Competing Mechanisms with Multi-Unit Consumer Demand: On-line Appendix

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1. Examples

In this section, examples with $I = 1$ and $I = 2$ are computed, to illustrate the results and to demonstrate that the number of firms need not be very large. Consider the class of examples in which utility is of the form, $u_i(x_i) = -\frac{a_i}{x_i}$, which implies $d_i(p) = (\frac{a_i}{p})^{1/2}$ and $d_i'(p) = -\frac{1}{2}(a_i)^{1/2}p^{-3/2}$.

1.1. Example 1: $I = 1, a_1 = 1, r_1 = 1$.

Since there is only one type, we omit the subscript denoting type. The competitive equilibrium price satisfies the market clearing condition,

$$(\frac{1}{p})^{1/2} = 1,$$

so $p^c = 1$. To see that the fixed-price-per-unit mechanism equivalent to setting $p^f = 1$ is part of an equilibrium of Γ, consider the optimization problem of firm f within the space of all mechanisms that fully allocate capacity (i.e., $x^f = 1/(n\beta^f)$, given that others are setting the competitive price. Clearly the firm will not set a

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price below 1, so we have

$$\max_{\beta^f \leq \frac{1}{n}, P^f} n\beta^f P^f$$
$$\text{subject to}$$
$$-n\beta^f - P^f = -(\frac{n(1-\beta^f)}{n-1}) - \frac{n-1}{n(1-\beta^f)}.$$

The constraint characterizes the equilibrium of all subgames following a unilateral deviation. Substituting the constraint into the objective, we have the unconstrained problem to maximize

$$\beta^f \left[(\frac{n(1-\beta^f)}{n-1}) + \frac{n-1}{n(1-\beta^f)} - n\beta^f \right].$$

Setting the derivative with respect to β^f equal to zero and solving the cubic equation (messy details omitted), there are three roots, but only the root $\beta^f = \frac{1}{n}$ lies between 0 and $\frac{1}{n}$. The second-order conditions are satisfied for any $n > 1$, because the second derivative is increasing in β^f and takes the value

$$-\frac{2(n-2)n^2}{(n-1)^2}$$

at $\beta^f = \frac{1}{n}$. Therefore, the best response for firm f is to set $\beta^f = \frac{1}{n}$, yielding the same profits as it receives by setting the constant price, $p^f = 1$.

Now consider the equilibrium of Γ in which all firms choose the fixed-price-per-share mechanism,

$$P^* = \frac{n}{n-1}.$$

The best response for firm f is the solution to

$$\max_{\beta^f \leq 1, P^f} n\beta^f P^f$$
$$\text{subject to}$$
$$-n\beta^f - P^f = -(\frac{n(1-\beta^f)}{n-1}) - \frac{n}{n-1}.$$

Substituting the constraint into the objective, we have the unconstrained problem to maximize

$$\beta^f \left[(\frac{n(1-\beta^f)}{n-1}) + \frac{n}{n-1} - n\beta^f \right].$$

Taking the derivative with respect to β^f yields the expression,

$$-\frac{2n(\beta^f n-1)}{n-1},$$

so the appropriate solution to the first-order condition is $\beta^f = \frac{1}{n}$, yielding the same profits as it receives by choosing the fixed-price-per-share mechanism, $P^f = \frac{n}{n-1}$. The second-order conditions are satisfied, because the second derivative of the objective is negative for $n > 1$.

Finally, consider the equilibrium of Γ in which for all f, firm f chooses the $\varepsilon - MAWEF$ mechanism given by

$$E^* = -\frac{d_i(p^c)}{(n-1)\sum_{h=1}^{I} r_h \frac{\partial d_h(p^c)}{\partial p}} = \frac{2}{(n-1)}.$$

