Department of Economics
The Ohio State University
Econ 817–Game Theory

Homework #1 Answers

1. Consider the following cooperative game involving 5 political parties. The number of votes controlled by party i is denoted by x_i, where we have $x_1 = 45$, $x_2 = 45$, $x_3 = 3$, $x_4 = 3$, and $x_5 = 4$. Any coalition receiving a majority of the votes will be able to form a government and divide the surplus. Thus, the characteristic function is given by

$$\begin{aligned} v(S) &= 1 \quad \text{if} \quad \sum_{i \in S} x_i \geq 51 \\ &= 0 \quad \text{otherwise.} \end{aligned}$$

(a) Find the core of this game.
(b) Find the Shapley value of this game.
(c) What would be the Shapley value of the game in which the three smaller parties (3, 4, and 5) merged into a single party? Is it in their interest to merge?

Answer: (a) (Sorry for using up the variable x for the votes, so that we will have to use a different variable to denote allocations.) The core of this game is empty. For example, if $(y_1, y_2, y_3, y_4, y_5)$ is a core allocation, at least one component, say y_1, must be positive. But then the remaining players can form a coalition to increase each of their payoffs by $y_1/4$, by dividing what was allocated to player 1 among themselves.

(b) There are $5! = 120$ possible orderings, and a party's marginal contribution is 1 if it converts a losing coalition into a winning one, 0 otherwise. Let us start with one of the smaller parties, say party 3. Party 3 has a marginal contribution of 1 in all orderings in which it follows parties (1,4), (1,5), (4,1), (5,1), (2,4), (2,5), (4,2), and (5,2). For each of these possibilities, there are 2 orderings, making a total of 16. For example party 3 has a marginal contribution of 1 for the orderings (1,4,3,2,5) and (1,4,3,5,2). Thus, the Shapley value for party 3 is 16/120 or 2/15. The other small parties also have a Shapley value of 2/15, and the two large parties have a Shapley value of 3/10.

(c) With a merged small party with 10 votes (call it party 3), its marginal contribution is 1 in the following orderings: (1,3,2) and (2,3,1). Thus, its Shapley value is 2/6=1/3. Merging gives more power to the large parties, so the small parties will not want to merge. With 5 parties, the total Shapley value allocated to the small parties is 2/5, which is more than 1/3.

2. O-R, exercise 56.4.

Answer: By the symmetry of the game, the set of rationalizable pure actions is the same for both players. Call it Z. Consider $m \equiv \inf(Z)$ and $M \equiv \sup(Z)$. Any best response of player i to a belief about player j (whose support is a subset of Z) maximizes $E(a_i(1 - a_i - a_j))$, or equivalently, it maximizes $a_i(1 - a_i - E(a_j))$. Thus, player i's best response to a belief about player j depends only on $E(a_j)$, which can be written as $B_i(E(a_j)) = (1 - E(a_j))/2$. Because $m \leq E(a_j) \leq M$ must hold, $a_i \in B_i(E(a_j))$ implies $a_i \in [(1-M)/2, (1-m)/2]$. By the best response property of the rationalizable set, we have $m \in [(1-M)/2, (1-m)/2]$ and $M \in [(1-M)/2, (1-m)/2]$. Therefore, we have

$$m \geq \frac{1-M}{2} \text{ and} \tag{1}$$

$$M \leq \frac{1-m}{2}. \tag{2}$$

It follows from (1) and (2) that $m \geq M$ holds, which can only occur if $m = M$. From (1) and (2), we have $m = M = 1/3$. Therefore, the only rationalizable strategy is the unique Nash equilibrium strategy, $a_i = 1/3$.

3. O-R, exercise 19.1.

Answer: There are n players, and each player i has the action set, $A_i = \{out\} \cup [0, 1]$. Each player prefers an action profile with more votes than any other player than one in which he/she ties for the most votes; prefers to tie than to be out; and prefers to be out rather than lose.

With two players, the unique equilibrium is for both players to choose the median of the distribution, $m = F^{-1}(\frac{1}{2})$. This is a NE, because both firms tie, and any deviation will cause that firm to lose or be out. To see that this NE is unique, there cannot be a NE in which one of the players is out, because that player could guarantee at least a tie by choosing the right position. There cannot be a NE in which the players choose different positions, because a player standing to lose could guarantee at least a tie, and a player standing to tie could move closer to the other player and thereby win. There cannot be a NE in which the players choose the same position other than the median, because they would stand to tie, but a player could deviate closer to the median and win.

With three players, there cannot be a NE. If all three are out, then one player could choose a position and win. If two are out, then one of those players could guarantee at least a tie by choosing the right position. If one player is out, then the other two must be choosing the median voter position; in that case, the player that is out could choose a position close to the median voter position and receive almost half the votes, thereby winning. Finally, suppose all three voters choose positions. They must tie for first, because otherwise being out is preferred by a loser. If the players choose three distinct points, then one of the outside players could move closer to the middle and win. If two of the players choose the same position, then the other player could move closer to them and win. If all three players choose the same position, then one of them could move slightly away in the proper direction, and receive nearly half the votes (or more), thereby winning.