Games in Strategic Form

Definition 11.1: A **strategic game** consists of:

1. a finite set N (the set of players),

2. for each player $i \in N$, a nonempty set A_i (the set of actions available to player i),

3. for each player $i \in N$, a preference relation $\succsim_i$ on $A = \times_{j \in N} A_j$.

If the set of actions for every player is finite, then the game is *finite*.

We refer to an action profile, $a = (a_j)_{j \in N}$, as an *outcome*.

Note: Equivalently, we can define preferences, not over outcomes, but over the *consequences* of those outcomes. (Sometimes it is more natural this way. In Cournot competition, firms receive payoffs based on profits rather than quantities.)

$$g : A \to C$$

If $\succsim_i^*$ is the preference relation over consequences, then $\succsim_i$ is defined by $a \succsim_i b$ if and only if $g(a) \succsim_i^* g(b)$.

Note: Sometimes there is randomness in determining the consequences that result from actions. We model this with a probability space, Ω, and a function, $g : A \times \Omega \to C$. Then a profile of actions induces a *lottery* on C, and preferences $\succsim_i^*$ must be defined over the space of lotteries.

We can model random consequences in Definition 11.1 by introducing nature as a player.

Often $\succsim_i$ can be represented by a payoff function (or utility function), $u_i : A \to \mathbb{R}$. Then we denote the game by $\langle N, (A_i), (u_i)\rangle$ rather than $\langle N, (A_i), (\succsim_i)\rangle$. We can describe finite strategic games with two players in a table or matrix.

Example: Prisoner's Dilemma.

		player 2	
		cooperate	defect
player 1	cooperate	$3, 3$	$0, 4$
	defect	$4, 0$	$1, 1$

Interpretations of the model: (1) The game is only played once, and players choose their actions simultaneously and independently.

(2) The game or a similar game has been played in the past. We observe the "history," but there are no strategic links between the plays. (Maybe different individuals played the game previously.)

(3) By simultaneous, it is only important that each player acts in ignorance of the other players' actions.

Nash Equilibrium

Definition 14.1: A **Nash equilibrium of a strategic game** $\langle N, (A_i), (\succsim_i)\rangle$ is a profile of actions, $a^* \in A$, such that, for every player $i \in N$, we have

$$(a^*_{-i}, a^*_i) \succsim_i (a^*_{-i}, a_i) \text{ for all } a_i \in A_i.$$

Given the others' strategies, no player can profitably deviate.

Each player is choosing an action in his/her best response correspondence, $a^*_i \in B_i(a^*_{-i})$ for all $i \in N$, where

$$B_i(a_{-i}) = \{a_i \in A_i : (a_{-i}, a_i) \succsim_i (a_{-i}, a'_i) \text{ for all } a'_i \in A_i\}.$$

Interpretations of Nash equilibrium

1. **If** a theory of rational play is to predict a unique outcome, then it must be a Nash equilibrium.

2. Self-enforcing **agreement**.

3. A **steady state** of a learning or evolutionary process.

4. A **stable** profile of strategies. Each player has *rational expectations* about how the others will play, and optimizes accordingly. (Form beliefs, which turn out to be correct.) Thus, N.E. does is not a prediction of how the game will be played, but it is a consistent theory of how the game might be played.

Existence of Nash Equilibrium

Not every game has a Nash equilbrium (in pure strategies): Matching Pennies

		player 2	
		heads	tails
player 1	heads	$1, -1$	$-1, 1$
	tails	$-1, 1$	$1, -1$

Proposition 20.3: The strategic game, $\langle N, (A_i), (\succsim_i) \rangle$, has a Nash equilibrium if for all $i \in N$,

1. A_i is a nonempty, compact, convex subset of Euclidean space,

2. Preferences are continuous on A, and quasi-concave on A_i.

Lemma 20.1 (Kakutani's fixed point theorem): Let X be a compact, convex subset of $\mathbb{R}^n$ and let $f : X \to X$ be a correspondence such that

(i) for all $x \in X$, the set $f(x)$ is nonempty and convex, and

(ii) the graph of f is closed. [For all sequences such that $x_n \to x$, $y_n \to y$, and $y_n \in f(x_n)$, we have $y \in f(x)$.]

Then there exists $x^* \in X$ such that $x^* \in f(x^*)$.

Proof of Prop. 20.3: Let $B(a) = \times_{i \in N} B_i(a_{-i})$. Then $B : A \rightarrow A$. Since preferences are continuous and defined over a compact set, $B(a)$ is nonempty. By quasi-concavity, $B_i(a_{-i})$ is a convex set.

