# A Diamond-Dybvig Model in which the Level of Deposits is Endogenous[*]

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### Abstract

We extend the Diamond-Dybvig model of bank runs to include a specification of *how much* to deposit. Contracts allow for partial or full suspension of convertibility. We prove that the efficient allocation can be achieved in equilibrium as long as the deposit level is above a threshold. Within this range, the lower the deposit level, the more tempted patient depositors are to withdraw early. We extend the baseline model so that non-bank investments yield a higher return than bank investments. Then when the propensity to run is small enough, it is common for runs to occur on the equilibrium path.

## 1. Introduction

In the many years and many published articles following the bank runs paper of Diamond and Dybvig (1983), only a few papers have modeled the decision of *whether* to deposit, much less the decision of *how much* to deposit. This is peculiar, considering the vast amount of wealth that is invested by financial firms that do not provide insurance against being an "impatient" consumer with

immediate consumption needs. The questions we address here are, how does the opportunity for consumers to invest in a mutual fund or otherwise invest outside the banking system (1) affect the nature of the final allocation, (2) affect the nature of the optimal deposit contract, and (3) affect the fragility of the banking system?

We consider a model with $N$ ex ante identical consumers, who invest some of their endowment in a mutual fund and deposit the rest of their endowment with the bank in period 0, before observing whether they will become impatient (requiring consumption in period 1) or patient (able to consume in period 2). In the baseline model, there is a single technology available to the bank and to the mutual fund. A contract specifies a deposit level, $d$, and period 1 withdrawals. Due to a sequential service constraint, withdrawals in period 1 depend on the consumer's place in line, but not on the number of consumers yet to withdraw in period 1.

We find that any allocation that the bank can induce (satisfying incentive compatibility and non-negativity constraints) with a given deposit level can also be induced with any higher deposit level. For the typical economy, the incentive compatibility constraint does not bind at the optimal contract when consumers deposit their full endowment. If so, we find that there is an interval of deposit levels yielding the efficient allocation in equilibrium. The intuition for this equivalence result is that the withdrawal amount augments the mutual fund investment of impatient consumers by exactly the amount of consumption needed to achieve the optimal allocation. Since the optimal contract provides insurance against being impatient, by allowing impatient consumers more consumption than their endowment on average, the bank responds to lower deposits by magnifying the ratio of period 1 withdrawals to deposits. As we consider lower and lower deposit levels, at some point the efficient allocation cannot be achieved, due to a failure of incentive compatibility or non-negativity.

We also find that, in equilibria achieving the efficient allocation, lower deposit levels make the banking system more fragile. The lower the deposit level, the more tempted a patient consumer is to withdraw early in the no-run equilibrium, and the more tempted she is to join a run. The reason is that withdrawals from the bank in period 1 are stored by a patient consumer until consuming in period 2, but she can keep her mutual fund investment until period 2, receiving a higher return. Due to the increased fragility, it is easy to construct equilibria in which, when consumers deposit significantly less than their entire endowment, both the no-run equilibrium yielding the efficient allocation and a run equilibrium are equilibria to

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the post-deposit subgame. This can occur even when the efficient allocation can be uniquely implemented if consumers must deposit their entire endowment.

Finally, we consider the full model, with the specification that mutual fund investment held until period 2 yields an $\varepsilon$ higher return than bank investment held until period 2. The equivalence result for the baseline model no longer holds when $\varepsilon$ is positive, because the optimal contract will economize on the deposit level, and full deposits are not optimal. When $\varepsilon$ is small, and when the propensity to run is small relative to $\varepsilon$, then at the optimal contract, the deposit level is so low that either incentive compatibility or non-negativity binds. This makes bank runs on the equilibrium path far more likely than in the previous literature where consumers deposit their entire endowment. When $\varepsilon$ is considerably greater than zero, we show by an example in the Appendix, that the same qualitative results hold.

Section 2 contains a literature review. Section 3 sets up the model, and Section 4 contains the main results for the baseline model. Section 5 and the Appendix contain the full model and results about bank runs occurring with positive probability on the equilibrium path.

## 2. Literature Review

Most of the papers in the Diamond-Dybvig literature simply assume that consumers have already deposited their entire endowment, and consider the optimal contract that would arise if a run is certain not to occur. There are only a few exceptions. Peck and Shell (2003) consider a model similar to the present model, except that (1) consumers decide whether or not to deposit their entire endowment with the bank, rather than how much to deposit, and (2) the utility functions of impatient and patient consumers can be different.[1] Examples are constructed in which any contract achieving the efficient allocation (highest welfare consistent with resource, incentive compatibility, and sequential service constraints) has a bank run equilibrium to the post-deposit subgame. Bank runs, triggered by sunspots, can occur on the equilibrium path if the probability of a run is small enough. Shell and Zhang (2018a) provide a complete characterization of the 2 consumer version of Peck and Shell (2003). The main insight of Shell and Zhang (2018a) is the following. When incentive compatibility binds

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[1]In their main model, only consumers wishing to withdraw arrive in period 1, but an example is presented in which all consumers arrive in random order to report their type, without observing their place in line.

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for the contract achieving the efficient allocation, then the optimal contract for the pre-deposit game, which takes into account a small propensity to run, is that contract. However, when incentive compatibility does not bind for the contract achieving the efficient allocation, the optimal contract for the pre-deposit game is different. Withdrawal levels depend on the sunspot probability, striking a balance between efficiency when a run does not occur and efficiency when a run occurs. The present paper models the decision of *how much* to deposit. Also, we provide a result about when a run occurs on the equilibrium path, but we do not provide a complete characterization of the optimal contract for a positive propensity to run, like Peck and Shell (2003) and Shell and Zhang (2018a) do. The connection is discussed further in Section 5.

In Peck and Shell (2010), there are two technologies, one liquid and one illiquid. Both patient and impatient consumers care about "left-over" consumption, with the difference being that an impatient consumer receives an urgent consumption opportunity in period 1 and a patient consumer receives an urgent consumption opportunity in period 2. It is shown that legal restrictions that prevent the bank from accessing the illiquid technology lead to overinvestment in the liquid asset and run equilibria in the post-deposit subgame. In the separated system in Peck and Shell (2010), investment in the illiquid technology is outside the banking system, and the result is greater instability, as in the present model. Shell and Zhang (2018b) model the pre-deposit game in a two-consumer version of Peck and Shell (2010), to encorporate runs on the equilibrium path and to characterize when the propensity to run affects the optimal contract (for the separated system). The main difference between these papers and the current paper is that these papers depart from the original Diamond-Dybvig model, in terms of utility functions and timing of impatient consumption. Similarly, Ennis and Keister (2003) consider an endogenous growth model with the Diamond-Dybvig timing within a generation. Banks invest in both storage and capital, and consumers can store income that is not deposited. Contracts are of a simple form that rules out suspension schemes. The authors show that much of the welfare cost of bank runs fall on future generations, which is not taken into account by banks competing for deposits from the current generation. Thus, although there is a decision of how much to deposit, the model and questions are very different from those of the current paper. In a recent paper by Gu et al. (2019), a dynamic model is studied in which long-lived agents endogenously set up banks and are willing to honor their obligations in order to maintain their reputations. The simple bank contract specifies how much of the endowment is deposited and specifies constant withdrawal amounts in each

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period. A smaller deposit level reduces the bank's incentive to misbehave, while in our setting, a smaller deposit level increases a patient consumer's incentive to withdraw early. Interestingly, in one of their parameter regions, an equivalence result reminiscent of our Proposition 1 obtains.

