Dynamic Competition with Random Demand and Costless Search: A Theory of Price Posting

Raymond Deneckere and James Peck*

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Abstract

This paper studies a dynamic model of perfectly competitive price posting under demand uncertainty. Firms must produce output in advance. After observing aggregate sales in prior periods, firms post prices for their unsold output. In each period the demand of a new batch of consumers is randomly activated. Existing customers that have not yet bought and then new customers arrive at the market in random order, observe the posted prices, and either purchase at the lowest available price or delay their purchase decision.

We construct a sequential equilibrium in which the output produced and its allocation across consumers is efficient. Thus consumers endogenously sort themselves efficiently, with the highest valuations purchasing first. Transactions prices in each period rise continuously, as firms become more optimistic about demand, followed by a market correction. By the last period, prices are market clearing.

1 Introduction

This paper studies markets characterized by three important features. First, the good is offered for sale for an extended, but limited length of time, called the "sales season." Consumers purchase at most once during the sales season, but can optimize in the timing of their purchases. At the end of the sales season the good loses most of its consumption value. Second, the market is competitive,

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and because demand is concentrated in the sales season, firms need to produce or schedule capacity in advance, so as to be able to satisfy this demand. Third, aggregate demand is highly uncertain.

Any type of good for which demand is seasonal, either because of customs in purchasing behavior (e.g. the Christmas season or graduation time), or because its use is related to the weather pattern (e.g. skis or lawn fertilizer), and for which there is a need to smooth production over time, fits this mold. Goods or services for which the variability in demand is of higher frequency, such as airline travel, hotels, vacation home rentals, and car or equipment rentals also exhibit these characteristics.

Because in these situations the strength of demand is only gradually revealed over the demand period, firms must solve two difficult problems. First, they must decide how much output to make available for the demand season. The wrong level of output may result in forgone sales, or force firms to mark down their prices excessively. In addition, firms face the difficult problem of how to price their output under partial knowledge of the demand realization. Prices may have to be adjusted upwards over time in order to respond to what appears to be strong demand, and when that guess turns out to be wrong, markdowns or sales may be needed. Consumers face the equally challenging task of deciding whether to purchase at the going price, or wait to purchase later in the season.

While the set-up of our model is non-standard, it is nevertheless highly relevant, as a substantial fraction of commerce is traded under the conditions set forth here. In the US, the Christmas shopping season alone accounts for over 18 percent of total retail sales during the year, making November and December the busiest months of the year for retailers.[1,2] Other significant sales seasons include the Back to School period, Graduation, Father's Day and Mother's Day, and the four apparel seasons. Furthermore, for products for which the variability in demand is of higher frequency, it should be noted that annual spending on travel in the US nearly equals the amount Americans spend during the Christmas season.[3]

Our assumptions of production in advance and the presence of significant demand uncertainty are well motivated. The gradual disappearance of low-priced seats for airline travel, sometimes followed by the sudden appearance of bargain tickets, is a phenomenon well-known both to airline customers and travel agents. Similarly, the occurrence of clearance sales near the end of the holiday shopping season is well documented.[4] Indeed, markdowns have become so rampant in retailing that they now

[1] "Retail Industry Indicators 2008," National Retail Federation, Washington, D.C.

[2] For many retail sectors, Christmas holiday sales are particularly important. For example, in 2004, 32.2% of jewelry stores' total annual sales and 24.4% of department stores' total annual sales occurred during the holiday season.

[3] "Economic Review of Travel in America, 2008," U.S. Travel Association, Washington, D.C.

[4] See e.g. "Wal-Mart Sparks War Among Big Toy Sellers," Wall Street Journal, October 9, 2008, or "Marks &

take up more than 35% of department stores' annual sales volume.[5] There is also a preponderance of evidence that consumers are strategic in the timing of their purchases,[6] and that firms spend significant resources to solve their difficult intertemporal pricing problem.[7]

Yet despite the importance of the phenomenon we study, economic theorists have written relatively little on the topic, presumably because of the complicated problem of solving the simultaneous intertemporal arbitrage problems of firms and consumers. The few models that do exist typically have firms producing to inventory (e.g., Danthine (1977), Caplin (1985)), and treat consumer demand in a simplistic fashion. Indeed, the literature invariably postulates either a time-invariant flow demand, or else assumes that consumers are myopic, again resulting in a static demand function. One of the major innovations of our paper is to introduce intertemporal substitutability in demand, by allowing consumers to optimize in the timing of their purchase decisions.

We develop a dynamic trading model, in which aggregate demand is uncertain, and production must occur before demand is realized. There are multiple trading periods, and in each period the demand of a new batch of consumers is randomly activated.[8] Market participants observe the sales made in prior periods, thereby updating their beliefs about the demand in future periods. Within each period, given the remaining output, firms start by posting prices for that period, after which any remaining active customers arrive in random order to the market. New customers then arrive in random order as well. All consumers have perfect information on product availability and current prices, and decide whether to purchase then or delay their purchase until a future round. The dynamic setting leads to a nontrivial decision problem for firms and consumers. In equilibrium, firms' pricing decisions and inferences about demand are based on perceived purchasing decisions of consumers, and purchasing decisions are based on currently available prices and expectations about future prices, which depend on the purchasing decisions of other consumers during the current round, as well as future pricing decisions. We demonstrate the existence of an equilibrium to our model

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Spencer fires new shot in Christmas price war," Times Online, December 8, 2008.

[5] "Retail Horizons 2008," Merchandising and Operating Results of Department and Specialty Stores, National Retail Federation, Washington, DC.

[6] "This Weekend's Shopping Deals: Another Black Friday?," Wall Street Journal, December 20, 2007.

[7] Airlines were the first to develop sophisticated and proprietary software allowing them to adjust prices frequently in response to changes in demand. Big retailers such as Home Depot and J.C. Penney now also use such "Revenue Management" or "Markdown Management" systems to manage demand uncertainty. According to a survey of the National Retail Federation fully 21% of retailers were using such software early in 2009, and 33% planned to have it installed within the next 18 months ("US Retailers Find New Ways To Fine-Tune Discounts", Wall Street Journal, July, 3, 2009).

[8] This feature of demand arriving in batches or "generations" within the market period also appears in Board (2008). However, that paper studies a monopoly problem, and assumes that the monopolist can precommit to a price path.

under regularity assumptions on the distribution of demand, and characterize its properties.

The model generates price dispersion within each period, because some firms will post a low price and be certain to sell. Others might post a higher price and only sell in that period when demand is sufficiently high; when they do not sell, they become more pessimistic about future demand. The support of the price distribution varies with what firms learned about demand in previous periods. Our equilibrium has the property that transaction prices rise within each demand period, reflecting firms' increasing optimism about demand in that period. Once demand for that period dries up, firms hold a sale. More precisely, the lowest priced unit offered for sale in the next period is priced below the highest transaction price in the current period. However, this lowest price may lie either above or below the lowest price offered in the previous period. On average, the lowest price in future periods equals the lowest available price in the current period, reflecting the general property that prices are a martingale.

The equilibria to our model also have a surprising theoretical implication: although consumers arrive in a random order, in equilibrium they sort themselves efficiently, with high-valuation consumers purchasing early, and lower-valuation ones postponing their purchase decision (thereby exploiting their option of refusing to purchase in the future should the sale price exceed their valuation). Rather than assuming efficient rationing, we thus show that it arises endogenously in equilibrium! As a consequence, despite the existence of price pre-commitment within each period, the presence of asymmetric information about demand, and the random arrival of customers to the market, the output produced and its distribution across consumers turn out to be efficient. In addition, the final price equals the market clearing price based upon the output produced and the state of demand. Thus our model illustrates how in the presence of demand uncertainty an economy can grope its way towards competitive equilibrium.

Our paper also significantly innovates in its modelling of uncertainty about consumer demand, and in its analysis of the factors that govern optimal consumer purchase decisions. In our model, consumers know their valuation for the good and whether or not they are active, and hence possess private information. Since there is aggregate demand uncertainty, this information is correlated amongst consumers. As a consequence, beliefs about the demand state differ across consumers, and generally also differ from the beliefs held by firms. We describe two novel effects that sort consumers' types into those that purchase at the currently available price, and those that delay their purchase decision: an option value effect, and an information effect. In our model, as a consequence of intertemporal arbitrage by firms, the lowest available current price is an unbiased estimate of the future transaction price, i.e. prices are martingales. If a consumer holds the same beliefs about future demand as firms, she will prefer to postpone purchasing whenever her valuation is low. Indeed, when the future transaction price exceeds her valuation, she has the option of not purchasing at that price. Second, there is an information effect: our regularity condition ensures that high valuation customers give relatively more weight to high-demand states than other customers, so they have more pessimistic beliefs over future prices, and are more inclined to purchase early.

