Part 2

Cooperative Game Theory

CHAPTER 3

Coalitional games

A coalitional game is a model of interacting decision makers that focuses on the behaviour of groups of players. Each group of players is called a *coalition* and the coalition of all players is called the *grand coalition*.

DEFINITION 3.1. Given a set of players $N = \{1, 2, \ldots, N\}$, a *coalition* is a nonempty subset of N. The set N itself is called the *grand coalition*. The collection of all coalitions for a given player set N is denoted $C(N)$.

		Player 2 Q	Player 2 F
Player 1	Q	$2, 2$	$0, 3$
	F	$3, 0$	$1, 1$

FIGURE 1. The prisoners' dilemma.

Consider for example the prisoners' dilemma game, repeated here in Figure 1. We already know that (F, F) with corresponding payoffs of 1 to each player is the only Nash equilibrium of this game. But what if the players could somehow cooperate and write a contract that specified how each was going to play? The solid dots in Figure 2 show the four possible contracts. If we allow for randomisations over all such contracts, the shaded area shows all possible expected utilities of each player, with Player 1's expected utility plotted horizontally and Player 2's expected utility plotted vertically.

In the prisoners' dilemma example we have $N = \{1, 2\}$. For coalitional games we also specify a *worth function*, $v(\cdot)$ which gives the payoffs that any coalition of players can get. For this example we have $v(\{1\}) = \{1\}$, $v(\{2\}) = \{1\}$ and $v(\{1, 2\}) =$ anything in the grey set in Figure 2. This is an example of a game without side payments, also called a game with non-transferrable utility (NTU). For the rest of this chapter we'll deal with games with side payments, or transferrable utility (TU) games. If side payments are allowed in the prisoners' dilemma game with contracts, all the points on the diagonal line from $(0, 4)$ to $(4, 0)$ in Figure 2 are possible.

DEFINITION 3.2. A *transferrable utility coalitional game* is a pair (N, v) where $N = \{1, 2, \ldots, N\}$ is a set of players and $v : C(N) \to \mathbb{R}$ is the worth function.

For example, in the prisoners' dilemma cooperative game with transferrable utility, $N = \{1, 2\}$, $v(1) = 1$, $v(2) = 1$, and $v(1, 2) = 4$. At this point, let us give some other example of transferrable utility coalitional games. We shall refer back to these examples.

(1) **Gloves**. $N = L \cup R$, i.e. L is the set of left-handed gloves that you have and R is the set of right-handed gloves. Then $v(S) = \min\{\#S \cap L, \#S \cap R\}$

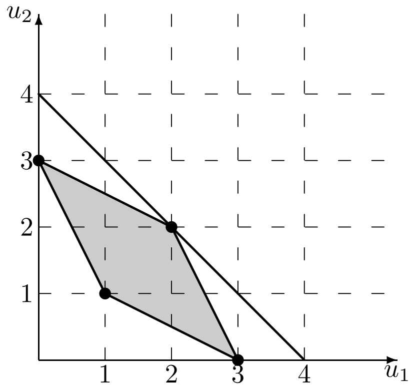

FIGURE 2. Possible expected utilities if contracts can be used in the Prisoners' Dilemma game. The four black dots show the possible contracts and if randomisations over these contracts are allowed the grey area shows all possible expected payoffs. If side payments are allowed then all the points on the diagonal line from $(0, 4)$ to $(4, 0)$ are possible.

for some coalition S. That is, only pairs of gloves are valuable. So for any given coalition of gloves, the value of the coalition is the minimum of the number of left-handed and right-handed gloves in the coalition.

(2) **Security Council**. $N = P \cup O$ where P is the set of permanent members of the security council and O is the number of other members. In this game, $v(S)$ equals 1 if $P \subset S$ and $\#S \geq 7$, and equals 0 otherwise. Thus $v(N) = 1$.

CHAPTER 4

The core

For coalitional games such as gloves and security council, we would like to be able to say what the outcome is likely to be. To do this, we need a solution concept. One such concept is called the *core*. Basically, an outcome is in the core of a game if no subset of players could make themselves all better off by breaking away and playing the game amongst themselves.

DEFINITION 4.1. An *allocation*, x, is a vector $(x_1, x_2, \ldots, x_N)$ that is feasible, i.e., $\sum_{i=1}^{N} x_i \leq v(N)$, and individually rational, i.e., $x_n \geq v(n)$.