Green and Lin (2003) consider a model in which all agents join the line in period 1 to report their type, and all agents know the "clock" time at which they join the line. It is shown that the efficient allocation is uniquely implemented, with no possibility of a bank run. Ennis and Keister (2009a) show that the Green and Lin (2003) implementation result is sensitive to observing the clock and the assumption of independent types. Ennis and Keister (2016) show that the efficient allocation may also have a bank run equilibrium in a model where only those seeking to withdraw in period 1 join the line, but where consumers know their place in line. Andolfatto et al. (2017) extend the space of possible reports, and they assume that all depositors visit the bank in period 1, without knowing their place in line. Essentially, a depositor reports whether she is patient or impatient, and also reports whether a run is in progress. With these more general mechanisms, the efficient allocation is uniquely implemented. A similar result is obtained by Cavalcanti and Monteiro (2016). While these implementation results are important, an essential function of demand deposit accounts is their convenience, so it is useful to study models in which mechanisms do not involve reports, and an arrival at the bank is a request to withdraw. It would also be interesting to see whether endogenizing the deposit level affects any of these implementation results, or the results regarding commitment (see Ennis and Keister (2009b)).

Let us be careful to distinguish what we are doing from the literature on the bank as a specialist in making loans. In Gertler and Kiyotaki (2015), consumers could make loans directly, but they are less efficient in making and monitoring loans than the bank, so they deposit in the bank instead. We do not imagine non-deposited endowments as being loaned by consumers to small businesses. In such a situation, our $\varepsilon$ should surely be negative! Rather, we are thinking about consumers as buying mutual funds. The rate of return on mutual funds is typically greater than the rate of return on bank assets. See, Fama and French (2010) for mutual fund returns, and Albertazzi and Gambacorta (2009) or Chronopoulos et al. (2015) for returns on bank assets. Our model does not explain the difference in returns between bank assets and mutual funds, but a positive $\varepsilon$ makes sense. One possible explanation is that banks face costs when issuing and monitoring their loans, and mutual fund managers do not face these costs. Another explanation is that the rate of return on bank assets is hampered by reserve requirements and

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other regulatory requirements.

## 3. The Baseline Model

We consider an otherwise standard Diamond-Dybvig economy, but where consumers decide how much of their endowment to deposit and how much to invest directly. There are three time periods and a single investment technology. Each unit of consumption invested in period 0, by either a consumer or the bank, yields 1 unit if harvested in period 1, and $R > 1$ units if harvested in period 2.

There is a finite number, $N$, of consumers. In period 0, each consumer receives an endowment of the consumption good, normalized to 1. In period 1, each consumer will privately observe whether she is impatient or patient. An impatient consumer only derives utility from consumption in period 1, and a patient consumer derives utility from consumption in period 2. A patient consumer can costlessly store consumption received in period 1 to period 2. Thus, denoting the consumption received in period 1 by $x_1$ and the consumption received in period 2 by $x_2$, an impatient consumer receives utility $u(x_1)$ and a patient consumer receives utility $u(x_1 + x_2)$. As is standard, we assume $u'(x) > 0$, $u''(x) < 0$, and $\lim_{x \to 0} u'(x) = \infty$. Although not needed for our results, the condition ensuring that the bank provides insurance against being impatient, by allowing some withdrawals in period 1 to exceed the deposit, is

$$-\frac{xu''(x)}{u'(x)} > 1.$$

The number of impatient consumers, denoted by $\alpha$, is a random variable with probability distribution $f(\alpha)$. Impatience can be i.i.d. or correlated across consumers. It is assumed that the distribution of $\alpha$, *conditional* on being patient, denoted by $f_p(\alpha)$, is the same for all consumers. From Bayes' rule, we have

$$f_p(\alpha) = \frac{(1 - \frac{\alpha}{N})f(\alpha)}{\sum_{a=0}^{N-1}(1 - \frac{a}{N})f(a)}.$$

Here is the timing of the game. At the beginning of period 0, the bank chooses a deposit contract, which specifies a deposit level, $d$, and period-1 and period-2 withdrawals, $C$, which we fully describe below. Next, consumers decide whether or not to deposit; any endowment not invested in the bank is invested in, say, a mutual fund that is independent of the bank. At the beginning of period 1,

consumers observe their type, impatient or patient, and a public "sunspot" variable, $\sigma$. Next, each consumer decides whether or not to withdraw in period 1. Those who withdraw in period 1 arrive in random order, with all orders treated as equally likely at the time of the withdrawal choice. Arriving in period 1 can be interpreted as a statement of impatience, but no formal reports are made. Those who do not withdraw in period 1 do not contact the bank until period 2.[2] We assume that a consumer can, if she wishes, liquidate her mutual fund in period 1, to provide consumption over and above whatever she may withdraw from the bank. However, as is standard in the literature, we rule out additional markets in which consumption and unharvested investments are traded in period 1.[3]

Based on the idea that the banking system is competitive, we model the bank as choosing a deposit contract to maximize the expected utility of their depositors. Due to a sequential service constraint, withdrawals in period 1 must be a function of the history of previous withdrawals. Withdrawals can be characterized by a pair of non-negative functions, $c_1(z, d)$ and $c_2(\alpha_1, d)$. For $z = 1, ..., N$, $c_1(z, d)$ is the withdrawal amount for the $z^{th}$ consumer to arrive in period 1, when all consumers deposit $d$ units.[4] For $\alpha = 0, ..., N-1$, $c_2(\alpha_1, d)$ is the withdrawal amount in period 2, when all consumers deposit $d$ units and the number of consumers withdrawing in period 1 is $\alpha_1$. The specification, that each consumer withdrawing in period 2 receives the same consumption, follows from expected utility maximization and consumption smoothing. Also due to expected utility maximization, the optimal contract will fully allocate the bank's resources:

$$c_1(N, d) = dN - \sum_{z=1}^{N-1} c_1(z, d), \tag{3.1}$$

$$c_2(\alpha_1, d) = \frac{[dN - \sum_{z=1}^{\alpha_1} c_1(z, d)]R}{N - \alpha_1}. \tag{3.2}$$

We now characterize the bank's optimal contract by solving the associated

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[2]For simplicity and realism, we do not allow mechanisms where the bank requires all consumers to contact the bank in period 1 to report their types, even patient consumers who do not want to withdraw. See Peck and Shell (2003) for a discussion. Also see Green and Lin (2003), Cavalcanti and Monteiro (2016), Andolfatto et al. (2017), and the appendix of Peck and Shell (2003) for analysis of more complicated mechanisms.

[3]Jacklin (1987) has shown that allowing such markets undermines the ability of the bank to provide insurance against being impatient.

[4]Our notation presumes that all $N$ consumers will choose to deposit, which must hold in equilibrium. Also, it will be convenient to keep track of the deposit level $d$ in notation for withdrawals.

planner's problem, of maximizing welfare subject to the resource constraints (3.1) and (3.2), and the incentive compatibility constraint, that a patient consumer weakly prefers to wait until period 2, given that other patient consumers also wait. This is the optimal contract if the no-run equilibrium, in which the patient consumers withdraw in period 2, is selected in the post-deposit subgame, for all realizations of $\sigma$.

Notice that an impatient consumer withdrawing $c_1$ from the bank in period 1 receives utility, $u(1 - d + c_1)$. Also notice that a patient consumer withdrawing $c_1$ from the bank in period 1 will store this consumption until period 2 and liquidate her mutual fund in period 2, thereby receiving utility, $u((1 - d)R + c_1)$. A patient consumer withdrawing $c_2$ from the bank in period 2 will liquidate her mutual fund in period 2, thereby receiving utility, $u((1 - d)R + c_2)$. Given the contract, it will be convenient to use the notation, $x_1(z, d) \equiv 1 - d + c_1(z, d)$ to denote the overall consumption of an impatient consumer who withdraws in position $z$ in period 1 and $x_2(\alpha_1, d) \equiv (1 - d)R + c_2(\alpha_1, d)$ to denote the overall consumption of a patient consumer who withdraws in period 2 when the number of consumers withdrawing in period 1 is $\alpha_1$. Then we can write the welfare associated with a contract, $(C, d)$, when the patient consumers withdraw in period 2, in which case $\alpha_1 = \alpha$ holds, as

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\begin{aligned}
\widehat{W}(C, d) &= \sum_{\alpha=0}^{N-1} f(\alpha) \left[ \sum_{z=1}^{\alpha} u(x_1(z, d)) + (N - \alpha)u(x_2(\alpha, d)) \right] \\
&+ f(N) \left[ \sum_{z=1}^{N} u(x_1(z, d)) \right].
\end{aligned}
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Given the contract, the incentive compatibility constraint for a patient consumer is given by

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\sum_{\alpha=0}^{N-1} f_p(\alpha) \left[ \frac{1}{1+\alpha} \sum_{z=1}^{\alpha+1} u((1 - d)(R - 1) + x_1(z, d)) \right] \leq \sum_{\alpha=0}^{N-1} f_p(\alpha)u(x_2(\alpha, d)).
$$

(3.3)

The reason for the term, $(1 - d)(R - 1)$, in (3.3) is that a patient consumer who withdraws in period 1 receives this additional consumption because her outside investment is harvested in period 2 rather than period 1.