It follows that, for a non-deviating firm f', we have $\beta^{f'} = \frac{1-\beta^f}{n-1}$, so the market clearing price at firm f' is $[\frac{n(1-\beta^f)}{n-1}]^2$. Therefore, consumption offered by firm f' is $\frac{n-1}{n(1-\beta^f)}$, so utility offered by firm f' is $-(\frac{2n(1-\beta^f)}{n-1}) - \frac{2}{n-1}$. The best response for firm f is the solution to

$$\max_{\beta^f \leq 1, P^f} n\beta^f P^f$$

$$\text{subject to}$$

$$-n\beta^f - P^f = -(\frac{2n(1-\beta^f)}{n-1}) - \frac{2}{n-1}.$$

Substituting the constraint into the objective, we have the unconstrained problem to maximize

$$\beta^f \left[\frac{2n(1-\beta^f)}{n-1} + \frac{2}{n-1} - n\beta^f\right].$$

Taking the derivative with respect to β^f yields the expression,

$$-\frac{2(\beta^f n-1)(n+1)}{n-1},$$

so the appropriate solution to the first-order condition is $\beta^f = \frac{1}{n}$, yielding the same profits as it receives by choosing the $\varepsilon - MAWEF$ mechanism. The second-order conditions are satisfied, because the second derivative of the objective is negative for $n > 1$.

It is interesting to compare outcomes in these three equilibria. In all three equilibria, per capita consumption of the good is 1 unit, but the total payment by consumers and firm profits differ. With fixed-price mechanisms, the price per unit is 1, and profits are at the competitive level, equal to 1. With fixed-price-per-share mechanisms, the effective price per unit and profit are equal to $\frac{n}{n-1}$. With $\varepsilon - MAWEF$ mechanisms, the effective price per unit and profit are equal to $1 + \frac{2}{n-1}$. Thus, the $\varepsilon - MAWEF$ mechanisms yield the highest profit, followed by the fixed-price-per-share mechanisms, with fixed-price mechanisms exhibiting no market power and yielding the lowest profit.

1.2. Example 2: $I = 2, a_1 = 4, a_2 = 1, r_1 = r_2 = \frac{1}{3}$.

The competitive equilibrium price satisfies the market clearing condition,

$$\frac{1}{3}(\frac{4}{p})^{1/2} + \frac{1}{3}(\frac{1}{p})^{1/2} = 1,$$

so $p^c = 1$. To see that the fixed-price-per-unit mechanism equivalent to setting $p^f = 1$ is part of an equilibrium of Γ, see the proof of Proposition 3, which shows that the best deviation for a firm is to attract only type 1 consumers. We can write (**??**), the derivative of profits, as

$$\begin{aligned} & -\frac{4n\beta_1^f}{3} + \frac{4n(1-\beta_1^f)}{3(\frac{2n}{3}-1)} + \frac{3(\frac{2n}{3}-1)}{n(1-\beta_1^f)} \\ & +\beta_1^f \left[-\frac{4n}{3} - \frac{4n}{3(\frac{2n}{3}-1)} + \frac{3(\frac{2n}{3}-1)}{n(1-\beta_1^f)^2} \right]. \end{aligned} \tag{1.1}$$

Setting expression (1.1) equal to zero and solving the cubic equation yields only one sensible root, $\beta_1^f = \frac{3}{2n}$, which yields firm f the same profit as it earns by setting the fixed price equal to the competitive price, 1. Second order conditions can be shown to be satisfied whenever $n > 2$ holds. It can also be shown that when $n = 2$ holds, profits are maximized at $\beta_1^f = \frac{3}{2n}$, even though profits are not globally concave in β_1^f.

Now consider the equilibrium of Γ in which for all f, firm f chooses the $\varepsilon - MAWEF$ mechanism given by

$$E_1^* = \frac{4}{(n-1)} \quad \text{and} \quad E_2^* = \frac{2}{(n-1)}.$$

The best response for firm f is the solution to

$$\begin{aligned}
& \max_{\beta^f, P_1^f, P_2^f, x_1^f, x_2^f} \beta_1^f P_1^f + \beta_2^f P_2^f \\
& \text{subject to} \\
\frac{1}{3} n \beta_1^f x_1^f + \frac{1}{3} n \beta_2^f x_2^f &= 1, \\
\frac{4}{x_1^f} + P_1^f &= 4\sqrt{\widetilde{p}} + \frac{4}{(n-1)}, \\
\frac{1}{x_2^f} + P_2^f &= 2\sqrt{\widetilde{p}} + \frac{2}{(n-1)}, \\
\beta^f &\geq 0,
\end{aligned}$$