Suppose we have sequences $(a)_n \rightarrow \overline{a}$ and $y_n \rightarrow \overline{y}$, such that $(y_i)_n \in B_i((a_{-i})_n)$, but $\overline{y}_i \notin B_i(\overline{a}_{-i})$. Then there exists $\widehat{a}_i \in A_i$ such that $(\widehat{a}_i, \overline{a}_{-i}) \succsim_i (\overline{y}_i, \overline{a}_{-i})$ holds strictly. By continuity of preferences, for sufficiently large n, we have $(\widehat{a}_i, (a_{-i})_n) \succsim_i ((y_i)_n, (a_{-i})_n)$ holding strictly, a contradiction. Thus, the graph of B is closed.

By KFPT, there exists $a^* \in A$ such that $a^* \in B(a^*)$, so a^* is a Nash equilibrium.

Strictly Competitive Games

Definition 21.1: A strategic game, $\langle \{1, 2\}, (A_i), (\succsim_i) \rangle$ (two players) is **strictly competitive** if for any $a \in A$ and $b \in A$, we have $a \succsim_1 b$ if and only if $b \succsim_2 a$.

Note: When preferences are represented by utility functions, it is without loss of generality to assume $u_1(a) + u_2(a) = 0$. (Zero Sum)

Definition 21.2: Let $\langle \{1, 2\}, (A_i), (u_i) \rangle$ be a strictly competitive strategic game. The action $x^* \in A_1$ is a **maxminimizer** for player 1 if

$$\min_{y \in A_2} u_1(x^*, y) \geq \min_{y \in A_2} u_1(x, y) \text{ for all } x \in A_1.$$

The action $y^* \in A_2$ is a **maxminimizer** for player 2 if

$$\min_{x \in A_1} u_2(x, y^*) \geq \min_{x \in A_1} u_2(x, y) \text{ for all } y \in A_2.$$

Intuition: A maxminimizer is an action that maximizes a player's guaranteed payoff.

Lemma 22.1: Let $\langle \{1,2\}, (A_i), (u_i) \rangle$ be a strictly competitive strategic game. Then

$$\max_{y \in A_2} \min_{x \in A_1} u_2(x,y) = -\min_{y \in A_2} \max_{x \in A_1} u_1(x,y).$$

Also, $y^* \in A_2$ solves $\max_{y \in A_2} \min_{x \in A_1} u_2(x,y)$ if and only if it solves $\min_{y \in A_2} \max_{x \in A_1} u_1(x,y)$.

Proof of Lemma 22.1: For every $y \in A_2$, we have

$-\min_{x \in A_1} u_2(x, y) = \max_{x \in A_1}(-u_2(x, y)) =$

$\max_{x \in A_1} u_1(x, y)$

[property of all functions, then def. of str. comp.]

Thus, $\max_{y \in A_2} \min_{x \in A_1} u_2(x, y) =$

$-\min_{y \in A_2}[-\min_{x \in A_1} u_2(x, y)] =$

$-\min_{y \in A_2} \max_{x \in A_1} u_1(x, y)$

[property of all functions, then above eq.]

Also, y^* solves $\max_{y \in A_2} \min_{x \in A_1} u_2(x, y)$ if and only if it solves

$$\min_{y \in A_2}[-\min_{x \in A_1} u_2(x, y)] =$$

$$\min_{y \in A_2}[\max_{x \in A_1} u_1(x, y)].$$

[property of all functions, then above eq.]

What does Lemma 22.1 tell us about Nash equilibrium?

Proposition 22.2: Let $G = \langle \{1, 2\}, (A_i), (u_i) \rangle$ be a strictly competitive strategic game.

(a) If (x^*, y^*) is a NE of G then x^* is a maxminimizer for player 1 and y^* is a maxminimizer for player 2.

(b) If (x^*, y^*) is a NE of G then

$\max_{x \in A_1} \min_{y \in A_2} u_1(x, y) =$

$\min_{y \in A_2} \max_{x \in A_1} u_1(x, y) = u_1(x^*, y^*)$,

so all NE yield the same payoffs.

(c) If we have

$\max_{x \in A_1} \min_{y \in A_2} u_1(x, y) = \min_{y \in A_2} \max_{x \in A_1} u_1(x, y)$,

and if x^* is a maxminimizer for player 1 and y^* is a maxminimizer for player 2, then (x^*, y^*) is a NE of G.

Proof of Prop 22.2: If (x^*, y^*) is a NE,

$u_2(x^*, y^*) \geq u_2(x^*, y)$ for all y, which implies

$u_1(x^*, y^*) \leq u_1(x^*, y)$ for all y.