The bank's optimal contract specifies a deposit level and withdrawals that

solve the following problem.

$$\max \widehat{W}(C, d)$$
$$\text{subject to} \qquad\qquad (3.4)$$
$$(3.1), (3.2), (3.3)$$
$$c_1(z, d) \;\geq\; 0 \quad \text{for all } z$$

## 4. Results for the Baseline Model

Sometimes it is useful to consider the withdrawals that solve (3.4) for fixed $d$. The solution to (3.4) for $d = 1$ is the solution to the planner's problem studied in the previous literature in which consumers deposit their entire endowment. See, for example, Peck and Shell (2003) and Shell and Zhang (2018a). Denote the correponding efficient allocation by $x^* = \{x_1^*(z)|_{z=1}^N, x_2^*(\alpha)|_{\alpha=0}^{N-1}\}$. Henceforth, we assume that $x^*$ is unique.[5]

When impatient and patient consumers have the same utility function, given by $u(\cdot)$ here, incentive compatibility typically does not bind when consumes deposit their entire endowment. Lemma 1, below, shows that any allocation satisfying the constraints in (3.4) with deposit level $d'$ must satisfy the constraints with deposit level $d'' > d'$. It follows that the allocation yielding the highest welfare across all PBE of the game is $x^*$. Proposition 1 shows that, if incentive compatibility does not bind when consumers deposit their entire endowment, then for any $d$ sufficiently close to 1, there is a PBE with deposit level $d$ yielding the efficient allocation, $x^*$. The intuition for this equivalence result is that the bank can negate the effect of mutual fund investment, by subtracting it from $x^*$ and allowing consumers to withdraw the difference.

**Lemma 1:** *Suppose that, for deposit level $d'$, there is a contract satisfying the constraints in (3.4) yielding the allocation $x = \{x_1(z)|_{z=1}^N, x_2(\alpha)|_{\alpha=0}^{N-1}\}$ if patient consumers withdraw in period 2. Then, for all deposit levels $d'' > d'$, there is a contract satisfying the constraints in (3.4) yielding the same allocation $x$.*

**Proof.** Since the contract for deposit level $d'$ yields the allocation $x$, it must be

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[5]We are not aware of any counterexamples.

given by

$$c_1(z, d') = x_1(z) - (1 - d'), \tag{4.1}$$
$$c_2(\alpha_1, d') = x_2(\alpha_1) - (1 - d')R. \tag{4.2}$$

The contract for the deposit level $d''$ yields the allocation $x$ if and only if it is given by

$$c_1(z, d'') = x_1(z) - (1 - d''), \tag{4.3}$$
$$c_2(\alpha_1, d'') = x_2(\alpha_1) - (1 - d'')R. \tag{4.4}$$

We first show that if (4.1) and (4.2) satisfy the resource constraints (3.1) and (3.2), then (4.3) and (4.4) satisfy the resource constraints. From (3.1) and (4.1), we have

$$c_1(N, d') = d'N - \sum_{z=1}^{N-1}[x_1(z) - (1 - d')] = x_1(N) - (1 - d'),$$

which can be simplified to

$$\sum_{z=1}^{N} x_1(z) = N. \tag{4.5}$$

For deposit level $d''$, (3.1) is satisfied if and only if we have

$$x_1(N) - (1 - d'') = d''N - \sum_{z=1}^{N-1}[x_1(z) - (1 - d'')],$$

which is equivalent to (4.5).

From (3.2) and (4.2), we have

$$c_2(\alpha_1, d') = \frac{d'NR - \sum_{z=1}^{\alpha_1}[x_1(z) - (1 - d')]R}{N - \alpha_1} = x_2(\alpha_1) - (1 - d')R,$$

which can be simplified to

$$NR = R\sum_{z=1}^{\alpha_1} x_1(z) + (N - \alpha_1)x_2(\alpha_1). \tag{4.6}$$

For deposit level $d''$, (3.2) is satisfied if and only if we have

$$x_2(\alpha_1) - (1 - d'')R = \frac{d''NR - \sum_{z=1}^{\alpha_1}[x_1(z) - (1 - d'')]R}{N - \alpha_1},$$

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which is equivalent to (4.6). Thus, (4.3) and (4.4) satisfy the resource constraints.

Since (4.1) and (4.2) satisfy the incentive compatibility constraint, we have

$$\sum_{\alpha=0}^{N-1} f_p(\alpha) \left[ \frac{1}{1+\alpha} \sum_{z=1}^{\alpha+1} u((1-d')(R-1)+x_1(z)) \right] \leq \sum_{\alpha=0}^{N-1} f_p(\alpha) u(x_2(\alpha)).$$

Because utility is strictly increasing and $d'' > d'$ holds, it follows that

$$u((1-d')(R-1)+x_1(z)) > u((1-d'')(R-1)+x_1(z))$$

holds, which implies

$$\sum_{\alpha=0}^{N-1} f_p(\alpha) \left[ \frac{1}{1+\alpha} \sum_{z=1}^{\alpha+1} u((1-d'')(R-1)+x_1(z)) \right] < \sum_{\alpha=0}^{N-1} f_p(\alpha) u(x_2(\alpha)).$$

Thus, the incentive compatibility constraint with deposit level $d''$ holds (and is not binding).

From (4.1), (4.2), (4.3), and (4.4), and from the fact that $d'' > d'$ holds, it follows immediately that the non-negativity constraints being satisfied with deposit level $d'$ implies that the non-negativity constraints must be satisfied with deposit level $d''$. Thus, we have constructed a contract with deposit level $d''$, satisfying non-negativity and incentive compatibility, yielding the allocation $x$ if the patient withdraw in period 2. ∎

**Proposition 1 (Equivalence):** *If IC is not binding when consumers deposit their entire endowment, so*

$$\sum_{\alpha=0}^{N-1} f_p(\alpha) \left[ \frac{1}{1+\alpha} \sum_{z=1}^{\alpha+1} u(x_1^*(z)) \right] < \sum_{\alpha=0}^{N-1} f_p(\alpha) u(x_2^*(\alpha)) \tag{4.7}$$

*holds, then there exists $d^* < 1$ such that, for all $d > d^*$, there is a contract satisfying the constraints in (3.4) yielding the allocation $x^*$.*

**Proof.** Consider the allocation $x^*$ and suppose that the incentive compatibility constraint for $d = 1$ holds strictly,

$$\sum_{\alpha=0}^{N-1} f_p(\alpha) \left[ \frac{1}{1+\alpha} \sum_{z=1}^{\alpha+1} u(x_1^*(z)) \right] < \sum_{\alpha=0}^{N-1} f_p(\alpha) u(x_2^*(\alpha)).$$

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By continuity of the utility function,

$$\sum_{\alpha=0}^{N-1} f_p(\alpha) \left[ \frac{1}{1+\alpha} \sum_{z=1}^{\alpha+1} u((1-d)(R-1) + x_1^*(z)) \right] < \sum_{\alpha=0}^{N-1} f_p(\alpha) u(x_2^*(\alpha)) \qquad (4.8)$$

holds for all $d$ close enough to 1. Also, since utility is strictly increasing in consumption, the left side of (4.8) is strictly decreasing in $d$. Let $d_{IC}^* < 1$ denote the value of $d$ such that (4.8) holds as an equality, if such a solution exists, and let $d_{IC}^* = 0$ if the right side always exceeds the left side.