Our paper adds to a sparse literature on dynamic competition under demand uncertainty. The one-period version of our model is a generalization of Prescott's (1975) static "hotels" model.[9] Eden (1990, 2009) and Lucas and Woodford (1993) present a quasi-dynamic version of the Prescott model in which firms change prices gradually over time, in response to observed increases in cumulative aggregate sales. However, unlike in our model, these papers assume that consumers cannot return to the market once they have chosen not to purchase.[10] Lazear (1986) presents a dynamic pricing model for a monopolist selling a fashion good (a "designer dress"). The firm faces a sequence of identical potential buyers over time, but is uncertain about their valuation for the good. The dress is initially offered at the static monopoly price, and price is lowered over time to reflect the seller's increasing pessimism regarding the buyer's valuation. Our model differs from Lazear's in a number of important respects: we have competition on the seller side of the market, there is uncertainty about the quantity to be sold, and consumers are fully optimizing.[11] Pashigian (1988) introduces competition into Lazear's model, and also obtains a declining pattern of prices. See also Nocke and Peitz (2007) on monopoly clearance sales.

Our model bears some resemblance to the literature on durable goods (Stokey (1981), Fudenberg, Levine and Tirole (1985), Gul, Sonnenschein and Wilson (1986), Ausubel and Deneckere (1989)), where consumers are also allowed to intertemporally substitute. However, in that literature there is no production in advance, firms end up saturating the entire market, and the important dimension of demand uncertainty is absent. Furthermore, product durability is not a necessary feature of our model. Our problem is also related to the literature on peak-load pricing under stochastic demand

[9] The static, one-period version of this model has been studied in previous work by Prescott (1975), Bryant (1980), Eden (1990), Dana (1993, 1998, 1999), and Deneckere, Marvel, and Peck (1996).

[10] An equivalent assumption is that firms cannot adjust prices downward once demand dries up.

[11] Lazear's model assumes that consumers are myopic.

(Crew and Kleindorfer (1986)) or priority pricing (Harris and Raviv (1981)). Unlike in our model, the demand state then is either known when the demand period arrives, or else all customers are present in the market at the same time, and private information on demand can be elicited.

Our paper is more closely related to the substantial operations research literature on 'revenue management', initiated by Gallego and van Ryzin (1994), and Bitran and Mondschein (1997). This literature considers the dynamic pricing problem of a monopolist whose customer arrivals follow a Poisson process. Customers are assumed to be myopic, i.e. purchase at the posted price whenever their valuation exceeds it. The main result is that on average price falls over the demand season, because the option value of unsold units decreases as the deadline approaches. Our work is distinguished from this literature in several important respects. First, we study competition between many producers, rather than the monopoly problem. Second, we allow for intertemporal substitutability in demand, so that consumers must decide when to purchase.[12] Finally, we allow for aggregate demand uncertainty, and consumers are negligible, justifying their price taking behavior.[13]

On the empirical end, the literature that comes closest to studying the type of markets we describe involves the pricing of airline tickets (e.g. Escobari and Gan (2007) and Escobari (2009)) and hotel rooms (Lee et al (2009)). We discuss this literature in Section 4, after having presented our theoretical results. There is also an empirical literature estimating demand elasticities and pricing during peak and off-peak demand periods. Warner and Barsky (1995) document that prices tend to be marked down during peak periods (on weekends and before Christmas). Chevalier et al (2003) use supermarket scanner data and find that average prices are lower during peak periods of demand, even though demand is not more price sensitive during such periods. Their econometric specification assumes that demand depends on the current period's price but no lagged variables, ruling out the intertemporal substitution that is the key to our analysis. Nevo and Hatzitaskos (2006) focus on product differentiation and substitution across brands, but again treat demand as exogenously tied to a period, with no intertemporal substitution. Hendel and Nevo (2006) estimate a dynamic model of consumer choice, allowing for intertemporal substitution and consumer stockpiling. They find strong evidence of intertemporal substitution. To overcome the lack of a good theory of both supply and demand, they assume that prices follow an exogenous first-order Markov process. Erdem et al

[12] Su (2007) considers strategic consumers, but his model lacks the important ingredient of demand uncertainty.

[13] With idiosyncratic uncertainty, as in the case with Poisson arrivals, aggregate demand becomes deterministic when the number of potential consumers is allowed to become large.

(2003) focus on the important issue of consumer expectations of future prices, but also assume that prices are exogenously determined. Although more suited to the airline industry than the market for supermarket products, our model provides a theory of both forward-looking firms and consumers, which is missing in this literature.

In Section 2, we set up the model. Section 3 studies the one-period version of our model. We study the dynamic model in Section 4, which contains our characterization and efficiency results. Section 4 also contains a computed example and a discussion of empirical implications of our model. Section 5 concludes. Except for Propositions 4 and 5, all proofs are contained in the Appendix.[14]

2 The Model

We consider a competitive market for which for which production must occur in advance of the demand season. There is a continuum of firms, of total measure M, each of whom can produce one unit of output at time $t = 0$, at a common cost of $c > 0$.

On the demand side, there is a continuum of potentially active customers of total measure $C < M$. Each active consumer buys at most one unit of the good.

Aggregate demand is random, and arrives sequentially over time. The demand season consists of T periods, indexed by $t = 1, ..., T$. The demand that is activated in a period $t \in \{1, ..., T-1\}$ is represented by a demand function $D_t(p, \alpha_t)$. For simplicity, no new demand is activated in the final period T.[15] We impose the following regularity conditions on the demand functions D_t:

Assumption 1 *There exist $\underline{v} \geq 0$ and $\bar{v} > \max\{c, \underline{v}\}$ such that (i) $D_t(p, \alpha_t) = D_t(\underline{v}, \alpha_t)$ for all $p \in [0, \underline{v})$; (ii) $D_t(p, \alpha_t) = 0$ for all $p \geq \bar{v}$; (iii) $D_t(p, \alpha_t)$ is strictly decreasing in p on $[\underline{v}, \bar{v}]$. Furthermore, for each t, the function D_t is C^1 on the interval $[\underline{v}, \bar{v}]$, and strictly increasing in α_t for all $p < \bar{v}$.*

The random variables $(\alpha_1, ..., \alpha_{T-1})$ are distributed according to the joint distribution function, $F(\alpha_1, ..., \alpha_{T-1})$, which has a continuous density function $f(\alpha_1, ..., \alpha_{T-1})$ whose support equals the rectangle $\prod_{t=1}^{T-1}[\underline{\alpha}_t, \bar{\alpha}_t]$. We assume that the random variables $(\alpha_1, ..., \alpha_{T-1})$ are affiliated:[16]

[14] The proofs of Propositions 4 and 5 are available as an on-line Appendix, at http://www.econ.ohio-state.edu/jpeck/proofprops4and5.pdf

[15] This formulation ensures that each consumer can delay her purchase for at least one period.

[16] In economics supermodularity of the logarithm of the density function is familar from auction theory.

Assumption 2 *For all $t \neq t'$, we have*

$$\frac{\partial^2 \ln f(\alpha_1, ..., \alpha_{T-1})}{\partial \alpha_t \partial \alpha_{t'}} > 0.$$

The process by which nature selects consumers to become active is as follows. First, nature draws a realization of $(\alpha_1, ..., \alpha_{T-1})$, according to the distribution F. Then each consumer is randomly and independently selected to become active in period $t \in \{1, ..., T-1\}$ with probability $\frac{D_t(0,\alpha_t)}{C}$. Finally, for each active consumer in period t, nature randomly and independently selects a valuation v from the distribution $1 - \frac{D_t(v,\alpha_t)}{D_t(0,\alpha_t)}$. The strong law of large numbers then guarantees that the number of consumers with valuation v or more that become active in period t equals $D_t(v, \alpha_t)$.[17]

We refer to a consumer who becomes active in period t and has valuation v as type (v, t). If a consumer of type (v, t) purchases at a price p in some period $t' \geq t$, she receives a net surplus of $v - p - (t' - t)\delta(v)$. Thus for each period of delay, type (v, t)'s surplus from transacting is reduced by the delay cost $\delta(v)$. If (v, t) never purchases, she receives a net surplus of 0. We make the following assumptions on the delay cost:

Assumption 3 *There exists $\hat{v} \in (\underline{v}, \bar{v})$ such that $\delta(v) = 0$ for all $v \leq \hat{v}$, and such that $\delta(\cdot)$ is strictly increasing and C^1 on $[\hat{v}, \bar{v}]$ with $\delta(v) \leq v - \hat{v}$.*

The timing of decisions is as follows. In an initial period, $t = 0$, firms first decide whether or not to produce. Nature then chooses the set of active consumers. Subsequently, in each period $t = 1, ..., T$, firms who still have output available decide whether to make this output available in period t, and if so, at what price to post it.

