &= u_1(\frac{n-1}{r_1 n(1-\beta_1^f)}) - P^*. \end{aligned}$$

Substituting the constraint into the objective, we have the equivalent unconstrained problem of choosing β_1^f to maximize

$$r_1 n \beta_1^f \left[u_1(\frac{1}{r_1 n \beta_1^f}) - u_1(\frac{n-1}{r_1 n(1-\beta_1^f)}) + P^* \right]. \tag{8.35}$$

The necessary first order conditions are given by

$$\begin{aligned} 0 &= r_1 n \left[u_1(\frac{1}{r_1 n \beta_1^f}) - u_1(\frac{n-1}{r_1 n(1-\beta_1^f)}) + P^* \right] \\ &\quad + \beta_1^f \left[u_1'(\frac{1}{r_1 n \beta_1^f})(\frac{-1}{(\beta_1^f)^2}) - u_1'(\frac{n-1}{r_1 n(1-\beta_1^f)})(\frac{n-1}{(1-\beta_1^f)^2}) \right]. \end{aligned} \tag{8.36}$$

Differentiating the right side of (8.36) with respect to β_1^f and simplifying, the

[23] For a deviation that attracts so many consumers that the consumption cap, X, is reached at the other firms, the utility received by consumers is so high that the deviation cannot be profitable. Therefore, it suffices to consider this simplified optimization problem.

second derivative of profits is given by

$$-2u_1'(\frac{n-1}{r_1 n(1-\beta_1^f)})(\frac{n-1}{(1-\beta_1^f)^3}) + u_1''(\frac{1}{r_1 n\beta_1^f})(\frac{1}{r_1 n(\beta_1^f)^3})$$
$$-u_1''(\frac{n-1}{r_1 n(1-\beta_1^f)})(\frac{(n-1)^2\beta_1^f}{r_1 n(1-\beta_1^f)^4}). \tag{8.37}$$

The second order conditions are satisfied if (8.37) is negative, which must be the case if the sum of the first and third terms is negative, given by

$$-(\frac{n-1}{(1-\beta_1^f)^3})\left[2u_1'(\frac{n-1}{r_1 n(1-\beta_1^f)}) + u_1''(\frac{n-1}{r_1 n(1-\beta_1^f)})(\frac{n-1}{r_1 n(1-\beta_1^f)})(\beta_1^f)\right]. \tag{8.38}$$

When n is sufficiently large, β_1^f must be close to zero, or else type 1 consumers would prefer one of the other firms even if firm f chose a share price of zero. Also, the consumption offered by other firms, $\frac{n-1}{r_1 n(1-\beta_1^f)}$, is bounded from above and below. Therefore, the expression in brackets in (8.38) must be positive, so the second order conditions are satisfied.

Substituting $\beta_1^f = \frac{1}{n}$ and $P^* = \frac{np^c}{(n-1)r_1} = u_1'(\frac{1}{r_1})\frac{1}{r_1}\frac{n}{n-1}$ into (8.36), we see that the first order conditions are satisfied, so firm f has no profitable deviation. From the constraint, the corresponding value of P_1^f is P^*, so the mechanism chosen by firm f is the same fixed-price-per-share mechanism chosen by the other firms. ■

Proof of Proposition 5. For any profile of mechanisms, the ensuing consumer subgame has a type-symmetric equilibrium, which follows from Lemma 1. Select an arbitrary type-symmetric equilibrium following a deviation by two or more firms.

Consider a potential deviation by a single firm, f. To show that there is no profitable deviation, we can restrict attention to mechanisms that fully allocate capacity, $x_1 = \frac{1}{r_1 n\beta_1^f}$, thereby allowing a higher total payment. Thus, the optimal deviation can be seen as choosing P_1^f and β_1^f to maximize profits, subject to the constraint of making type 1 consumers indifferent between firm f and the other firms (i.e., that consumers adjust their arrival probabilities to form a Nash equilibrium of the subgame).

If $\beta_1^f \leq \frac{1}{n}$ holds, consumers at other firms receive zero surplus, so the optimization problem for firm f is given by

$$\begin{aligned} & \max r_1 n\beta_1^f P_1^f \\ & \text{subject to} \\ u_1(\frac{1}{r_1 n\beta_1^f}) - P_1^f &= u_1(0). \end{aligned}$$

Substituting the constraint into the objective, we equivalently have the unconstrained problem of maximizing

$$\beta_1^f[u_1(\frac{1}{r_1 n\beta_1^f}) - u_1(0)].$$

The derivative of this function is

$$\begin{aligned} & u_1(\frac{1}{r_1 n\beta_1^f}) - u_1(0) - \beta_1^f u_1'(\frac{1}{r_1 n\beta_1^f})\frac{1}{r_1 n(\beta_1^f)^2} \\ = \; & u_1(x_1) - u_1(0) - u_1'(x_1)x_1, \end{aligned}$$

which is strictly positive due to the strict concavity of $u_1(x_1)$. Thus, the objective is increasing in β_1^f, and the highest payoff within this range is to choose $\beta_1^f = \frac{1}{n}$.

If $\beta_1^f \geq \frac{1}{n}$ holds, we have $\rho_1^j = \frac{(1-\beta_1^f)r_1 n}{n-1} < r_1$ at firms $j \neq f$, so the optimization problem for firm f is given by

$$\begin{aligned} & \max r_1 n\beta_1^f P_1^f \\ & \text{subject to} \\ u_1(\frac{1}{r_1 n\beta_1^f}) - P_1^f &= u_1(\frac{n-1}{(1-\beta_1^f)r_1 n}) - Ar_1\left[\frac{1-n\beta_1^f}{n-1}\right] - u_1(\frac{1}{r_1}) + u_1(0), \end{aligned}$$

which is equivalent to the unconstrained problem of maximizing

$$\beta_1^f\left[u_1(\frac{1}{r_1 n\beta_1^f}) - u_1(\frac{n-1}{(1-\beta_1^f)r_1 n}) + Ar_1\left[\frac{1-n\beta_1^f}{n-1}\right] + u_1(\frac{1}{r_1}) - u_1(0)\right]. \quad (8.39)$$

Differentiating (8.39) with respect to β_1^f yields

$$\begin{aligned} & -Ar_1\left[\frac{(n+1)\beta_1^f - 1}{n-1}\right] + u_1(\frac{1}{r_1 n\beta_1^f}) - u_1(\frac{n-1}{(1-\beta_1^f)r_1 n}) + u_1(\frac{1}{r_1}) \quad (8.40) \\ & -u_1(0) + \beta_1^f\left[-u_1'(\frac{1}{r_1 n\beta_1^f})\frac{1}{r_1 n(\beta_1^f)^2} - u_1'(\frac{n-1}{(1-\beta_1^f)r_1 n})\frac{n-1}{(1-\beta_1^f)^2 r_1 n}\right]. \end{aligned}$$

Since $\beta_1^f \geq \frac{1}{n}$ holds, the first expression in brackets in (8.40) is greater than $\frac{1}{n(n-1)}$, so the first overall term can be made arbitrarily negative for sufficiently large A. Also, the optimal β_1^f must be bounded well below 1 for sufficiently large A, or else satisfying the indifference constraint would require P_1^f to be negative. Therefore, (8.40) is strictly negative. Since profits for firm f are decreasing in β_1^f, it follows that the optimal choice within this range is $\frac{1}{n}$, and the indifference constraint implies $P_1^f = u_1(\frac{1}{r_1}) - u_1(0)$. Firm f receives the same profit as it would by adopting the mechanism specified in the statement of Proposition 6, so there is no profitable deviation. ■

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