Since $\lim_{x \to 0} u(x) = \infty$ holds, we have $x_1^*(z) > 0$ for all $z$ and $x_2^*(\alpha) > 0$ for all $\alpha$. The non-negativity constraints are given by

$$\begin{aligned} c_1(z,d) &= x_1^*(z) - (1-d) \geq 0 \text{ for all } z \text{ and} \\ c_2(\alpha_1, d) &= x_2^*(\alpha_1) - (1-d)R \geq 0 \text{ for all } \alpha_1. \end{aligned}$$

The left side of each inequality is increasing in $d$ and strictly positive for $d = 1$. It follows that there is a deposit level, $d_{NN}^* < 1$, above which all non-negativity constraints are satisfied. Define $d^* = \max[d_{IC}^*, d_{NN}^*]$. $\blacksquare$

**Definition 1:** *Consider an economy in which incentive compatibility is not binding at $x^*$ when consumers deposit their entire endowment, so (4.7) holds. For $d \geq d^*$, consider the contract achieving the optimal allocation $x^*$ (when the patient consumers wait). The **temptation to join a run** is defined to be the utility advantage of withdrawing in period 1, relative to withdrawing in period 2, when all other consumers withdraw in period 1, given by*

$$\frac{1}{N} \sum_{z=1}^{N} u((1-d)(R-1) + x_1^*(z)) - u(x_2^*(N-1)). \qquad (4.9)$$

*An equilibrium of the post-deposit subgame in which all consumers withdraw in period 1 is called a **run equilibrium**.*

It is easy to see that the post-deposit game has a run equilibrium if and only if the temptation to join a run is non-negative. Consider contracts yielding the optimal allocation $x^*$, for various deposit levels, $d \geq d^*$. Since utility is increasing, expression (4.9) is strictly decreasing in $d$, so the lower the deposit level the

higher the temptation to join a run. Define $d^*_{NR}$ to be the minimum deposit level above which the optimal allocation, $x^*$ can be uniquely implemented. There are three possible mutually exclusive cases. First, if there exists a threshold, $d^*_{NR} \in [d^*, 1)$, for which expression (4.9) equals zero, then the post-deposit subgame has a run equilibrium for $d \leq d^*_{NR}$, and the optimal allocation $x^*$ can be uniquely implemented for any $d > d^*_{NR}$. Second, if the temptation to join a run is non-negative for $d = 1$, then the post-deposit subgame has a run equilibrium for all $d \geq d^*$; in this case, define $d^*_{NR} = 1$. Third, if the temptation to join a run is negative for $d = d^*$, then the optimal allocation $x^*$ can be uniquely implemented for any $d \in [d^*, 1]$; in this case, define $d^*_{NR} = d^*$. We have shown the following.

**Proposition 2:** *Consider an economy in which incentive compatibility is not binding at $x^*$ when consumers deposit their entire endowment, so (4.7) holds. Then for all $d', d''$ such that $d^* < d' < d''$ holds, the temptation to join a run is higher with deposit level $d'$ than with deposit level $d''$. The post-deposit subgame has a run equilibrium for $d < d^*_{NR}$, and the optimal allocation $x^*$ can be uniquely implemented for any $d > d^*_{NR}$.*

Consider the typical economy, where incentive compatibility is not binding at $x^*$ when consumers deposit their entire endowment. Ennis and Keister (2016) have shown that, for CRRA utility, $x^*_1(z)$ is strictly decreasing in $z$. This captures the idea that the optimal contract offers liquidity insurance to the impatient, so it would be surprising not to see this pattern more generally. The following Corollary to Proposition 2 establishes that, if this monotonicity holds and non-negativity binds before incentive compatibility, then there is a range of deposit levels for which the post-deposit game has a run equilibrium (the optimal contract is "fragile" according to the terminology of Ennis and Keister).

**Corollary to Proposition 2:** *Consider an economy in which (4.7) holds. Also assume that $x^*_1(z)$ is strictly decreasing in $z$, and that $d^*_{IC} \leq d^*_{NN}$ holds. Then we have $d^* < d^*_{NR}$, and the post-deposit subgame has a run equilibrium for deposit levels between $d^*$ and $d^*_{NR}$.*

**Proof.** At deposit level $d^*$, non-negativity binds. Since $x^*_1(z)$ is strictly decreasing in $z$, it follows that $c_1(z, d^*)$ is strictly decreasing in $z$, so we must have $c_1(N, d^*) = 0$. Expression (4.9) must be positive, because if all other consumers withdraw in period 1, we have $c_1(N, d^*) = 0$, and therefore, $c_2(N - 1, d^*) = 0$. Waiting until period 2 ensures a withdrawal of zero, so joining a run must yield higher utility.

By continuity, the post-deposit subgame has a run equilibrium for deposit levels greater than, but sufficiently close to, $d^*$. $\blacksquare$

The following proposition shows that there are economies in which the optimal allocation $x^*$ can be uniquely implemented for $d = 1$, but for smaller deposit levels, the post-deposit subgame has two equilibria: there is an equilibrium yielding $x^*$ when the patient consumers wait, and there is also a run equilibrium. The proof is by construction. For some parameter values, we have $d^*_{IC} \leq d^*_{NN}$, and the Corollary to Proposition 2 applies, and for other parameter values, we have $d^*_{IC} > d^*_{NN}$, so incentive compatibility is the binding constraint determining $d^*$.

**Proposition 3:** *For some economies, we have $d^* < d^*_{NR} < 1$. That is, (i) there is a contract yielding the allocation $x^*$ with $d = 1$, such that the ensuing post-deposit subgame does not have a run equilibrium, and (ii) there is a contract yielding the allocation $x^*$ with $d < d^*_{NR}$, such that the ensuing post-deposit subgame has a run equilibrium (in addition to the equilibrium yielding $x^*$).*

**Proof.** The proof is by construction, based on the following example. $\blacksquare$

**An Example.**
Consider the CRRA utility function with risk aversion parameter, 2, given by

$$u(x) = -\frac{1}{x}.$$

There are two consumers, $N = 2$, and the probability of being impatient is i.i.d. with probability $\pi$. The benefits of having only two consumers are (i) it makes the existence of a run equilibrium at the optimal contract difficult, as there are no such examples in the literature when patient and impatient have the same utility function, $u$, and (ii) the calculations are simple, with the optimal contract characterized by the consumption offered to the first consumer to withdraw in period 1, for each $d$.

First we calculate the allocation, $x^*$, as characterized by $x_1^*(1)$, which is the solution to

$$\max_x \pi^2[u(x) + u(2-x)] + 2\pi(1-\pi)[u(x) + u((2-x)R)] + 2(1-\pi)^2 u(R)$$

subject to

$$\pi[\frac{u(x)}{2} + \frac{u(2-x)}{2}] + (1-\pi)u(x) \leq \pi u((2-x)R + (1-\pi)u(R). \qquad (4.10)$$

When the incentive compatibility constraint, (4.10), is not binding, $x_1^*(1)$ is computed by differentiating the objective with $u(x) = -\frac{1}{x}$ substituted, setting the expression equal to zero, and solving for the correct root, yielding

$$x_1^*(1) = \frac{R(2 - \pi) - \sqrt{R(2 - \pi)(R\pi - 2\pi + 2)}}{(1 - \pi)(R - 1)}. \qquad (4.11)$$

Given $d > d^*$ and a contract that yields the allocation $x^*$ in the no-run equilibrium, the incentive compatibility constraint is given by

$$\pi\left[\frac{u((1 - d)(R - 1) + x_1^*(1))}{2} + \frac{u((1 - d)(R - 1) + 2 - x_1^*(1))}{2}\right] \quad (4.12)$$
$$+(1 - \pi)u((1 - d)(R - 1) + x_1^*(1))$$
$$\leq \quad \pi u((2 - x_1^*(1))R) + (1 - \pi)u(R).$$

The relevant non-negativity constraint is for the second consumer to withdraw in period 1,

$$[2 - x_1^*(1)] - [1 - d] \geq 0 \text{ or}$$
$$x_1^*(1) \leq 1 + d. \qquad (4.13)$$

Thus, we have $d_{NN}^* = x_1^*(1) - 1$, and $d_{IC}^*$ is the solution to (4.12), after substituting (4.11) and the functional form of the utility function. The condition for the existence of a run equilibrium, that a patient consumer prefers to withdraw in period 1 when all other patient consumers withdraw in period 1, is

$$\frac{u((1 - d)(R - 1) + x_1^*(1))}{2} + \frac{u((1 - d)(R - 1) + 2 - x_1^*(1))}{2} \geq u((2 - x_1^*(1))R).$$
$$(4.14)$$

Numerical computations,[6] for $\pi = \frac{1}{2}$ and $R = 3$, yield the efficient allocation $x^*$ given by

$$x_1^*(1) = 1.145898$$
$$x_1^*(2) = 0.854102$$
$$x_2^*(0) = 3$$
$$x_2^*(1) = 2.562306.$$

---

[6] Computations were performed using Maple, and are available upon request.