All remaining active customers (those who have not purchased in prior periods) are then put in a queue, in random order. Consumers that become active in period t are put at the end of this queue, also in random order.[18] After the queue has been formed, consumers are released sequentially into the market. When a consumer is released into the market, she observes a_t, the amount of sales that have been made so far in period t, after which she decides whether or not to purchase at the lowest

[17] It is well known that with a continuum of independent random variables, thorny measure theoretic problems arise in guaranteeing that the sample paths of this process are measurable, and that the strong law of large numbers applies (Judd (1985), Sun (2006)). Sun and Zhang (2009) have recently resolved this difficulty.

[18] This timing reflects the idea that the demand from period t consumers is randomly activated in the course of period t.

remaining price. At the end of period t, all market participants who have not yet transacted observe a_t^r, the aggregate sales realized in period t, and play proceeds to the next period, unless no output remains to be sold, in which case the game is over.

Given the period t queue, the purchase strategies of consumers in the queue, and the posted prices in period t, $p_t(a_t)$, this determines the measure of output that sells in period t, which we denote as $\bar{a}_t$. Define $p^0 \equiv \lim_{a_t \uparrow \bar{a}_t} p_t(a_t)$. The outcome function then specifies that a firm posting $p < p^0$ sells, and a firm posting $p > p^0$ does not sell. For a firm posting $p = p^0$, (i) if the measure of consumers who would purchase at this price is greater than or equal to the measure of output posted at this price, the firm sells, and (ii) otherwise, the firm sells with a probability equal to the ratio of the measure of consumers who would purchase to the measure of output posted. Then the highest posted price such that output posted at that price is sold is p^0.

It is important to note that in this game, consumers have private information about the state of demand, information that is not shared by the rest of the market. Specifically, a consumer of type (v, t) who is selected by nature knows that she is active in period t. Since such a consumer is more likely to be active when α_t is high, Bayesian updating will shift her beliefs toward higher realizations of α_t, and through positive correlation between α_t and $\alpha_{t'}$ for $t' \neq t$, towards higher realizations of the demand state $(\alpha_1, ..., \alpha_{T-1})$. Furthermore, because conditional of α_t the likelihood of becoming active in period t generally differs with valuation, these posterior beliefs will vary across active consumers. A consumer of type (v, t) also observes in period τ her private history of when she was released from the queue in prior periods, which we denote by $a_\tau^p \equiv (a_t, ..., a_{\tau-1})$.

Our description of the market therefore defines a dynamic Bayesian Game, and we are interested in sequential equilibria in pure strategies (extended to allow for a continuum of players, and a continuum of player strategies). We look for equilibria in which all consumers of the same type use the same strategy. Since in equilibrium the identities of the producing firms, and the identities of firms posting prices in each period will be indeterminate, we will identify all sequential equilibria with the same aggregate behavior.

Accordingly, on the firm side, a sequential equilibrium will specify an aggregate production quantity q^*, and for each $t = 1, ..., T$, and each history of sales in prior periods $(a_1^r, ..., a_{t-1}^r)$ such that the market remains active in period t (i.e. such that $\sum_{\tau=1}^{t-1} a_\tau^r < q^*$), an aggregate quantity

$a_t^{\max}(a_1^r,...,a_{t-1}^r)$ offered for sale in period t,[19] and a nondecreasing function $p_t(a_t;a_1^r,...,a_{t-1}^r)$: $[0,a_t^{\max}] \to \mathbf{R}_+$ indicating the price of the a-th lowest priced unit offered for sale in period t. On the consumer side, a sequential equilibrium specifies for each period t and each $(a_1^r,...,a_{t-1}^r)$ such that the market remains active in period t, for each consumer type (v,t') with $t' \leq t$, and for each private history a_t^p, a function $\psi_t^{v,t'}(a_t;a_1^r,...,a_{t-1}^r,a_t^p) : [0,a_t^{\max}] \to \{0,1\}$, indicating whether or not she will accept the price $p_t(a_t;a_1^r,...,a_{t-1}^r)$ if this is the lowest priced unit remaining on the market when she arrives at the market, and a measure a_t of output has already been sold in period t.[20] The decision $\psi_t^{v,t'} = 1$ indicates that a consumer of type (v,t') chooses to transact in period t.

3 The Static Model

In this section, we consider the one-period version of the model, in which there is no terminal period T. In essence, this is the model of Prescott (1975), extended to allow for heterogeneous consumers, and arbitrary demand uncertainty. Dana (1999) and Eden (1990) also characterize equilibrium for variants of the model. We significantly generalize their assumption of multiplicative uncertainty and provide a formal proof of inefficiency, but our main contribution lies in the dynamic model analyzed Section 3. The static model provides a benchmark against which we can evaluate the performance of the full-fledged dynamic model.

To simplify notation, we will drop the time subscript in this section. Also, we will only consider consumer strategies in which a consumer purchases if and only if her valuation exceeds the lowest available price, since this is the unique sequentially rational strategy.[21] Since in equilibrium all produced output will be offered for sale, and since firms must be able to recoup their ex-ante production cost of c, the price schedule is a non-decreasing function $p(\cdot) : [0,q^*] \to \mathbb{R}_+$ satisfying $p(0) \geq c$. Given a price schedule $p(\cdot)$, let $\beta(a)$ denote the lowest demand state in which the a-th

[19] Without loss of generality, we assume that any output that would never be sold in period t following the history $(a_1^r,...,a_{t-1}^r)$ is just not offered for sale in that period.

[20] As is common in the literature (see e.g. Gul, Sonnenschein and Wilson (1986)), we do not specify players' strategies following histories that involve simultaneous deviations. Such histories can never arise following single deviations from a candidate equilibrium profile, and hence do not affect the incentive to deviate.

A single deviation by a firm cannot shrink the range of prices offered in equilibrium, but could lead the consumer to observe a price below $p_t(0;a_1^r,...,a_{t-1}^r)$, or above $p_t(a_t^{\max};a_1^r,...,a_{t-1}^r)$. For the latter case we may set $\psi_t^{v,t'} = 0$ for all (v,t') with $t' \leq t$, and for the former case, we would have $\psi_t^{v,t'} = 1$ for all (v,t') with $t' \leq t$ for which $\psi_t^{v,t'}(0;a_1^r,...,a_{t-1}^r,a_t^p) > 0$, so such a price would be sure to be accepted. Firms would therefore prefer to set $p_t(0;a_1^r,...,a_{t-1}^r)$ instead.

[21] Consumers ignore any information learned about the demand state, either through observation of the lowest remaining price or their private information (their valuation, and the fact that they are active), because the market is only open for a single period.

lowest priced unit sells. A sequential equilibrium is then a triple $\{q^*, p(\cdot), \beta(\cdot)\}$ characterized as follows:

Proposition 1 *In any sequential equilibrium, the price function $p(\cdot)$ and the marginal state $\beta(\cdot)$ are continuous nondecreasing functions, satisfying:*

$$p(a)(1 - F(\beta(a)) = c, \text{ for all } a \leq q^* \tag{1}$$

$$p(a) = c, \text{ for all } a \leq a^{\min} \equiv D(c, \underline{\alpha}) \tag{2}$$

$$p(q^*) = \bar{v}, \tag{3}$$

$$\begin{aligned} \beta(a) &= \underline{\alpha}, \text{ for all } a \leq a^{\min} \\ \int_0^a \frac{dz}{D(p(z), \beta(a))} &= 1, \text{ for all } a^{\min} \leq a \leq q^*. \end{aligned} \tag{4}$$

Conversely, any triple satisfying (1)-(4) is a sequential equilibrium. Furthermore, a sequential equilibrium exists, and is uniquely determined.