The definition of an allocation says that what we give to everyone cannot be greater than what is available in total, and what everyone gets in the allocation must make them at least as well off as they could be by themselves.

DEFINITION 4.2. An allocation $x = (x_1, x_2, \ldots, x_N)$ is in the *core* if for every $S \subseteq N$,

$$\sum_{n \in S} x_n \geq v(S).$$

The definition of allocations that are in the core says that every such allocation must make all the members of every possible coalition at least as well off as they would be in the coalition itself.

As an illustration of the core, consider the gloves game with $N = \{1, 2, 3\}$, $L = \{1, 2\}$ and $R = \{3\}$. Thus $v(1) = v(2) = v(1, 2) = v(3) = 0$, and $v(1, 3) = v(2, 3) = v(1, 2, 3) = 1$. Thus if an allocation (x_1, x_2, x_3) is in the core, we must have $x_1 \geq 0$, $x_2 \geq 0$, $x_3 \geq 0$, $x_1 + x_2 + x_3 = v(N) = 1$, $x_1 + x_3 \geq 1$, $x_2 + x_3 \geq 1$, and $x_1 + x_2 \geq 0$. Note that $x_1 + x_3 \geq 1$ and $x_2 + x_3 \geq 1$ together imply that $x_1 + x_2 + 2x_3 \geq 2$ and since $x_1 + x_2 + x_3 = 1$, we must have $x_3 \geq 1$, $x_1 \geq 0$ and $x_2 \geq 0$. Thus the only allocation that can be in the core is $(0, 0, 1)$.

As another example, consider the following *simple majority game*. Let $N = \{1, 2, 3\}$, then

$$v(S) = \begin{cases} 1 & \text{if } \#S \geq 2 \\ 0 & \text{otherwise} \end{cases}.$$

In this game if an allocation (x_1, x_2, x_3) is in the core, we must have $x_1 \geq 0$, $x_2 \geq 0$, $x_3 \geq 0$, $x_1 + x_2 + x_3 = 1$, $x_1 + x_2 \geq 1$, $x_1 + x_3 \geq 1$, and $x_2 + x_3 \geq 1$. Note that the last three of these inequalities together imply $2(x_1 + x_2 + x_3) \geq 3$ but since $x_1 + x_2 + x_3 = 2$, we have $2 \geq 3$, which is clearly not true. Thus the core of this game is the empty set, i.e., there are no allocations in the core. This example with the empty core suggests that the core may be a good solution concept for economic problems but perhaps not for political problems.

DEFINITION 4.3. A *simple game* is a game (N, v) with $v(N) = 1$ and $v(S)$ either 0 or 1 for all coalitions S.

In a simple game, a coalition S for which $v(S) = 1$ is called a *winning coalition*. A player who is in every winning coalition is called a *veto player*.

Examples:

(1) $N = \{1, 2, 3\}$ and let $v(1, 2, 3) = 1$ and $v(S) = 0$ for all other coalitions S. This is a simple game and in this game every player is a veto player. The core of this game is the set $\{(x_1, x_2, 1 - x_1 - x_2) \,|x_n \geq 0\}$.
(2) $N = \{1, 2, 3\}$, $v(1, 2, 3) = v(1, 2) = v(1, 3) = v(2, 3) = 1$, and $v(1) = v(2) = v(3) = 0$. This is a simple game and in this game there are no veto players. The core of this game is empty.
(3) $N = \{1, 2, 3\}$, $v(1, 2, 3) = v(1, 2) = v(1, 3) = 1$, and $v(2, 3) = v(1) = v(2) = v(3) = 0$. This is a simple game and Player 1 is the only veto player. The core of this game is $(1, 0, 0)$.

You may have noticed that in the above three examples, only the games in which there is at least one veto player have non-empty cores. In fact this is a general result for simple games.

THEOREM 4.1. *A simple game has a non-empty core if and only it has veto players.*

PROOF. It's easy, try to do it yourself. □

Note also that if a simple game has veto players then the core is the set of individually rational allocations that give everything to the veto players.