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The value of $d$ below which incentive compatibility binds is $d_{IC}^* = 0.1514666$, and the value $d$ below which non-negativity binds is $d_{NN}^* = 0.145898$, so we have $d^* = 0.1514666$. Given $d > d^*$, the contract achieving the allocation $x^*$ in the no-run equilibrium is given by

$$
\begin{aligned}
c_1^*(1, d) &= 0.145898 + d \\
c_1^*(2, d) &= -0.145898 + d \\
c_2^*(0, d) &= 3d \\
c_2^*(1, d) &= -0.437694 + 3d.
\end{aligned}
$$

For this example, we can compute $d_{NR}^* = 0.2147067$. That is, the contract achieving the allocation $x^*$ in the no-run equilibrium also has a run equilibrium, for deposit levels $d \in [0.1514666, 0.2147067]$, and does not have a run equilibrium for $d \in (0.2147067, 1]$.

Notice that, for $\pi = \frac{1}{2}$ and $R = 3$, we have $d_{IC}^* > d_{NN}^*$, so incentive compatibility binds first as $d$ is reduced. For other parameter values, such as $\pi = \frac{1}{2}$ and $R = 4$, non-negativity binds first, in which case the Corollary to Proposition 2 applies.[7]

## 4.1. The Baseline Model with a Positive Propensity to Run

In this subsection, we consider the baseline model with a positive propensity to run. It is without loss of generality to assume that the sunspot variable is uniformly distributed over the unit interval. Then, given a contract that satisfies (3.1), (3.2), and (3.3), a propensity to run of $s$ means that, if the post-deposit subgame has a run equilibrium, all patient consumers withdraw in period 1 whenever $\sigma \leq s$ holds, and all patient consumers withdraw in period 2 whenever $\sigma > s$ holds. If the post-deposit subgame does not have a run equilibrium, all patient consumers withdraw in period 2 for all realizations of $\sigma$.

Welfare, conditional on a run not taking place, is $\widehat{W}(C, d)$. Welfare, conditional on a run taking place, is denoted by $W^R(C, d)$, given by

$$
W^R(C, d) = \sum_{\alpha=0}^{N} f(\alpha) \left[ \frac{\alpha}{N} \sum_{z=1}^{N} u(x_1(z, d)) + \frac{N - \alpha}{N} \sum_{z=1}^{N} u((1 - d)(R - 1) + x_1(z, d)) \right].
$$

---

[7]For $\pi = \frac{1}{2}$ and an investment return close to 1, $R = 1.05$, the situation is qualitatively similar to the example with $R = 3$, except that the range of deposit levels admitting the allocation $x^*$, but for which there is also a run equilibrium, is narrower. This range is $d \in [0.164736, 0.168117]$.

The last term in brackets is due to the fact that a patient consumer who runs does not liquidate her mutual fund until period 2, and therefore receives the additional consumption, $(1 - d)(R - 1)$. Overall welfare is given by $W(C, d, s)$, where we have

$$
\begin{aligned}
W(C, d, s) &= (1 - s)\widehat{W}(C, d) + sW^R(C, d) \quad \text{if } (C, d) \text{ allows a run equilibrium} \\
&= \widehat{W}(C, d) \quad \text{if } (C, d) \text{ does not allow a run equilibrium.}
\end{aligned}
$$

Following Peck and Shell (2003), the optimal contract, taking into account the propensity to run, is the solution to the problem of maximizing $W(C, d, s)$, subject to (3.1), (3.2), and (3.3). We refer to this contract as the $s$-optimal contract.

If $d^*_{NR} < 1$ holds, the situation is quite simple for the baseline model. Any deposit level, $d > d^*_{NR}$, along with the withdrawal schedule yielding the allocation $x^*$, is an $s$-optimal contract for $s > 0$. That is, the optimal allocation $x^*$ can be implemented, without risk of a run equilibrium for the post-deposit subgame. Deposit levels between $d^*$ and $d^*_{NR}$, where the post-deposit subgame also has a run equilibrium, are optimal if the propensity to run is zero, but they cannot be $s$-optimal for $s > 0$.

If $d^*_{NR} = 1$ holds, the situation for the baseline model is more interesting. Now the $s$-optimal contract depends on $s$, and welfare will be strictly less than the welfare associated with the allocation, $x^*$.[8] If $s$ is large, then the $s$-optimal contract will eliminate the possibility of a run, by reducing the amount of insurance offered against being impatient. For smaller $s$, then the $s$-optimal contract will tolerate a positive probability of runs on the equilibrium path. Shell and Zhang (2018a) perform an elegant and thorough analysis of economies with two consumers and full deposits, characterizing the optimal contract that takes into account a positive propensity to run. An important takeaway from Shell and Zhang (2018a) is that, when the incentive compatibility constraint is not binding at the optimal contract with a zero propensity to run, then for a small positive propensity to run, the optimal overall contract balances welfare when a run occurs and welfare when a run does not occur. Thus, the probability of a run affects the contract. Here, for $d > d^*$, incentive compatibility does not bind, and we would expect to see the Shell and Zhang (2018a) finding that the run probability affects the optimal

---

[8]Welfare will be strictly less, because either (i) a run will occur on the equilibrium path, and when a run occurs, welfare is even less than welfare under autarky, or (ii) the contract will be altered so that the post-deposit subgame does not have a run equilibrium, in which case welfare is below the level associated with $x^*$.

contract. We do not attempt a general characterization, but we can show the following limiting result.

**Proposition 4:** *Suppose that the s-optimal contract with $s = 0$ and deposit $d = 1$ (yielding allocation $x^*$) has a run equilibrium in the post-deposit subgame. Consider a convergent sequence of s-optimal contracts, $(C^s, d^s)$, as $s > 0$ converges to zero along the sequence. Denote $\lim_{s \to 0}(C^s, d^s) = (C^0, d^0)$. Then we have $d^0 = d^*$ and the allocation in the no-run equilibrium converges to $x^*$.*

Proposition 4 shows that, as the positive propensity to run converges to zero, the $s$-optimal contract tolerates runs on the equilibrium path, and requires the minimum deposit level to achieve $x^*$. Intuitively, there is no harm in reducing $d$ when a run does not occur, but reducing $d$ is strictly beneficial when a run occurs, due to the fact that the patient have more investments that do not have to be liquidated. The proof of Proposition 4 follows the techniques developed in the next section for the full model, so to avoid duplication the proof appears at that point.

## 5. The Full Model

For the interesting case, in which $d^* < d^*_{NR} < 1$ holds, full deposits are enough to implement the allocation $x^*$, so runs do not occur on the equilibrium path. It would seem, then, that there is no reason for the bank not to require consumers to deposit their entire endowments. In this section, we consider the full model, which includes the feature that long-term investments made by banks might yield a lower net return than mutual funds. We do not endogenize this difference in returns by modeling monitoring costs or reserve requirements. Instead, we introduce a non-negative parameter, $\varepsilon$. Bank investment held until period 2 yields the return, $R$, but outside investment held until period 2 now yields the return, $R + \varepsilon$. For $\varepsilon > 0$, the equivalence result given in Proposition 1 no longer holds. Indeed, allocations yielding welfare higher than $x^*$ (associated with $d = 1$) are feasible and incentive compatible, since the patient benefit from the higher return on mutual funds.