According to Proposition 1 there is price dispersion, because firms "bet on the demand state". Since $(1 - F(\beta(a))$ is decreasing in a, firms that charge higher prices will be less likely to sell. Furthermore, since competition drives expected profits down to zero, the expected revenue of unit a is just sufficient to recoup the ex-ante cost of production c. Units that are sure to sell, i.e. $a \leq D(c, \underline{\alpha})$, are priced at marginal cost. Over the remainder of the range $(D(c, \underline{\alpha}), q^*]$ the price function is strictly increasing. Equally, $\beta(\cdot)$ is strictly increasing on $[a^{\min}, q^*]$, so sales in state α equal $\bar{a}(\alpha) = \beta^{-1}(\alpha)$.

Equation (4) says that residual demand for unit a vanishes when the demand state equals $\beta(a)$. To see why, note that our assumption that consumers are randomly ordered in a queue is equivalent to proportional rationing (Davidson and Deneckere, 1986). As a consequence, the residual demand in state α at prices $p \geq p\,(a)$ is a fraction of the demand $D(p, \alpha)$. We claim that this fraction equals

$$\lambda(a, \alpha) = 1 - \int_0^a \frac{dz}{D(p(z), \alpha)}. \tag{5}$$

Thus when $\alpha = \beta(a)$ no residual demand remains, i.e. the queue is exhausted.

We can understand (5) as follows. Suppose that the lowest priced unit currently available is unit a. Then all consumer types with valuations $v \geq p(a)$ who remain are equally likely to be able to purchase this unit. Thus the probability that such a consumer receives a unit between a and Δa equals

$$\frac{\Delta a}{\lambda(a,\alpha)D(p(a),\alpha)}.$$

Residual demand decreases by this percentage. Since residual demand equals $\lambda(a,\alpha)D(p(a),\alpha)$, this means that

$$\frac{\partial\lambda}{\partial a} = -\frac{1}{D(p(a),\alpha)},$$

implying (5).

We now address the question of efficiency. Of course, we require feasible allocations to respect the constraint that output must be chosen before demand is realized.

Definition 1 *A feasible allocation is an aggregate quantity, q^*, and consumption probabilities $\phi(v,\alpha)$ for each active consumer type v in each state α such that:*

$$-\int_{\underline{v}}^{\bar{v}} \phi(v,\alpha)D_p(v,\alpha)dv \leq q^*, \textit{ for all } \alpha$$

A feasible allocation is efficient if there is no other feasible allocation yielding higher surplus, where surplus is given by

$$-\int_{\underline{\alpha}}^{\bar{\alpha}}\int_{\underline{v}}^{\bar{v}} v\phi(v,\alpha)D_p(v,\alpha)f(\alpha)dvd\alpha - cq^*.$$

We then have:

Proposition 2 *The equilibrium of the static model is not efficient.*

There are three sources of inefficiency. For low realizations of α, i.e. $\alpha < \beta(q^*)$, there will be unsold goods at the end of the period, and low valuation consumers who did not purchase. This is possible because the market price cannot adjust to dispose of the excess supply once it becomes known that $\alpha < \beta(q^*)$. The second source of inefficiency is that there will be demand states in which, due to the random arrival of consumers, some lower-valuation consumers receive the good while some higher-valuation consumers do not. The third source of inefficiency is that the aggregate quantity of output produced is generally not the quantity that a planner would produce, if the planner could also distribute output as desired.

The proof of Proposition 2 relies on the first source of inefficiency described above, based on the fact that prices cannot adjust to allow low valuation consumers to purchase when demand turns out to be low. As we show in the next section, the dynamic model allows firms to adjust prices so that unsold output can be allocated to later periods. More surprisingly, under some conditions, consumers will choose to sort themselves efficiently over time.

4 The Dynamic Model

Equilibrium in the dynamic model is considerably more complicated than in the static model. The presence of multiple sales periods gives consumers intertemporal substitution possibilities and firms intertemporal arbitrage opportunities that are absent in the static model. When faced with the lowest remaining price $p_t(a_t; a_1^r, ..., a_{t-1}^r)$ in some period $t < T$, a consumer whose valuation exceeds $p_t(a_t; a_1^r, ..., a_{t-1}^r)$ may decide to postpone purchasing, in the hopes of clinching a lower future price. The benefits of such a strategy depends on the distribution of prices that will prevail in the future. As explained in the model section, a consumer's assessment of the likelihood of these events is based upon private information about the state of demand, given the available public information $(a_1^r, ..., a_{t-1}^r, a_t)$. Firms no longer need to put up for sale all the available output in the first period, instead reserving some of it for sale in future periods. Furthermore, firms can dynamically adjust the pricing of unsold output upon the basis of publicly observed information.

Indeed, solving for equilibrium becomes a very arduous task. Prices in future periods will depend upon the residual demand and supply remaining at the end of the current period. Both of these in turn depend upon consumers' purchase behavior in the current period, and this behavior is governed by expectations of future prices. In principle, it is possible to simplify the computation of equilibrium by using backward induction. However, backward induction is made difficult by the high dimensionality of the state variable, the distribution of remaining consumer types in every possible demand state.

Our approach to solving for equilibrium is therefore based upon conjecturing the nature of equilibrium, and then provide conditions under which the proposed candidate equilibrium is indeed the outcome of a sequential equilibrium of the game. The candidate equilibrium has the intuitive property that in each period t the purchase behavior during the period reveals α_t, so that the state of demand $(\alpha_1, ..., \alpha_{T-1})$ is gradually revealed over time. To guarantee the existence and efficiency of such an equilibrium, we make three additional regularity assumptions. To state these assumptions, define aggregate demand over the demand season as

$$D(p,\alpha_1,....,\alpha_{T-1}) = \sum_{t=1}^{T-1} D_t(p,\alpha_t),$$

and let $P(q,\alpha_1,...,\alpha_{T-1})$ denote the inverse demand function associated with $D(p,\alpha_1,....,\alpha_{T-1})$. Next, let q^e denote the efficient quantity, solving

$$E[P(q^e,\alpha_1,...,\alpha_{T-1})] = c.$$

Finally, let $\bar{p}$ denote the highest possible market clearing price

$$\bar{p} = P(q^e,\bar{\alpha}_1,...,\bar{\alpha}_{T-1}).$$

Assumption 4 *$\hat{v} \geq \bar{p}$.*

Assumption 4 says that market frictions due to the delay cost δ are sufficiently low. If Assumption 4 did not hold, then there might exist types (v,t) with $v < \bar{p}$ that have an option value of waiting, but nevertheless purchase in period t because their delay cost $\delta(v)$ is positive. If the demand state turned out to be high, i.e. equal to $(\bar{\alpha}_1,...,\bar{\alpha}_{T-1})$, this purchase behavior would be inconsistent with efficiency, as efficiency dictates that only consumers with valuations above $\bar{p}$ purchase.

Assumption 5 *The logarithm of the hazard rate of any consumer with valuation v in period t, i.e.*

$$\ln\left(-\frac{\frac{\partial D_t}{\partial p}(v,\alpha_t)}{D_t(v,\alpha_t)}\right)$$

is (i) strictly decreasing in α_t; and (ii) strictly supermodular in (v,α_t).

Assumption 5 requires the hazard rate to be increasing in α_t and supermodular in (v,α_t). Its role is to guarantee that for every period t and every history of realized sales $(a_1^r,...,a_{t-1}^r)$ there exists a marginal valuation $v_t^*(a_t;a_1^r,...,a_{t-1}^r) \in (\hat{v},\bar{v})$ such that when a consumer of type (v,t) is released from the queue, and finds that the lowest remaining price equals $p_t(a_t;a_1^r,...,a_{t-1}^r)$, she purchases if and only if $v \geq v_t^*(a_t; a_1^r, ..., a_{t-1}^r)$. Assumption 5(i) has the plausible economic interpretation that D_t becomes less elastic in higher demand states.

Finally, we use a tie breaking rule to uniquely determine the purchasing behavior of consumer types that have no delay cost:

Assumption 6 *Whenever a consumer is indifferent between purchasing in period t and delaying trade, and derives nonnegative surplus from doing so, she purchases in period t.*

Assumption 6 can be thought of as resulting from vanishingly small delay costs.