Thus,

$$u_1(x^*, y^*) = \min_{y \in A_2} u_1(x^*, y) \leq \max_{x \in A_1} \min_{y \in A_2} u_1(x, y).$$

Similarly, $u_1(x^*, y^*) \geq u_1(x, y^*)$ for all x, so

$u_1(x^*, y^*) \geq \min_{y \in A_2} u_1(x, y)$ for all x. Thus,

$$u_1(x^*, y^*) \geq \max_{x \in A_1} \min_{y \in A_2} u_1(x, y).$$

It follows that

$$u_1(x^*, y^*) = \max_{x \in A_1} \min_{y \in A_2} u_1(x, y)$$

holds. From $u_1(x^*, y^*) = \min_{y \in A_2} u_1(x^*, y)$, we have

$$\min_{y \in A_2} u_1(x^*, y) = \max_{x \in A_1} \min_{y \in A_2} u_1(x, y),$$

so x^* is a maxminimizer for player 1. An analogous argument for player 2 establishes

$$\min_{y \in A_2} \max_{x \in A_1} u_1(x, y) = u_1(x^*, y^*)$$

and that y^* is a maxminimizer for player 2. Thus, (a) and (b) hold.

For part (c), let

$$
\begin{aligned}
v^* &= \max_{x \in A_1} \min_{y \in A_2} u_1(x,y) = \\
&\quad \min_{y \in A_2} \max_{x \in A_1} u_1(x,y).
\end{aligned}
$$

By Lemma 22.1, we have

$$\max_{y \in A_2} \min_{x \in A_1} u_2(x,y) = -v^*.$$

Since x^* is a maxminimizer, we have

$$
\begin{aligned}
\min_{y \in A_2} u_1(x^*,y) &\geq \min_{y \in A_2} u_1(x,y) \quad \text{for all } x \\
\min_{y \in A_2} u_1(x^*,y) &\geq \max_{x \in A_1} \min_{y \in A_2} u_1(x,y) = v^* \\
u_1(x^*,y) &\geq v^* \quad \text{for all } y.
\end{aligned}
$$

Since y^* is a maxminimizer, we have

$$
\begin{aligned}
\min_{x \in A_1} u_2(x,y^*) &\geq \min_{x \in A_1} u_2(x,y) \quad \text{for all } y \\
\min_{x \in A_1} u_2(x,y^*) &\geq \max_{y \in A_2} \min_{x \in A_1} u_2(x,y) = -v^* \\
u_2(x,y^*) &\geq -v^* \quad \text{for all } x.
\end{aligned}
$$

Setting $x = x^*$ and $y = y^*$ in these inequalities, and using $u_1 = -u_2$, we have $u_1(x^*, y^*) = v^*$. Therefore, we can rewrite $u_1(x^*, y) \geq v^* \quad$ for all y as

$$u_1(x^*, y) \geq u_1(x^*, y^*) \quad \text{for all } y,$$

which implies

$$\begin{aligned} -u_2(x^*, y) &\geq -u_2(x^*, y^*) \quad \text{for all } y, \text{ or} \\ u_2(x^*, y) &\leq u_2(x^*, y^*) \quad \text{for all } y. \end{aligned}$$

Thus, y^* is a best response to x^*.

We can rewrite $u_2(x, y^*) \geq -v^* \quad$ for all x as

$$\begin{aligned} -u_1(x^*, y) &\geq -u_1(x^*, y^*) \text{ for all } x, \text{ or} \\ u_1(x^*, y) &\leq u_1(x^*, y^*) \text{ for all } x, \end{aligned}$$

so x^* is a best response to y^*. It follows that (x^*, y^*) is a NE.

Comments:

For any game, the payoff that player 1 can guarantee herself is at most the amount that player 2 can guarantee that she is held to.

$$\max_{x \in A_1} \min_{y \in A_2} u_1(x, y) \leq \min_{y \in A_2} \max_{x \in A_1} u_1(x, y). \qquad (1)$$

[Intuitively, when player 2 is holding player 1 to the lowest payoff on the right side of (1), player 2 "chooses first." The payoff player 1 can guarantee herself on the left side of (1) requires player 1 to "choose first."]

A NE exists if and only if (1) holds as an equality and maxminimizers exist. In that case, we can solve for a NE.

If (1) holds as an equality, we call

$v^* = \max_{x \in A_1} \min_{y \in A_2} u_1(x, y)$ the **value** of the game. This is as close to a decision problem as it gets in game theory.