For the full model, welfare, conditional on a run not taking place, is denoted

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by $\widehat{W}(C, d, \varepsilon)$, given by

$$
\begin{aligned}
\widehat{W}(C, d, \varepsilon) \;=\; & \sum_{\alpha=0}^{N-1} f(\alpha) \left[ \sum_{z=1}^{\alpha} u(x_1(z, d)) + (N - \alpha) u(x_2(\alpha, d, \varepsilon)) \right] \\
& + f(N) \left[ \sum_{z=1}^{N} u(x_1(z, d)) \right].
\end{aligned}
$$

The dependence on $\varepsilon$ is due to the fact that $x_2(\alpha_1, d, \varepsilon) = (1-d)(R+\varepsilon) + c_2(\alpha_1, d)$ depends on $\varepsilon$. The incentive compatibility constraint is now given by

$$
\sum_{\alpha=0}^{N-1} f_p(\alpha) \left[ \frac{1}{1+\alpha} \sum_{z=1}^{\alpha+1} u((1-d)(R+\varepsilon-1) + x_1(z, d)) \right] \leq \sum_{\alpha=0}^{N-1} f_p(\alpha) u(x_2(\alpha, d, \varepsilon)).
$$

$$(5.1)$$

Welfare, conditional on a run taking place, is denoted by $W^R(C, d, \varepsilon)$, given by

$$
W^R(C, d, \varepsilon) = \sum_{\alpha=0}^{N} f(\alpha) \left[ \frac{\alpha}{N} \sum_{z=1}^{N} u(x_1(z, d)) + \frac{N-\alpha}{N} \sum_{z=1}^{N} u((1-d)(R+\varepsilon-1) + x_1(z, d)) \right].
$$

Overall welfare is given by $W(C, d, s, \varepsilon)$, where

$$
\begin{aligned}
W(C, d, s, \varepsilon) \;=\; & (1-s)\widehat{W}(C, d, \varepsilon) + s W^R(C, d, \varepsilon) \quad \text{if } (C, d, \varepsilon) \text{ allows a run equilibrium} \\
\;=\; & \widehat{W}(C, d, \varepsilon) \qquad\qquad\qquad\qquad\quad \text{if } (C, d, \varepsilon) \text{ does not allow a run equilibrium.}
\end{aligned}
$$

The $s$-optimal contract is the solution to the following problem

$$
\begin{aligned}
& \max_{C, d} W(C, d, s, \varepsilon) \\
& \text{subject to} \\
& (3.1), (3.2), (5.1) \\
& c_1(z, d) \;\geq\; 0.
\end{aligned}
$$

$$(5.2)$$

It will be useful to fix the deposit level and consider the optimal withdrawal schedule that solves (5.2).

**Proposition 5:** *Fix the deposit level, $d$, and consider the set of withdrawal schedules, $C_d^*(d, s, \varepsilon)$, that solve (5.2) where $d$, $s$, and $\varepsilon$ are all treated as parameters*

*within parameter space denoted by* $\Theta$. *Then the correspondence,* $C_d^*$, *is upper hemi-continuous and compact-valued, and the associated welfare is continuous, in* $\Theta$.

**Proof.** For fixed $(d, s, \varepsilon)$, the set of contracts satisfying the constraints is continuous and compact-valued. Also, $W(C, d, s, \varepsilon)$ is a continuous function. By the Theorem of the Maximum, the correspondence $C_d^*$ is upper hemi-continuous in $\Theta$, and the associated welfare is a continuous function of $\Theta$. ∎

**Remark 1:** Although $C_d^*$ is not necessarily single-valued, we know from the uniqueness of $x^*$ that $C_d^*$ is single-valued when $s = 0$ and $\varepsilon = 0$ hold, for all $d \geq d^*$. The reason is that the withdrawal schedule,

$$
\begin{aligned}
c_1(z, d) &= x_1^*(z) - (1 - d) \text{ for all } z \text{ and} \\
c_2(\alpha_1, d) &= x_2^*(\alpha_1) - (1 - d)R \text{ for all } \alpha_1,
\end{aligned}
\tag{5.3}
$$

is the unique schedule yielding the allocation $x^*$, and non-negativity and incentive compatibility constraints are satisfied, since we have $d \geq d^*$. The schedule (5.3) must be in $C_d^*$, because if there were another withdrawal schedule that yielded higher welfare than (5.3), the allocation could be achieved with $d = 1$, by Proposition 1, contradicting the definition of $x^*$. Similarly, if there were a second schedule in $C_d^*$, it must yield an allocation distinct from $x^*$, and this allocation could be achieved with $d = 1$, contradicting the uniqueness of $x^*$.

The following proposition compares the optimal withdrawal schedules associated with deposit levels $d'$ and $d''$, where $d'' < d'$ holds, where $s$ and $\varepsilon$ are small. Welfare under $d''$ is strictly higher than welfare under $d'$ whenever we have $d^* < d'' < d' < d_{NR}^* < 1$ or $d^* < d_{NR}^* < d'' < d' < 1$.

**Proposition 6:** *Assume that* $d'' < d'$, $s > 0$, *and* $\varepsilon > 0$ *hold. Let* $C' \in C_d^*(d', s, \varepsilon)$ *and* $C'' \in C_d^*(d'', s, \varepsilon)$ *hold. Then for* $s$ *and* $\varepsilon$ *sufficiently small, and if we have either (i)* $d^* < d'' < d' < d_{NR}^* < 1$ *or (ii)* $d^* < d_{NR}^* < d'' < d' < 1$, *welfare is strictly higher under the lower deposit level,* $W(C'', d'', s, \varepsilon) > W(C', d', s, \varepsilon)$.

**Proof.** Denote the withdrawals under $C'$ by $c_1^*(z, d')$ and $c_2^*(\alpha_1, d')$. Consider the withdrawal schedule for deposit level $d''$ given by

$$
\begin{aligned}
c_1(z, d'') &= c_1^*(z, d') + d'' - d' \\
c_2(\alpha_1, d'') &= c_2^*(\alpha_1, d') + (d'' - d')R.
\end{aligned}
\tag{5.4}
$$

(i) Suppose that $d^* < d'' < d' < d^*_{NR} < 1$ holds. For sufficiently small $s$ and $\varepsilon$, we know from Proposition 5 that $C'$ must yield an allocation close to $x^*$ in the no-run equilibrium of the post-deposit subgame, and the post-deposit subgame also has a run equilibrium. Note that the schedule, (5.4), is not necessarily in $C^*_d(d'', s, \varepsilon)$, but it gives the same consumption to the impatient consumers in the no-run equilibrium as does $C'$ when the deposit level is $d'$. Since $C'$ satisfies the resource constraint, so does the schedule, (5.4), using the logic in the proof of Proposition 1.

When $s$ and $\varepsilon$ are small enough, by Proposition 5, the allocation under $(C', d')$ in the no-run equilibrium will be arbitrarily close to $x^*$. Since $d^* < d''$ holds, and since the schedule, (5.4), with deposit level $d''$, also yields an allocation arbitrarily close to $x^*$, we know that the schedule, (5.4), will satisfy non-negativity and incentive compatibility.

With probability $1 - s$, the no-run equilibrium occurs in the post-deposit subgame. Welfare conditional on the no-run equilibrium is strictly higher under (5.4) than under $C'$, due to the higher consumption of the patient consumers. With probability $s$, a run equilibrium occurs in the post-deposit subgame. Welfare conditional on the run equilibrium is strictly higher under (5.4) than under $C'$, because a patient consumer's consumption is higher under (5.4) than under $C'$ by exactly $(d' - d'')(R + \varepsilon - 1)$. Therefore, welfare is strictly higher under (5.4) than under $C'$. Welfare under $C'''$, an optimal schedule with deposit $d''$, is weakly higher than under (5.4), so the result follows.