Under these assumptions, Theorem 1 below establishes the existence of an equilibrium that is revealing over time. The equilibrium has the following properties. In each period t, given any sales history $(a_1^r, ..., a_{t-1}^r)$, firms set aside some of the remaining output for future sales,[22,23] and jointly offer a non-decreasing price schedule on the remaining output $a_t^{\max}(a_1^r, ..., a_{t-1}^r)$. The units that are sure to sell in period t, of which there are $a_t^{\min}(a_1^r, ..., a_{t-1}^r)$ all carry the same price $p_t(0; a_1^r, ..., a_{t-1}^r)$. Some of the remaining consumers from previous generations, and a fraction of the generation t customers purchase at this price. The remainder of the generation t customers face a strictly increasing price schedule $p_t(a_t; a_1^r, ..., a_{t-1}^r)$ for $a_t \in [a_t^{\min}, a_t^{\max}]$. The amount of output sold in period t is strictly increasing in α_t, so that by the end of period t the state α_t is revealed. In period $t+1$, the cheapest available units will be marked down from $p_t(a_t^r; a_1^r, ..., a_{t-1}^r)$ to $p_{t+1}(0; a_1^r, ..., a_{t-1}^r, a_t^r)$, with the markdown percentage decreasing in a_t^r, and reaching zero when a_t^r equals $a_t^{\max}$. The amount sold in period t to previous generation customers is a deterministic function of the sales history.

Theorem 1 *Suppose that Assumptions 1 - 6 hold. Then there exists a revealing equilibrium, in which the output produced and its allocation across consumers is efficient.*[24] *More precisely, $q^* = q^e$, and there exist unique functions $\bar{a}_t(\alpha_t; a_1^r, ..., a_{t-1}^r)$ and $v_t^*(a_t; a_1^r, ..., a_{t-1}^r) \in (\widehat{v}, \bar{v})$, where $\bar{a}_t$ is strictly increasing in α_t, such that:*

(i) If the state equals $(\alpha_1, ..., \alpha_{T-1})$, then $a_t^r = \bar{a}_t(\alpha_t; a_1^r, ..., a_{t-1}^r)$ for all $t = 1, ..., T-1$;

(ii) In state $(\alpha_1, ..., \alpha_{T-1})$ all period T output is offered at the price $P(q^e, \alpha_1, ..., \alpha_{T-1})$;

[22] Thus our model predicts that stockouts never occur, and that regardless of the state of demand $(\alpha_1, ...\alpha_{T-1})$ a positive amount is sold in every period.

[23] Of course, in period T, no output is reserved for future sales, as the market disappears beyond T.

[24] The only inefficiency remaining is that some generation t consumers with positive delay cost purchase in period t+1.

(iii) Prices are martingales, i.e.

$$p_t(a_t; a_1^r, ..., a_{t-1}^r) = E[P(q^e, \alpha_1, ..., \alpha_{T-1}) | \alpha_1 = \alpha_1^r, ..., \alpha_{t-1} = \alpha_{t-1}^r, \alpha_t \geq \beta_t(a_t; a_1^r, ..., a_{t-1}^r)]$$
$$= \frac{\int_{\beta_t(a_t;a_1^r,...,a_{t-1}^r)}^{\bar{\alpha}_t} \int_{\underline{\alpha}_{t+1}}^{\bar{\alpha}_{t+1}} \cdots \int_{\underline{\alpha}_{T-1}}^{\bar{\alpha}_{T-1}} P(q^e, \alpha_1^r, ..., \alpha_{t-1}^r, \alpha_t, ..., \alpha_{T-1}) f(\alpha_1^r, ..., \alpha_{t-1}^r, \alpha_t, ..., \alpha_{T-1}) d\alpha_{T-1} ... d\alpha_t}{\int_{\beta_t(a_t;a_1^r,...,a_{t-1}^r)}^{\bar{\alpha}_t} \int_{\underline{\alpha}_{t+1}}^{\bar{\alpha}_{t+1}} \cdots \int_{\underline{\alpha}_{T-1}}^{\bar{\alpha}_{T-1}} f(\alpha_1^r, ..., \alpha_{t-1}^r, \alpha_t, ..., \alpha_{T-1}) d\alpha_{T-1} ... d\alpha_t},$$

where $(\alpha_1^r, ..., \alpha_{t-1}^r)$ are the revealed demand states in periods $(1, ..., t-1)$ and $\beta_t = \bar{a}_t^{-1}$;

(iv) A consumer of type (v, t) purchases in period t if and only if $v \geq v_t^*(a_t; a_1^r, ..., a_{t-1}^r)$;

(v) Independent of private history, a consumer of type (v, t') with $t' < t$ purchases in period t if and only if $v \geq \bar{p}_t(a_1^r, ..., a_{t-1}^r)$, where $\bar{p}_t(a_1^r, ..., a_{t-1}^r) = P(q^e; \alpha_1^r, ..., \alpha_{t-1}^r, \bar{\alpha}_t, ..., \bar{\alpha}_{T-1})$;

(vi) For all $t = 1, ..., T-1$ the function $\bar{a}_t(\alpha_t; a_1^r, ..., a_{t-1}^r)$ is the solution in a_t to the equation

$$1 = \int_{q_{t-1}^L(a_1^r,...,a_{t-1}^r)}^{a_t} \frac{dz}{D_t(v_t^*(z; a_1^r, ..., a_{t-1}^r), \alpha_t)}, \tag{6}$$

where

$$q_{t-1}^L(a_1^r, ..., a_{t-1}^r) \equiv \sum_{\tau=1}^{t-1} D_\tau(P(q^e, \alpha_1^r, ..., \alpha_{t-1}^r, \bar{\alpha}_t, ..., \bar{\alpha}_{T-1}), \alpha_\tau^r) - \sum_{\tau=1}^{t-1} a_\tau^r. \tag{7}$$

(vii) For each t, the function $v_t^*(a_t; a_1^r, ..., a_{t-1}^r)$ is the solution in v to $\Delta_t(v, a_t; a_1^r, ..., a_{t-1}^r) = 0$, where

$$\Delta_t = \delta(v) - \frac{\int_{\beta_t(a_t;a_1^r,...,a_{t-1}^r)}^{\bar{\alpha}_t} \int_{\underline{\alpha}_{t+1}}^{\bar{\alpha}_{t+1}} \cdots \int_{\underline{\alpha}_{T-1}}^{\bar{\alpha}_{T-1}} P(q^e, \alpha_1^r, ..., \alpha_{t-1}^r, \alpha_t, ..., \alpha_{T-1}) f(\alpha_1^r, ..., \alpha_{t-1}^r, \alpha_t, ..., \alpha_{T-1}) d\alpha_{T-1} ... d\alpha_t}{\int_{\beta_t(a_t;a_1^r,...,a_{t-1}^r)}^{\bar{\alpha}_t} \int_{\underline{\alpha}_{t+1}}^{\bar{\alpha}_{t+1}} \cdots \int_{\underline{\alpha}_{T-1}}^{\bar{\alpha}_{T-1}} f(\alpha_1^r, ..., \alpha_{t-1}^r, \alpha_t, ..., \alpha_{T-1}) d\alpha_{T-1} ... d\alpha_t}$$
$$+ \frac{\int_{\beta_t(a_t;a_1^r,...,a_{t-1}^r)}^{\bar{\alpha}_t} \int_{\underline{\alpha}_{t+1}}^{\bar{\alpha}_{t+1}} \cdots \int_{\underline{\alpha}_{T-1}}^{\bar{\alpha}_{T-1}} P(q^e, \alpha_1^r, ..., \alpha_{t-1}^r, \alpha_t, ..., \alpha_{T-1}) \frac{-\frac{\partial}{\partial p} D_t(v, \alpha_t)}{D_t(v_t^*(a_t; a_1^r, ..., a_{t-1}^r), \alpha_t)} f(\alpha_1^r, ..., \alpha_{t-1}^r, \alpha_t, ..., \alpha_{T-1}) d\alpha_{T-}}{\int_{\beta_t(a_t;a_1^r,...,a_{t-1}^r)}^{\bar{\alpha}_t} \int_{\underline{\alpha}_{t+1}}^{\bar{\alpha}_{t+1}} \cdots \int_{\underline{\alpha}_{T-1}}^{\bar{\alpha}_{T-1}} \frac{-\frac{\partial}{\partial p} D_t(v, \alpha_t)}{D_t(v_t^*(a_t; a_1^r, ..., a_{t-1}^r), \alpha_t)} f(\alpha_1^r, ..., \alpha_{t-1}^r, \alpha_t, ..., \alpha_{T-1}) d\alpha_{T-1} ... d\alpha_t}$$

Since $\bar{a}_t(\alpha_t; a_1^r, ..., a_{t-1}^r)$ is strictly increasing in α_t, condition (i) says that the total measure of sales in period t, a_t^r, is strictly increasing in α_t, and hence reveals the demand state α_t. Revelation of α_t occurs by the end of period t because: (a) for any valuation v the number of newly arriving customers with valuation v or more is strictly increasing in α_t; and (b) the strong law of large numbers guarantees that conditional on $(\alpha_1, ..., \alpha_t)$ the measure of sales in period t is deterministic.