where $\widetilde{p}$ satisfies market clearing condition,

$$1 = \frac{\frac{1}{3} n (1 - \beta_1^f)}{n-1} \frac{2}{\sqrt{\widetilde{p}}} + \frac{\frac{1}{3} n (1 - \beta_2^f)}{n-1} \frac{1}{\sqrt{\widetilde{p}}},$$

solved to be

$$\widetilde{p} = \left[\frac{(3 - 2\beta_1^f - \beta_2^f) n}{3(n-1)} \right]^2.$$

Imposing the necessary condition that marginal utilities are equated yields $x_1^f = 2x_2^f \equiv x$. Then, defining $B^f \equiv 2\beta_1^f + \beta_2^f$, we can rewrite the constraints as

$$\begin{aligned}
\frac{6}{nB^f} &= x, \\
P_1^f &= 4 \left[\frac{(3 - B^f) n}{3(n-1)} \right] + \frac{4}{(n-1)} - \frac{4}{x}, \\
P_2^f &= 2 \left[\frac{(3 - B^f) n}{3(n-1)} \right] + \frac{2}{(n-1)} - \frac{2}{x}.
\end{aligned}$$

From the two indifference constraints, we see that $P_1^f = 2P_2^f \equiv P^f$ must hold, so we can simplify the optimization problem to the unconstrained problem,

$$\max_{B^f} B^f \left[\left[\frac{(3 - B^f) n}{3(n-1)} \right] + \frac{1}{(n-1)} - \frac{nB^f}{6} \right].$$

Setting the derivative of this objective equal to zero and solving, we have $B^f = \frac{3}{n}$. The second derivative of this objective is

$$-\frac{n(n+1)}{3(n-1)},$$

so the second order conditions are satisfied. Firm f has no incentive to deviate, since following the equilibrium satisfies the resource and indifference constraints and achieves $\beta_1^f = \beta_2^f = \frac{1}{n}$ and $B^f = \frac{3}{n}$.

2. Reserve Price Equilibrium with Unit Demands

With unit demands, a type i consumer is characterized by his valuation, v_i. Suppose all other firms are choosing a uniform price auction with reserve price of zero. Consider a deviation for firm f, which induces an auction price at the other firms of $\widetilde{p}$. In the resulting consumer equilibrium, the indifference condition requires all consumers who purchase at firm f to pay $\widetilde{p}$. Also, firms other than firm f will sell all of their capacity, so the quantity sold by firm f is the residual demand at price $\widetilde{p}$.

First consider the model with a finite number of valuation types. We continue to denote the aggregate measure of type i consumers as $r_i n$. Because the market demand curve is a step function, p^c is equal to one of the valuations, say v_h, and in the competitive equilibrium allocation, only a fraction of the type v_h consumers purchase. We have

$$\sum_{i=1}^{h} r_i > 1 \quad \text{and} \quad \sum_{i=1}^{h-1} r_i < 1 \ .$$

Suppose firm f considers inducing a higher auction price at the other firms, say the next highest valuation, $\widetilde{p} = v_{h-1}$. Aggregate market demand, less the quantity sold by the other firms, is given by

$$\begin{aligned} & \sum_{i=1}^{h-1} r_i - (n-1) \\ = \;& 1 - n\left(1 - \sum_{i=1}^{h-1} r_i\right), \end{aligned}$$

so profits for firm f are the maximum of

$$v_{h-1}\left[1-n\left(1-\sum_{i=1}^{h-1} r_i\right)\right] \tag{2.1}$$

and zero.[1] In expression (2.1), the term in parentheses is strictly positive, except for knife-edge parameter values. Therefore, for large enough n, firm f cannot benefit by deviating from a reserve price of zero. In this case, and even if expression (2.1) is positive but less than v_h, then Γ has an equilibrium in which all firms choose a zero reserve price.