(ii) Suppose that $d^* < d^*_{NR} < d'' < d' < 1$ holds. For sufficiently small $s$ and $\varepsilon$, we know from Proposition 5 that schedules $C'$ and (5.4) must yield allocations close to $x^*$ in the no-run equilibrium of the post-deposit subgame. Since we have $d^*_{NR} < d'' < d'$, the post-deposit subgames associated with these contracts do not have a run equilibrium. By the argument given in part (i), the resource, non-negativity, and incentive compatibility constraints are satisfied for the schedule (5.4). The no-run equilibrium occurs with probability one under $C'$ and (5.4), but welfare is strictly higher under (5.4), due to the fact that a patient consumer receives higher consumption, by exactly $(d' - d'')\varepsilon$. Welfare under $C'''$, an optimal schedule with deposit $d''$, is weakly higher than under (5.4), so the result follows. ∎

Denote the set of fully optimal contracts solving (5.2) as $C^*(s, \varepsilon)$. We are now ready to show that, if $s$ and $\varepsilon$ are positive but sufficiently small, and if $d^* < d^*_{NR} < 1$ holds, then any optimal contract $(d, C) \in C^*(s, \varepsilon)$ will have a deposit level either close to $d^*$ or close to $d^*_{NR}$. The intuition is that the allocation

will be close to $x^*$; either the (small) benefit of the higher mutual fund return justifies tolerating runs, in which case the deposit is close to $d^*$, or the (small) propensity to run justifies reducing the deposit level only to the minimum level that avoids a run equilibrium to the post-deposit subgame, in which case the deposit level is close to $d^*_{NR}$.

**Proposition 7:** *Assume that $d^* < d^*_{NR} < 1$ holds. Consider a sequence of strictly positive propensities to run and mutual fund return parameters, indexed by $\nu$ and a scalar, $\lambda$, such that $s^\nu \to 0$, $\varepsilon^\nu \to 0$, and $s^\nu = \lambda \varepsilon^\nu$ hold. Assume without loss of generality that the sequence of $s^\nu$-optimal contracts converges, $(C^\nu, d^\nu) \in C^*(s^\nu, \varepsilon^\nu)$ converges to $(C^0, d^0)$. Then we have either $d^0 = d^*$ or $d^0 = d^*_{NR}$. If $\lambda$ is sufficiently small, we have $d^0 = d^*$, and along the sequence, there is a positive probability of bank runs at the $s^\nu$-optimal contract.*

**Proof.** Suppose $d^0 > d^*_{NR}$ holds. For $\nu$ close to the limit, the optimal contract must have a deposit level $d^\nu$ close to $d^0$, so $d^\nu > d^*_{NR}$ holds. By Proposition 6, there is a $d' \in (d^*_{NR}, d^\nu)$ and a withdrawal schedule $C' \in C^*_d(d', s^\nu, \varepsilon^\nu)$ for which $W(C', d', s^\nu, \varepsilon^\nu) > W(C^\nu, d^\nu, s^\nu, \varepsilon^\nu)$ holds, contradicting the fact that $(C^\nu, d^\nu) \in C^*(s^\nu, \varepsilon^\nu)$ must hold. Intuitively, the deposit level can be reduced while still avoiding a bank run equilibrium to the post-deposit subgame, yielding higher welfare.

Suppose $d^0 \in (d^*, d^*_{NR})$ holds. For $\nu$ close to the limit, the optimal contract must have a deposit level $d^\nu$ close to $d^0$, so $d^\nu \in (d^*, d^*_{NR})$ holds. By Proposition 6, there is a $d' \in (d^*, d^\nu)$ and a withdrawal schedule $C' \in C^*_d(d', s^\nu, \varepsilon^\nu)$ for which $W(C', d', s^\nu, \varepsilon^\nu) > W(C^\nu, d^\nu, s^\nu, \varepsilon^\nu)$ holds, contradicting the fact that $(C^\nu, d^\nu) \in C^*(s^\nu, \varepsilon^\nu)$ must hold. Intuitively, the deposit level can be reduced without affecting the probability of a run taking place, yielding higher welfare conditional on the run equilibrium being selected and conditional on the no-run equilibrium being selected.

For $\nu$ close to the limit and fixed $(s^\nu, \varepsilon^\nu)$, denote the contract that solves (5.2)), among contracts for which a run equilibrium does not exist for the post-deposit subgame, by $(C', d')$, and denote the corresponding allocation (including consumption from mutual funds) by $x'_1(z, d)$) and $x'_2(\alpha_1, d)$. We know that $d'$ is close to $d^*_{NR}$ and the allocation is close to $x^*$. Denote the contract that solves (5.2)), among contracts for which a run equilibrium exists for the post-deposit subgame, by $(C'', d'')$. We know that $d''$ is close to $d^*$ and the allocation is close to $x^*$.

From the argument in the proof of Proposition 6, the difference in welfare, between contract $(C'', d'')$ and contract $(C', d')$, is at least what would be obtained if $C''$ offered the same consumption to the impatient consumers as $(C', d')$. This difference in welfare is given by

$$(1-s)\sum_{\alpha=0}^{N-1} f(\alpha)(N-\alpha)\left[u(x_2(\alpha, d') + (d' - d'')\varepsilon^\nu) - u(x_2(\alpha, d'))\right] \quad (5.5)$$

$$-s\left[W^R(C', d', \varepsilon^\nu) - \widehat{W}(C', d', \varepsilon^\nu)\right].$$

Expression (5.5) captures the tradeoff, between having more investment earning the higher return when a run does not take place, vs. the loss of welfare when a run takes place. Substituting $s^\nu = \lambda\varepsilon^\nu$ and dividing by $\varepsilon^\nu$, we can write the welfare advantage, per unit of $\varepsilon^\nu$, as

$$\frac{(1-\lambda\varepsilon^\nu)}{\varepsilon^\nu}\sum_{\alpha=0}^{N-1} f(\alpha)(N-\alpha)\left[u(x_2(\alpha, d') + (d' - d'')\varepsilon^\nu) - u(x_2(\alpha, d'))\right] (5.6)$$

$$-\lambda\left[\widehat{W}(C', d', \varepsilon^\nu) - W^R(C', d', \varepsilon^\nu)\right].$$

Taking limits as $\varepsilon^\nu$ approaches zero, the first term in expression (5.6) approaches

$$(d_{NR}^* - d^*)\sum_{\alpha=0}^{N-1} f(\alpha)(N-\alpha)\left[u'(x_2^*(\alpha))\right],$$

which is strictly positive. The second term in expression (5.6) approaches zero as $\lambda$ approaches zero, which implies that welfare associated with tolerating runs (and deposit level approaching $d^*$) exceeds welfare associated with eliminating runs (and deposit level approaching $d_{NR}^*$). ∎

The proof of Proposition 4 now follows easily by slightly modifying the arguments given above and imposing $\varepsilon = 0$.

**Proof of Proposition 4.** Clearly, $d^0 < d^*$ is impossible, because near the limit, either incentive compatibility fails or the allocation provides lower welfare than $x^*$. Suppose we have $d^0 > d^*$. Consider an optimal contract near the limit as $s$ approaches 0, denoted by $(C^s, d^s)$, and denote the corresponding withdrawals by $c_1^*(z, d^s)$ and $c_2^*(\alpha_1, d^s)$. The continuity result of Proposition 5 implies $d^s > d^*$.