CHAPTER 5

The Shapley value

Consider again example 3 from the previous section. In that game, the only allocation in the core is $(1, 0, 0)$, i.e., Player 1 gets everything. In other words, the marginal worth of what players 2 and 3 contribute is zero. But you wouldn't expect in such a game that Player 1 would get everything, since Player 1 by himself is worth zero, i.e., $v(1) = 0$. This kind of problem leads us to an alternative solution concept called the *Shapley value*. The Shapley value is an axiomatic solution concept. That is, we define some things (axioms) that we think a solution concept should satisfy. Then we show that the Shapley value satisfies these axioms (in fact, it's the only thing that satisfies them).

Before giving the axioms, let us explain what the Shapley value is. Consider again example 3 from above. A *pivotal player* is a player who, when added to a coalition, turns that coalition from a losing coalition into a winning one. In example 3, let's list all the possible orderings of players and list who is the pivotal player in each case. This is shown in Figure 1.

The Shapley value of player n is then defined to be

$$\phi_n = \frac{\#\text{ orders in which } n \text{ is pivotal}}{\text{total } \#\text{ orders}}.$$

Thus for this game we have $\phi_1 = \frac{4}{6} = \frac{2}{3}$, $\phi_2 = \frac{1}{6}$ and $\phi_3 = \frac{1}{6}$. So, the Shapley value for the game is $\phi(v) = \left(\frac{2}{3}, \frac{1}{6}, \frac{1}{6}\right)$.

Order	**Pivotal Player**
123	2
213	1
231	1
132	3
312	1
321	1

FIGURE 1. The possible player orderings and the pivotal player in each case from example 3 in the previous section.

Now let us give the axioms that the Shapley value satisfies. First, suppose in a game (N, v) it's true that for any coalition S, $v(S) = v(T \cap S)$ for some coalition T, and that $v(\emptyset) = 0$. That is, all the action in the game happens in the coalition T. Then we call coalition T a *carrier* of the game (N, v). Now we specify our first axiom that we think solution concepts should satisfy.

AXIOM 5.1. If S is a carrier of (N, v) then

$$\sum_{n \in S} f_n(v) = v(S)$$

where f_n is the solution function, which tells what each player gets in the solution.

Axiom 5.1 says that if S is a carrier of the game then what the members of S get in the solution should be the same as what they could get as a group by themselves.

A *permutation* of the player set N is a one-to-one and onto function $\pi : \{1, 2, \ldots, N\} \to \{1, 2, \ldots, N\}$.[1] That is, the function π just re-labels the players with different numbers. Given a game (N, v) we define the *permuted game* , $(N, \pi v)$ by $\pi v(S) = v\left(\pi^{-1}(S)\right)$ where π^{-1} 'undoes' the permutation, i.e.,

$$\pi^{-1}(S) = \{m \in N | m = \pi(n) \text{ for some } n \in S\}.$$

Axiom 5.2. For any permutation π and any $n \in N$, $f_{\pi(n)}(\pi v) = f_n(v)$.

Axiom 5.2 says that changing the labels of the players (e.g., calling player '1' player '2' and calling player '2' player '1') should not essentially change the solution, i.e., the players should get the same payoff in the solution even when their 'names' are swapped around.

Axiom 5.3. For any two games (N, v) and (N, u) with $(N, (u + v))$ defined by $(u + v)(S) = u(S) + v(S)$ and $f_n(u + v) = f_n(u) + f_n(v)$.

Axiom 5.3 is a technical thing and we won't try to explain it. Anyway, amazingly, it can be proved that the solution concept called the Shapley value is the only solution that satisfies Axioms 5.1 – 5.3.

At this point, there are some useful observations that we can make from the axioms. First, axiom 5.1 implies that if n is a dummy player, that is if $v(S \cup \{n\}) = v(S)$ for any coalition S, then $\phi_n(v) = 0$. That is, dummy players do not get any payoff in the Shapley value. Second, axiom 5.1 also implies that $\sum_{n \in N} \phi_n(v) = v(N)$, that is, that there is no wastage in the Shapley value, i.e., all of the available surplus is allocated. Finally, axiom 5.2 implies that if players n and m are substitutes, that is, that $v(S \cup \{m\}) = v(S \cup \{n\})$ for any S such that $n \notin S$ and $m \notin S$ then $\phi_n(v) = \phi_m(v)$. That is, if any two players add the same value to any given coalition, then both of these players will get the same payoff in the Shapley value.