Now consider the model with an infinite number of types, represented by the continuously differentiable aggregate demand curve, $D(p)$. The competitive equilibrium price satisfies $D(p^c) = n$. A deviation by firm f, which induces an auction price at the other firms of $\widetilde{p}$, gives rise to profits of

$$\pi^f(\widetilde{p}) = \widetilde{p}\left[D(\widetilde{p}) - (n-1)\right].$$

Differentiating with respect to $\widetilde{p}$ yields

$$(\pi^f)'(\widetilde{p}) = D(\widetilde{p}) - (n-1) + \widetilde{p}D'(\widetilde{p}).$$

Firm f does not have a profitable deviation, and Γ has an equilibrium in which all firms choose a zero reserve price, whenever $(\pi^f)'(p^c) < 0$ holds, or

$$D(p^c) - (n-1) + p^c D'(p^c) < 0. \tag{2.2}$$

Substituting $D(p^c) = n$ into (2.2) yields the condition,

$$-\frac{p^c D'(p^c)}{n} > \frac{1}{n}. \tag{2.3}$$

Because we have $D(p^c) = n$, the left side of (2.3) is the absolute value of the price elasticity of demand. Whenever demand is sufficiently elastic, Γ has an equilibrium in which all firms choose a zero reserve price.

[1] If expression (2.1) is negative, this means that the firm f is unable to raise the auction price at other firms even if it withholds all of its capacity, in which case there is no profitable deviation.

3. Robustness of the Rationing Rule

Here we evaluate the robustness of the result, that fixed-price-per-unit mechanisms at p^c are consistent with equilibrium, by considering two alternative rationing rules.

First, suppose that firms continue to impose a maximum quantity per customer if there is excess demand at their firm, but that consumers can purchase from multiple firms. Analyzing the competing mechanisms game when consumers can choose any subset of firms is beyond the scope of this paper, but we consider the n (symmetric) firm version of Osborne and Pitchik (1986), with one consumer type having aggregate measure rn and per capita demand function $d(p)$. Each firm f selects a price, p^f, and consumers purchase, starting at the lowest price firm. If there is excess demand at a firm, consumers are rationed by a maximum quantity per customer.

The competitive equilibrium price satisfies $d(p^c) = 1/r$. If all firms set the price p^c, clearly there is no profitable deviation to a lower price. If firm f deviates to a higher price, then the quantity it is able to sell is the residual demand, $nrd(p^f) - (n-1)$. Thus, the profit of firm f is given by

$$\pi^f(p^f) = p^f\left(nrd(p^f) - (n-1)\right).$$

The derivative of profits with respect to p^f gives the concave function,

$$(\pi^f)'(p^f) = nrd(p^f) - (n-1) + p^f nrd'(p^f). \tag{3.1}$$

All firms setting the competitive price is an equilibrium if and only if (3.1) is non-positive when evaluated at p^c. Using $d(p^c) = 1/r$, this condition can be written as

$$1 + \frac{p^c nd'(p^c)}{d(p^c)},$$

or in other words, the absolute value of the price elasticity of demand is greater than or equal to $\frac{1}{n}$.

The second rationing rule we want to consider is first-come-first-served rationing. That is, suppose the set of allowable mechanisms is enlarged to allow first-come-first-served rationing, which treats two consumers of the same type asymmetrically. Can we have an equilibrium of Γ with all firms choosing the fixed-price-per-unit mechanism with price p^c and first-come-first-served rationing? If the utility of zero consumption is negative infinity, then clearly this profile of mechanisms cannot be consistent with equilibrium. Firm f could deviate to a higher price, without fear of losing customers. The reason is that, if firm f were to lose customers, there would be excess demand at the other firms, so consumers at the other firms face a strictly positive probability of zero consumption, in which case their expected utility is negative infinity.