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Based on the construction given in the proof of Proposition 6, we can find a lower deposit level, $d'' \in (d^*, d^s)$, and a withdrawal schedule, $C'''$, such that the impatient consumers receive the same consumption, in the no-run equilibrium, that they receive in the no-run equilibrium under $(C^s, d^s)$. That is, we have

$$
\begin{aligned}
c_1(z, d'') &= c_1^*(z, d^s) + d'' - d^s \\
c_2(\alpha_1, d'') &= c_2^*(\alpha_1, d^s) + (d'' - d^s)R.
\end{aligned}
$$

In the no-run equilibrium, which occurs with probability $(1-s)$, consumption is the same across both contracts, since we have $\varepsilon = 0$. In the run equilibrium, which occurs with probability $s$, the patient consumers consume more under $(C''', d'')$ than under $(C^s, d^s)$, by exactly $(d^s - d'')(R - 1)$. This contradicts the optimality of $(C^s, d^s)$. Therefore, we have $d^0 = d^*$. ∎

**Remark 2:** What if $d^* < d_{NR}^* < 1$ does not hold? First, suppose that the post-deposit subgame has a run equilibrium with full deposits at the optimal contract yielding the allocation $x^*$, so we have $d_{NR}^* = 1$. Then if $s$ and $\varepsilon$ are positive but sufficiently small, any optimal contract $(C, d) \in C^*(s, \varepsilon)$ will have a deposit level close to $d^*$. There is no tradeoff between run probability and investment return, since the post-deposit subgame always has a run equilibrium. Second, suppose the post-deposit subgame does not have a run equilibrium at deposit level $d^*$, so we have $d_{NR}^* = d^*$. Then, again, if $s$ and $\varepsilon$ are positive but sufficiently small, any optimal contract $(C, d) \in C^*(s, \varepsilon)$ will have a deposit level close to $d^*$. There is no tradeoff between run probability and investment return, because the post-deposit subgame never has a run equilibrium when the allocation is close to $x^*$.

The propositions of this section assume that $\varepsilon$ is small, to take advantage of the fact that the optimal allocation will be close to the $x^*$ of the baseline model, corresponding to $\varepsilon = 0$. However, when $\varepsilon$ is significantly positive, there is even *more* of an incentive to reduce bank deposits to the point that incentive compatibility or non-negativity binds, thereby increasing the fragility of the banking system. To illustrate this point, consider again the example of the previous section, with $\pi = \frac{1}{2}$ and $R = 3$. If we impose $d = 1$, then there is no run equilibrium, but this deposit level is not optimal when the propensity to run is small. Table 1 in the Appendix shows the optimal contract when the propensity to run is zero, and the optimal contract for which there is no run equilibrium, for $\varepsilon = \frac{1}{8}$ and $\varepsilon = \frac{1}{4}$. In both cases, the optimal contract will tolorate a positive probability of a run when the propensity is sufficiently small.

## 6. Summary and Discussion

We have extended the Diamond-Dybvig model to include a choice of *how much* to deposit. For typical economies studied in the literature in which impatient and patient consumers have the same utility function and consumers deposit their entire endowments, incentive compatibility does not bind in the no-run equilibrium under the optimal contract. If this is the case, then the efficient allocation, $x^*$, can also be achieved in an equilibrium of the present model, where consumers deposit only a fraction of their endowments with the bank, according to the equivalence result given in Proposition 1. The bank offers a contract that magnifies the extent to which impatient consumers can withdraw more than their deposit (on average), in order to provide the allocation $x^*$ once the non-deposited investments are considered. However, the less consumers deposit, the more tempted patient consumers are to withdraw early. It is easy to construct examples in which there does not exist a run equilibrium (to the subgame after the bank chooses the optimal contract) when consumers deposit their entire endowment, but there does exist a run equilibrium when consumers deposit only a fraction of their endowment.

When there is a positive propensity to run, $s$, and if the optimal allocation $x^*$ can be implemented as the unique equilibrium of the post-deposit subgame with full deposits, then there is no reason to tolerate runs on the equilibrium path in the baseline model. However, if achieving the allocation $x^*$ entails the existence of a run equilibrium in the post-deposit subgame, even with full deposits, then our equivalence result no longer holds. If $s$ is small enough, the optimal contract yields an allocation close to $x^*$ in the no-run equilibrium, but the deposit level must be close to $d^*$.

In the full model, there is a tradeoff between the enhanced stability associated with higher deposit levels and the higher return associated with investing more in mutual funds, $\varepsilon$. Many other extensions, such as a small probability of embezzlement by the bank, would exhibit a similar tradeoff and yield similar results. For the "typical" case in which $d^* < d^*_{NR} < 1$ holds, and when $\varepsilon$ and $s$ are small, the optimal contract will either economize on deposits to the point of tolerating runs ($d$ near $d^*$), or it will economize on deposits within the range that eliminates runs ($d$ near $d^*_{NR}$). When $s$ is small relative to $\varepsilon$, there will be a positive probability of runs on the equilibrium path. Shell and Zhang (2018a) perform an elegant and thorough analysis of a model with full deposits and two consumers, characterizing the optimal contract that takes into account a positive propensity to run. An important takeaway from Shell and Zhang (2018a) is that, when the

incentive compatibility constraint is not binding at the optimal contract with a zero propensity to run, then for a small positive propensity to run, the optimal overall contract balances welfare when a run occurs and welfare when a run does not occur. Thus, the probability of a run affects the contract. Here, for $d > d^*$, incentive compatibility does not bind, and we would expect to see the Shell and Zhang (2018a) finding that the run probability affects the optimal contract.

In this paper, our equilibrium selection is based on a single parameter, the propensity to run, $s$. It is important to note that $s$ is not a parameter of the economic environment, such as $\varepsilon$. More generally, the propensity to run could depend on the deposit level. Ennis and Keister (2005) have an interesting paper exploring optimal policies when the policy choice affects the equilibrium selection of the subgame, using the notion of risk dominance. In the context of our full model, it would be interesting to study the case in which the propensity to run is higher if the temptation to join a run (Definition 1) is higher. For example, suppose that non-negativity never binds, so $d^* = d^*_{IC} > d^*_{NN}$ holds. Then it is reasonable to require the propensity to run to be 1 at $d = d^*$, because there is no advantage of waiting even if all other patient consumers wait. Similarly, it is reasonable to require the propensity to run to be 0 at $d = d^*_{NR}$, because there is no advantage of running even if everyone else runs. If the propensity to run is a continuous function of $d$, then the fully optimal contract will specify a deposit level strictly between $d^*$ and $d^*_{NR}$.

## 7. Appendix

Consistent with the results derived for small $\varepsilon$, we show here that the properties of the optimal contract extend to large epsilon. We consider the example with 2 consumers, CRRA utility with risk aversion parameter 2, $\pi = \frac{1}{2}$, $R = 3$, and $\varepsilon$ taking the values of $\frac{1}{8}$ and $\frac{1}{4}$. For these parameters, $d^* < d^*_{NR} < 1$ holds, and the thresholds appear close to those for $\varepsilon = 0$. Table 1 presents the optimal contract, the optimal contract for which there is no run equilibrium, the corresponding allocation, and the corresponding welfare.

|  | $\varepsilon = \frac{1}{8}$ | No-Run, $\varepsilon = \frac{1}{8}$ | $\varepsilon = \frac{1}{4}$ | No-Run, $\varepsilon = \frac{1}{4}$ |
|---|---|---|---|---|
| $d$ | 0.15192 | 0.21142 | 0.15256 | 0.20878 |
|  |  |  |  |  |
| $c_1(1,d)$ | 0.29842 | 0.35493 | 0.29984 | 0.35038 |
| $c_1(2,d)$ | 0.00542 | 0.06791 | 0.00529 | 0.06719 |
| $c_2(0,d)$ | 0.45575 | 0.63425 | 0.45769 | 0.62635 |
| $c_2(1,d)$ | 0.01625 | 0.20372 | 0.01586 | 0.20157 |
|  |  |  |  |  |
| $x_1(1)$ | 1.14650 | 1.14351 | 1.14728 | 1.14159 |
| $x_1(2)$ | 0.85350 | 0.85649 | 0.85272 | 0.85841 |
| $x_2(0)$ | 3.10601 | 3.09857 | 3.21186 | 3.19780 |
| $x_2(1)$ | 2.66651 | 2.66804 | 2.77003 | 2.77303 |
|  |  |  |  |  |
| $W$ | $-1.29557$ | $-1.29653$ | $-1.28308$ | $-1.28488$ |

Table 1

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