As a consequence of the revelation over time of the demand state $(\alpha_1, ..., \alpha_{T-1})$, the period T price will both be a deterministic function of $(\alpha_1, ..., \alpha_{T-1})$, and be market clearing given the residual demand and supply entering period T. Consumer purchase behavior implies that this price must equal $P(q^e, \alpha_1, ..., \alpha_{T-1})$, the market clearing price based upon actual demand and the output produced for the demand season, as is indicated by (ii). To see why, note that generation t consumers who purchase in period t have valuations above $P(q^e, \alpha_1, ..., \alpha_{T-1})$. This is because for all $a_t \leq \bar{a}_t(\alpha_t; a_1^r, ..., a_{t-1}^r)$ any such consumer type has a valuation v satisfying $v \geq v_t^*(a_t; a_1^r, ..., a_{t-1}^r) \geq \widehat{v} \geq \bar{p} = P(q^e, \bar{\alpha}_1, ..., \bar{\alpha}_{T-1}) \geq P(q^e, \alpha_1, ..., \alpha_{T-1})$. Meanwhile, condition (v) guarantees that any consumer (v, t') with $t' < t$ who purchases in period t is also inframarginal. This is because then we have $v \geq \bar{p}_t(a_1^r, ..., a_{t-1}^r) = P(q^e; \alpha_1, ..., \alpha_{t-1}, \bar{\alpha}_t, ..., \bar{\alpha}_{T-1}) \geq P(q^e, \alpha_1, ..., \alpha_{T-1})$. Thus in equilibrium all available output is allocated efficiently, and the final price equals $P(q^e, \alpha_1, ..., \alpha_{T-1})$.

Firms who post a price $p_t(a_t; a_1^r, ..., a_{t-1}^r)$ in period t know that this unit will sell in period t if and only if $\alpha_t \geq \beta_t(a_t; a_1^r, ..., a_{t-1}^r)$. Since a firm posting this price has the option to withhold sale of this unit until period T, and then sell at the price $P(q^e, \alpha_1, ..., \alpha_{T-1})$, arbitrage requires that $p_t(a_t; a_1^r, ..., a_{t-1}^r)$ is the expectation of the final price, conditional on the information available at the moment in period t that $p_t(a_t; a_1^r, ..., a_{t-1}^r)$ is the lowest remaining price on the market, as required by condition (iii). Thus from the viewpoint of any observer holding only public information, such as firms, posted prices have to be martingales for all values of a_t.

In general, the conditional expectation $p_t(a_t; a_1^r, ..., a_{t-1}^r)$ need not be monotone in a_t. The role of Assumption 2 is to ensure that higher values of α_t shift expectations towards higher realizations of $(\alpha_{t+1}, ..., \alpha_{T-1})$, ensuring the required monotonicity:

Proposition 3 *Suppose that Assumption 2 holds. Then for all t and $(a_1^r, ..., a_{t-1}^r)$ the posted price function $p_t(a_t; a_1^r, ..., a_{t-1}^r)$ is constant in a_t for $a_t \leq a_t^{\min}$, and strictly increasing in a_t for $a_t \in [a_t^{\min}, a_t^{\max}]$.*

Condition (vii) requires that the cut-off valuation v_t^* be indifferent between purchasing in period t and delaying trade by one period. Indeed, if type (v, t) purchased in period t, she would receive a net surplus of $v - p_t(a_t; a_1^r, ..., a_{t-1}^r)$. If type (v, t) declined to purchase, and the realized period t measure of sales turned out to equal a_t^r, then next period she would face the price $p_{t+1}(0; a_1^r, ..., a_{t-1}^r, a_t^r)$.[25] Her net incentive for purchasing is therefore given by

$$\Delta_t = \delta(v) + E[p_{t+1}(0; a_1^r, ..., a_{t-1}^r, a_t^r) | \alpha_1 = \alpha_1^r, ..., \alpha_{t-1} = \alpha_{t-1}^r, \alpha_t \geq \beta_t(a_t^r), \text{ type } (v, t)] - p_t(a_t; a_1^r, ..., a_{t-1}^r)$$

[25] Because existing customers arrive at the beginning of the period $(t+1)$ queue, and because the equilibrium is revealing over time, all such consumers will face the same deterministic price $p_{t+1}(0; a_1^r, ..., a_{t-1}^r, a_t^r)$ in period $(t+1)$.

To calculate the expected period $(t+1)$ price, consider the density over demand states $(\alpha_t, ..., \alpha_{T-1})$, given that one is a (v, t) consumer released from the period t queue when a_t units have been sold in period t:

$$\frac{\frac{-\frac{\partial}{\partial p} D_t(v,\alpha_t)}{D_t(v_t^*(a_t;a_1^r,...,a_{t-1}^r),\alpha_t)} f(\alpha_1^r, ..., \alpha_{t-1}^r, \alpha_t, ..., \alpha_{T-1})}{\int_{\beta_t(a_t;a_1^r,...,a_{t-1}^r)}^{\bar{\alpha}_t} \int_{\underline{\alpha}_{t+1}}^{\bar{\alpha}_{t+1}} \cdots \int_{\underline{\alpha}_{T-1}}^{\bar{\alpha}_{T-1}} \frac{-\frac{\partial}{\partial p} D_t(v,\alpha_t')}{D_t(v_t^*(a_t;a_1^r,...,a_{t-1}^r),\alpha_t')} f(\alpha_1^r, ..., \alpha_{t-1}^r, \alpha_t', ..., \alpha_{T-1}') d\alpha_{T-1}' ... d\alpha_t'} \tag{8}$$

To derive equation (8) from Bayes' rule, notice that given $(\alpha_1^r, ..., \alpha_{t-1}^r)$ the limiting probability of a consumer in the period t queue of new arrivals, taken at random, having a valuation between v and $v + \triangle v$ is proportional to $-\frac{\partial}{\partial p} D_t(v, \alpha_t)/D_t(\underline{v}, \alpha_t)$, and the probability of arriving in the queue when the transactions are between a_t and $a_t + \triangle a_t$ is inversely proportional to the rate at which transactions are occurring, $D_t(v_t^*(a_t; a_1^r, ..., a_{t-1}^r), \alpha_t)/D_t(\underline{v}, \alpha_t)$.

It follows from the martingale property that

$$\begin{aligned} & E[p_{t+1}(0; a_1^r, ..., a_{t-1}^r, a_t)|\alpha_1 = \alpha_1^r, ..., \alpha_{t-1} = \alpha_{t-1}^r, \alpha_t \geq \beta_t(a_t), \text{ type } (v,t)] \\ & \quad = E[E[P(q^e, \alpha_1, ..., \alpha_{T-1})|\alpha_1 = \alpha_1^r, ..., \alpha_{t-1} = \alpha_{t-1}^r] \mid \alpha_t \geq \beta_t(a_t), \text{ type } (v,t)] \\ & \quad = E[P(q^e, \alpha_1, ..., \alpha_{T-1})|\alpha_1 = \alpha_1^r, ..., \alpha_{t-1} = \alpha_{t-1}^r, \alpha_t \geq \beta_t(a_t), \text{ type } (v,t)] \end{aligned} \tag{9}$$

Thus the net incentive for purchasing to type v is given by condition (vii), i.e. we must have $\Delta_t(v_t^*(a_t; a_1^r, ..., a_{t-1}^r), a_t) = 0$.