Now suppose that, for each i, $u_i(0)$ is finite. We will consider a deviation by firm f such that, in the ensuing consumer equilibrium, type 1 consumers visit firm f with probability β_1^f and the other types visit firm f with probability zero. We will think of firm f as allocating its entire capacity evenly across is customers and choosing β_1^f to be consistent with a consumer equilibrium, where the payment P_1^f is such that type 1 consumers are indifferent between firm f and the other firms. Thus, in the consumer equilibrium, consumption satisfies $x_1^f = \frac{1}{r_1 n \beta_1^f}$.[2] The probability of being able to consume at the other firms, denoted by μ, satisfies

$$n - 1 = \mu \sum_{i=1}^{I} r_i n (1 - \beta_i^f) d_i(p^c)$$

and, using the market clearing condition, is therefore given by

$$\mu = \frac{n-1}{n - r_1 n \beta_1^f d_1(p^c)}. \tag{3.2}$$

We can also compute

$$\frac{\partial \mu}{\partial \beta_1^f} = \frac{(n-1) r_1 d_1(p^c)}{n(1 - r_1 \beta_1^f d_1(p^c))^2}.$$

Note that, for "small" deviations that induce a small probability of being rationed at other firms, types $i > 1$ will strictly prefer not to visit firm f and receive (approximately) what type 1 receives in the competitive equilibrium allocation. Therefore, for small deviations of this form, the consumer equilibrium will be unique and there is no loss in allowing firm f to pick it.[3]

The indifference condition for a type 1 consumer is given by

$$u_1\left(\frac{1}{r_1 n \beta_1^f}\right) - P_1^f = \mu \left[u_1(d_1(p^c)) - p^c d_1(p^c)\right] + (1 - \mu) u_1(0).$$

[2] The analysis assumes for convenience that there are enough type 1 consumers in the economy to demand all of firm f's capacity at the price p^c, so we have $r_1 n d_1(p^c) > 1$. This will be true if n is large enough.

[3] We are omitting the tedious task of extending continuously the mechanism of firm f to surprise reports, but this can be done.

Using the indifference condition to solve for P_1^f, we derive an expression for the profits of firm f, as a function of β_1^f, given by

$$\pi^f(\beta_1^f) = r_1 n \beta_1^f \left[u_1(\frac{1}{r_1 n \beta_1^f}) - \mu\left(u_1(d_1(p^c)) - p^c d_1(p^c)\right) - (1-\mu) u_1(0) \right],$$

where the term in brackets is P_1^f. Differentiating yields

$$(\pi^f)'(\beta_1^f) = r_1 n P_1^f + r_1 n \beta_1^f \left[u_1'(\frac{1}{r_1 n \beta_1^f})(-\frac{1}{r_1 n (\beta_1^f)^2}) - K \frac{\partial \mu}{\partial \beta_1^f} \right], \tag{3.3}$$

where K denotes strictly positive difference between the competitive equilibrium utility and no-trade utility for type 1,

$$K \equiv u_1(d_1(p^c)) - p^c d_1(p^c) - u_1(0).$$

Now let us evaluate (3.3) at the value of β_1^f giving rise to the competitive equilibrium allocation (and profit). Then we substitute

$$\begin{aligned} \beta_1^f &= \frac{1}{r_1 n d_1(p^c)}, \\ P_1^f &= p^c d_1(p^c) \\ u_1'(\frac{1}{r_1 n \beta_1^f}) &= p^c \end{aligned}$$

into (3.3), which yields, after a lot of manipulation,

$$-\frac{n}{n-1} K r_1 d_1(p^c).$$

Since this expression is negative, we conclude that there is a profitable deviation to β_1^f smaller than $\frac{1}{r_1 n d_1(p^c)}$. Therefore, all firms choosing a fixed-price-per-unit mechanism with price p^c and first-come-first-served rationing is inconsistent with equilibrium.

4. Equilibrium in which Not All Capacity is Allocated

Suppose there are two firms, one consumer type with utility function of consumption and money given by $u(x) + M$, and a total measure of consumers of 1 (that is, $r_1 = \frac{1}{2}$ holds).