Let us do another example that we'll call "Australia". In the Australian federal government each of the 6 states have one vote each, and the federal government itself has two votes plus one extra vote if there is a tie. Thus $N = \{1, 2, 3, 4, 5, 6, 7\}$. Let's say that player 1 is the federal government and the other 6 players are the states. Then S is a winning coalition if $1 \in S$ and $\#S \geq 3$, or if $1 \notin S$ and $S \geq 5$. It is a bit tedious to solve this game directly, so let's see if we can use some of the above observations to help us out. First, when is player 1 pivotal? In any ordering, player 1 is pivotal if it is in positions 3, 4 or 5, and player 1 is in one of these positions in $\frac{3}{7}$ of the possible orders. Thus $\phi_1(v) = \frac{3}{7}$. Now, since players $2 \ldots 7$ are substitutes, we must have $\phi_2(v) = \phi_2(v) = \cdots = \phi_7(v)$. Furthermore, efficiency requires that $\sum_{n=1}^{7} \phi_n(v) = 1$. Thus $\phi_2(v) = \phi_2(v) = \cdots = \phi_7(v) = \frac{4}{7}/6 = \frac{2}{21}$.

As yet another example, suppose that $N = \{1, 2, 3\}$, $v(1) = v(2) = v(3) = 0$, $v(1, 2) = 6$, $v(1, 3) = 7$, $v(23) = 2$, and $v(1, 2, 3) = 8$. This is no longer a simple game. However the idea of the Shapley value is still similar. In this case we calculate the marginal contribution of each player in each possible ordering and then average these over all possible orderings to get the Shapley value. The marginal contributions of each player in this game for each ordering are shown in Figure 2. Thus we get $\phi_1(v) = \frac{25}{6}$, $\phi_2(v) = \frac{10}{6}$ and $\phi_3(v) = \frac{13}{6}$.

Before doing a final example, let us note some useful facts about orderings:

[1] A function f is *one-to-one* if for every $x \neq y$, $\pi(x) \neq \pi(y)$. A function f is *onto* if every element of the range of f is the image of some element of the domain of f.

	Marginal Contribution		
Order	**Player 1**	**Player 2**	**Player 3**
123	0	6	2
132	0	1	7
213	6	0	2
231	6	0	2
312	7	1	0
321	6	2	0
Total	25	10	13

FIGURE 2. Marginal contributions of each player in our non-simple Shapley value sample.

(1) If you have T objects and T ordered boxes to put them in there are $T \cdot (T-1) \cdot (T-2) \cdot \cdots \cdot 2 \cdot 1 = T!$ ways of doing this.
(2) A particular object j is in the kth box $\frac{1}{T}$ fraction of the times, or $(T-1)!$ times.

Now consider the following scenario. A farmer (F) owns some land that is worth \$1,000,000 to her to use as a farm. It's worth \$2,000,000 to an industrialist (I) and it's worth \$3,000,000 to a developer (D). So, $N = \{F, I, D\}$ and $v(F) = 1$, $v(I) = v(D) = 0$, $v(F, I) = 2$, $v(F, D) = 3$, $v(F, D, I) = 3$, $v(D, I) = 0$ and $v(\emptyset) = 0$. Now let R be some ordering and let S_{Rn} be the coalition of all players coming before n in the order R. Then, for example, if the ordering is $R = FID$ then $v(S_{RF} \cup v(F)) - v(S_{RF}) = 1$ is the value that F adds to the empty coalition. Or, if $R = FDI$ then $v(S_{RD} \cup v(D)) - v(S_{RD}) = 2$ is the value that D adds to the coalition consisting of F. Then the calculation of the Shapley value is shown in Figure 3. We have $\phi_F(v) = \frac{13}{6}$, $\phi_I(v) = \frac{1}{6}$, and $\phi_D(v) = \frac{4}{6}$.

R	$v(S_{RF} \cup v(F)) - v(S_{RF})$	$v(S_{RI} \cup v(I)) - v(S_{RI})$	$v(S_{RD} \cup v(D)) - v(S_{RD})$
FID	1	1	1
FDI	1	0	2
IFD	2	0	1
IDF	3	0	0
DFI	3	0	0
DIF	3	0	0
Total	13	1	4

FIGURE 3. Calculating the Shapley value in the Farmer-Industrialist-Developer example.

EXERCISE 5.1. In the farmland example described above, is the Shapley value $\phi(v) = \left(\frac{13}{6}, \frac{1}{6}, \frac{4}{6}\right)$ in the core of the game?