Condition (iv) says that conditional on the pre-history $(a_1^r, ..., a_{t-1}^r)$ and a_t units having sold in period t, when a generation t consumer is released from the queue, purchase behavior is monotone in the valuation v. The existence of a cut-off rule can be understood as follows. In Lemma 7 of the Appendix, we show that Assumption 5 implies that

$$\ln\left(\frac{-\frac{\partial D_t}{\partial p}(v, \alpha_t)}{D_t(v_t^*(a_t; a_1^r, ..., a_{t-1}^r), \alpha_t)}\right)$$

is supermodular in (v, α_t), and this in turn means that (8) is strictly increasing in v. Intuitively, Assumption 5 ensures that higher valuation generation t consumer types give relatively more weight to high-demand states than lower valuation types, so they have more pessimistic beliefs over future prices. Thus higher valuation types are more likely to purchase.

To understand the incentives to delay trade more precisely, we define the *information effect* to be the difference between the expected final price from the viewpoint of a consumer of type (v, t) who is released from the queue when the measure of units sold in period t equals a_t, and prior demand states were revealed to equal $(\alpha_1^r, ..., \alpha_{t-1}^r)$, and the expected final price from the viewpoint of the firms at the same moment in time, i.e. the last two terms in the expression for Δ_t. Our proof uses Assumption 5(i) to establish that a consumer whose valuation equals the cut-off valuation, i.e. type $(v_t^*(a_t; a_1^r, ..., a_{t-1}^r), t)$, assigns more weight to low demand states than firms, so she has higher expectations of future prices than firms. Thus for this type the information effect is negative. It follows that if consumer $v_t^*(a_t; \alpha_1^r, ..., \alpha_{t-1}^r)$ is to be indifferent, she must have positive delay cost, i.e. $v_t^*(a_t; a_1^r, ..., a_{t-1}^r) > \widehat{v}$. At the same time, we show that in the limit as $v_t^* \to \bar{v}$, type $(v_t^*(a_t; a_1^r, ..., a_{t-1}^r), t)$ assigns the same relative weight to all demand states as do firms. Since $\delta(\bar{v}) > 0$, this implies that $v_t^*(a_t; a_1^r, ..., a_{t-1}^r) < \bar{v}$. The monotonicity of the expected period $(t+1)$ price in valuation then implies that there exists a unique valuation $\widetilde{v}_t \in (v_t^*(a_t; a_1^r, ..., a_{t-1}^r), \bar{v})$ such that the information effect is positive for all (v, t) with $v > \widetilde{v}_t$, and negative for all $v < \widetilde{v}_t$. Since delay costs are strictly increasing in v for $v \geq \widehat{v}$, this explains why all types (v, t) with $v < v_t^*(a_t; \alpha_1^r, ..., \alpha_{t-1}^r)$ choose to delay trade, and all types with $v > v_t^*(a_t; a_1^r, ..., a_{t-1}^r)$ purchase.

Consumer types (v, t) whose valuation is below $\bar{p}_t(a_1^r, ..., a_{t-1}^r) = P(q^e, \alpha_1^r, ..., \alpha_{t-1}^r, \bar{\alpha}_t, ..., \bar{\alpha}_{T-1})$ have an additional reason to delay trade, as there is then a positive probability that the final price will strictly exceed their valuation. Such consumer types face no cost of delay, and have the option to wait to purchase until period T, and purchase only if their valuation exceeds the final price. This strategy would yield higher expected surplus than would be obtained by purchasing at the price $p_t(a_t; a_1^r, ..., a_{t-1}^r)$. To see why, note that competition by firms ensures that for any history $(a_1^r, ..., a_{t-1}^r)$ and any a_t, the current price $p_t(a_t; a_1^r, ..., a_{t-1}^r)$ equals the expected final price, conditional on the currently available public information. Thus a consumer type (v, t) that only knows the market information (i.e. does not condition on her type when she is able to purchase at the current price) is indifferent between paying the current price and always purchasing in period T. Consumer types (v, t) with $v < \bar{p}_t(a_1^r, ..., a_{t-1}^r)$ strictly prefer to wait to purchase until period T, no matter what price they face in period t, due to the option value of not purchasing in period T. We refer to this effect as the *option-value effect.*

In our model, the information effect is a short-lived phenomenon, as it is only operational for newly arriving consumers in any given period. Indeed, since the equilibrium is revealing over time, consumers who arrived in previous periods and have not yet purchased will hold the same information as the market, so their purchase behavior is solely guided by the option value effect. This explains condition (v). Condition (v) also implies that the number of previous generation consumers who purchase in period t is given by $q_{t-1}^L(a_1^r, ..., a_{t-1}^r)$. The quantity $q_{t-1}^L(\alpha_1^r, ..., \alpha_{t-1}^r)$ consists of: (a) all consumers of type $(v, t-1)$ with

$$v \geq P(q^e, \alpha_1^r, ..., \alpha_{t-1}^r, \bar{\alpha}_t, ..., \bar{\alpha}_{T-1})$$

who did not purchase in period $t-1$, because their valuation was below the relevant cutoff when they arrived at the market, $v_t^*(a_{t-1}; \bar{a}_1, ..., \bar{a}_{t-2})$, and (b) all consumers born in periods 1 through $t-2$ for whom we have

$$P(q^e, \alpha_1^r, ..., \alpha_{t-1}^r, \bar{\alpha}_t, ..., \bar{\alpha}_{T-1}) < v < P(q^e, \alpha_1^r, ..., \alpha_{t-2}^r, \bar{\alpha}_{t-1}, ..., \bar{\alpha}_{T-1}).$$

This second group of consumers has $\delta(v) = 0$, but they strictly preferred not to purchase in previous periods because of the option value effect. Because the highest possible market clearing price conditional on knowing that $(\alpha_1, ..., \alpha_{t-1}) = (\alpha_1^r, ..., \alpha_{t-1}^r)$ is now below their valuation, and because the price they face in period t equals the conditional expectation of the price they will face in period T, $P(q^e, \alpha_1, ..., \alpha_{T-1})$, they are optimally choosing to purchase in period t.

Condition (vi) generalizes condition (4) to the dynamic model, and says that when the demand state in period t equals α_t, the queue of newly arriving customers is exhausted when the number of units sold in period t equals $\bar{a}_t(\alpha_t; a_1^r, ..., a_{t-1}^r)$. To see why, consider a unit priced at $p_t(a_t; a_1^r, ..., a_{t-1}^r)$. Since consumers of type (v, t) will purchase unit a_t if and only if $v \geq v_t^*(a_t; a_1^r, ..., a_{t-1}^r)$, the density of demand that can be satisfied by that unit equals $1/D_t(v_t^*(a_t; \alpha_1^r, ..., \alpha_{t-1}^r), \alpha_t)$. Integrating over all a_t between $q_{t-1}^L(a_1^r, ..., a_{t-1}^r)$ and $\bar{a}_t(\alpha_t; a_1^r, ..., a_{t-1}^r)$ then produces the condition that the queue is exhausted in state α_t when sales in period t reach the level $a_t = \bar{a}_t(\alpha_t; a_1^r, ..., a_{t-1}^r)$.

The equation $\Delta_t = 0$ in condition (vii) and equation (6) in condition (vi) is a coupled system of functional equations in the unknown functions $v_t^*(a_t; a_1^r, ..., a_{t-1}^r)$ and $\bar{a}_t(\alpha_t; a_1^r, ..., a_{t-1}^r)$, or equivalently in $v_t^*(a_t; a_1^r, ..., a_{t-1}^r)$ and $\beta_t(a_t; a_1^r, ..., a_{t-1}^r)$. Our proof uses Assumption 5 (ii) to show that given $\beta_t(a_t; a_1^r, ..., a_{t-1}^r)$ there exists a unique cutoff valuation $v_t^*(a_t; a_1^r, ..., a_{t-1}^r)$. This reduces the system to a single functional equation in the function $\beta_t(a_t; a_1^r, ..., a_{t-1}^r)$. We then set up a map whose fixed points correspond to the solutions of this functional equation, and iteratively use the Banach fixed point theorem to show that the map has a fixed point, and that this fixed point is unique.