On the equilibrium path, firm 1 receives a measure of consumers, $\rho^1 = 0.6$, and we have

$$x^1(0.6) = 1.5 \quad \text{and} \quad P^1(0.6) = u(1.5).$$

That is, firm 1 utilizes 90% of its capacity and consumers receive zero surplus. Let us now continuously extend the mechanism to other values of ρ^1. As ρ^1 moves away from 0.6, we continuously but "quickly" increase the capacity allocated to 1 and reduce the payment to 0. Thus, for $\rho^1 < 0.599$ and $\rho^1 > 0.601$, we have[4]

$$x^1(\rho^1) = \min[\frac{1}{\rho^1}, 1000] \quad \text{and} \quad P^1(\rho^1) = 0.$$

On the equilibrium path, firm 2 receives a measure of consumers, $\rho^2 = 0.4$, and we have

$$x^2(0.4) = 2.5 \quad \text{and} \quad P^2(0.4) = u(2.5).$$

That is, firm 2 utilizes all of its capacity and consumers receive zero surplus. Let us now continuously extend the mechanism to other values of ρ^2. As ρ^2 moves away from 0.4, we utilize all of firm 2's capacity and continuously but "quickly" reduce the payment to 0. Thus, for $\rho^2 < 0.399$ and $\rho^2 > 0.401$ we have

$$x^2(\rho^2) = \min[\frac{1}{\rho^2}, 1000] \quad \text{and} \quad P^2(\rho^2) = 0.$$

Since all consumers are receiving the same utility of zero, $(\rho^1, \rho^2) = (0.6, 0.4)$ is a consumer equilibrium when these mechanisms are chosen.

We now show that neither firm has an incentive to deviate to a new mechanism. If firm 1 deviates, consumers will choose a consumer equilibrium such that $\rho^1 < 0.599$ holds. To see that such a consumer equilibrium must exist, at $(\rho^1, \rho^2) = (0.599, 0.401)$, firm 1 can offer its customers at most utility of $u(\frac{1}{0.599})$, while firm 2 is offering its customers utility of $u(\frac{1}{0.401})$, which is more. At $(\rho^1, \rho^2) = (0, 1)$, firm 1 must be offering its customers (if there were any), utility strictly greater than the utility offered by firm 2, $u(1)$, or else $(\rho^1, \rho^2) = (0, 1)$ would be a consumer equilibrium. By continuity, there must be some ρ^1 between 0 and 0.599 such that the utilities offered by the two firms are equal, thereby constituting a consumer equilibrium (in fact, it must be below 0.5).

[4] To ensure continuity at $\rho^1 = 0$, we impose a maximum consumption per capita, say, 1000, which binds for $\rho^1 < 1/1000$.

At the consumer equilibrium following firm 1's deviation, (ρ^1, ρ^2), the highest payment firm 1 can extract is based on allocating its entire capacity and matching the utility offered by firm 2, $u(\frac{1}{1-\rho^1})$. These profits are

$$\rho^1(u(\frac{1}{\rho^1}) - u(\frac{1}{1-\rho^1})). \tag{4.1}$$

If, for all $\rho^1 < 0.5$, expression (4.1) is less than firm 1's profits without the deviation, $0.6u(1.5)$, then there is no incentive to deviate. This will be the case for many utility functions, including $u(x) = x^{1/2}$.

If firm 2 deviates, at the consumer equilibrium following firm 2's deviation, (ρ^1, ρ^2), the highest payment firm 2 can extract is based on allocating its entire capacity and matching the utility offered by firm 1. Deviations that result in (ρ^1, ρ^2) very close to $(0.6, 0.4)$ cannot be profitable, because the marginal effect of increasing the measure of firm 2's customers is much less than the marginal effect of having to reduce the per capita revenue in order to match the rapidly increasing utility offered by firm 1. For deviations that result in substantially more customers, firm 1 is offering its customers utility of $u(\frac{1}{\rho^1})$, so we can write firm 2's maximum deviation profits as

$$\rho^2(u(\frac{1}{\rho^2}) - u(\frac{1}{1-\rho^2})). \tag{4.2}$$

If expression (4.2), which can only be positive if $\rho^2 < 0.5$ holds, is less than firm 2's profits without the deviation, $0.4u(2.5)$, then there is no incentive to deviate. This will be the case for many utility functions, including $u(x) = x^{1/2}$.