In our model asymmetric information is not a source of equilibrium multiplicity, as indicated in our next result:

Proposition 4 *Suppose that the regularity condition*

$$\frac{\partial^2}{\partial p \partial \alpha} D_t(p, \alpha_t) < 0$$

holds for all α_t *and all* $c \leq p \leq \alpha_t$. *Then if there is a single demand parameter, i.e.* $T = 2$, *there is a unique sequential equilibrium.*

Proposition 4 thus provides additional justification for focusing on sequential equilibria that are revealing over time.[26]

4.1 A Computed Example

The following example illustrates our characterization of equilibrium. There are two periods, $T = 2$, so we will drop the period subscript when no ambiguity arises. Demand in period 1 is given by

$$D(p, \alpha) = 1 - p^{\alpha},$$

so we have $\underline{v} = 0$ and $\bar{v} = 1$. There are no newly born consumers in period 2. Inverse demand is given by

$$P(q, \alpha) = (1 - q)^{1/\alpha}.$$

We assume that α is distributed uniformly on $[1, 2]$. Thus, we have $f(\alpha) = 1$ for all $\alpha \in [1, 2]$.

[26] We have no uniqueness proof for $T \geq 3$. The difficulty in extending the argument is that in the residual economy in period $T - 1$, old consumers are present. Such consumers might have an option value of waiting, but nevertheless purchase in period $T - 1$ because of nonzero delay costs. In turn, $\delta(v) > 0$ is possible because inefficient purchases in previous periods could cause the residual economy to have a market clearing price greater than $\hat{v}$.

Then the equilibrium quantity $q^* = q^e$ solves

$$\int_1^2 (1-q)^{1/\alpha} d\alpha = c.$$

Select c such that $q^* = q^e = \frac{1}{4}$; thus $c \approx 0.819876$. Since $\bar{\alpha} = 2$, we have

$$\bar{p} = (1 - \frac{1}{4})^{1/2} = \frac{\sqrt{3}}{2} \simeq 0.866025.$$

The delay-cost function $\delta(v)$ is of the form $\delta(v) = A(v - \hat{v})$, with $\hat{v} = \frac{\sqrt{3}}{2}$. In the on-line appendix, we verify that all of our assumptions are satisfied for this example.

Suppose that $A = \frac{1}{100}$. The first equilibrium condition is that, for each a_1, $v^*(a_1)$ solves

$$\Delta(v, a_1) = \frac{1}{100}(v - \hat{v}) - \frac{\int_{\beta(a_1)}^2 (\frac{3}{4})^{1/\alpha} d\alpha}{2 - \beta(a_1)} + \frac{\int_{\beta(a_1)}^2 (\frac{3}{4})^{1/\alpha} (\frac{\alpha v^{\alpha-1}}{1-v^\alpha}) d\alpha}{\int_{\beta(a_1)}^2 (\frac{\alpha v^{\alpha-1}}{1-v^\alpha}) d\alpha}. \tag{10}$$

The second equilibrium condition is that $\bar{a}_1(\alpha)$, which is the inverse of $\beta(a_1)$, satisfies

$$1 = \int_0^{\bar{a}_1(\alpha)} \frac{dz}{1 - v^*(z)^\alpha}. \tag{11}$$

No closed-form solutions are available, but we have numerically approximated the equilibrium. For $a_1 \leq a_1^{\min}$, $\beta(a_1) = \underline{\alpha} = 1$ holds, so from (10) we see that $v^*(a_1)$ is constant over this range, taking the value $v^*(a_1^{\min}) = 0.91063$. Then from (11), we can compute $a_1^{\min} = 0.08937$. Thus, for $a_1 \leq 0.08937$, the cutoff valuation above which a consumer purchases is 0.91063, nothing is revealed yet about the state, and the price is constant at $p_1(a_1) = c = 0.819876$.

As a_1 continues beyond $a_1^{\min}$, the cutoff valuation, $v^*(a_1)$, falls; the lowest state consistent with sales being a_1, $\beta(a_1)$, rises; and the price, $p_1(a_1)$, also rises. Figure 1 illustrates the functions $v^*(a_1)$, $p_1(a_1)$, and $\beta(a_1)$. The maximum possible measure of sales in period 1, $a_1^{\max}$, is approximately 0.2. When a_1 is close to $a_1^{\max}$, the state is known to be close to $\bar{\alpha}$, so the information effect is small. Therefore, almost all consumers with valuation greater than $\hat{v}$ are willing to purchase and $v^*(a_1)$ is close to $\hat{v} = 0.866025$. Notice that the maximum quantity sold in period 1 is less than the total output. Some output is always withheld, because there are consumers with valuation exceeding $\bar{p}$ who do not purchase in period 1.

The total measure of transactions in period 1, a_1^r, reveals α. In period 2, all firms post the price,

$$p_2(a_2; a_1^r) = P(q^e, \alpha^r) = (\frac{3}{4})^{1/\alpha^r}.$$

If we were to consider a smaller delay cost parameter A, then $v^*(a_1^{\min})$ would be closer to $\bar{v}$ (we have $v^*(a_1^{\min}) = 0.976645$ when $A = \frac{1}{1000}$), but for all delay cost parameters $v^*(a_1)$ declines to $\hat{v}$ as a_1 approaches $a_1^{\max}$. As A becomes smaller, $a_1^{\min}$ and $a_1^{\max}$ decrease. However, the example shows that even for relatively small delay cost parameters, a large fraction of total output can be sold in period 1.

4.2 No Information Effect

The consumer behavior derived in our model is the only behavior that is rational given the available information. At the same time, inferences associated with the information effect are rather sophisticated, and it is interesting to inquire under which conditions this effect is absent, so that consumers' and firms' predictions about future prices coincide. In order for this to be possible, we allow Assumption 5(i)-(ii) to hold in the weak sense. We have:

Lemma 1 *The information effect is absent if and only if for each $t = 1, ..., T-1$ demand is multiplicatively separable in α_t , i.e. we have $D_t(p, \alpha_t) = H(\alpha_t)D_t(p)$ for some strictly increasing function $H(\cdot)$.*

After appropriate rescaling of α_t, the demand from newly arriving customers in period t is then of the form $D_t(p, \alpha_t) = \alpha_t D_t(p)$, so that overall demand is given by $D(p, \alpha_1, ..., \alpha_{T-1}) = \sum_{t=1}^{T-1} \alpha_t D_t(p)$. With multiplicative uncertainty, in updating their beliefs about future demand states, consumers only need to condition on market information. As a consequence, their decision of whether or not to purchase is entirely governed by the option value effect. More precisely, a consumer of type (v, τ) purchases in period $t \geq \tau$ if and only if $v \geq \bar{p}_t(a_1^r, ..., a_{t-1}^r)$. In particular, we have $v_t^*(a_t;_t ; a_1^r, ..., a_{t-1}^r) = \bar{p}_t(a_1^r, ..., a_{t-1}^r)$; all generation t customers with valuations above $\widehat{v}$ therefore purchase in period t, since $\widehat{v} \geq \bar{p} \geq \bar{p}_t(a_1^r, ..., a_{t-1}^r)$. The equilibrium is fully efficient, as the allocation is efficient and no consumer incurs a utility reduction from delaying purchase. Because the purchase strategy is so simple, we may also explicitly solve for the sales function $\bar{a}_t(\alpha_t; a_1^r, ..., a_{t-1}^r)$, and sales made to previous generation consumers, $q_{t-1}^L(a_1^r, ..., a_{t-1}^r)$. We have:

Proposition 5 *Suppose that Assumptions 1-4 and 6 hold. Then the model with multiplicative uncertainty has an equilibrium that is Pareto optimal, characterized as follows:*

(i) $q^* = q^e$

(ii) If the state equals $(\alpha_1, ..., \alpha_{T-1})$*, then* $a_t^r = \bar{a}_t(\alpha_t; a_1^r, ..., a_{t-1}^r)$ *for all* $t = 1, ..., T-1$*, where*

$$\bar{a}_t(\alpha_t; a_1^r, ..., a_{t-1}^r) = q_{t-1}^L(a_1^r, ..., a_{t-1}^r) + \alpha_t D_t(\bar{p}_t(a_1^r, ..., a_{t-1}^r))$$

(iii) For all $t = 1, ..., T$*, and all histories* $(a_1^r, ..., a_{t-1}^r)$*, a consumer of type* (v, t') *with* $t' \leq t$ *purchases if and only if* $v \geq \bar{p}_t(a_1^r, ..., a_{t-1}^r)$

(iv) For all $t = 1, ..., T$*, and all histories* $(a_1^r, ..., a_{t-1}^r)$ *we have*

$$p_t(a_t; a_1^r, ..., a_{t-1}^r) = E[P(q^e, \alpha_1, ..., \alpha_{T-1}) | \alpha_1 = \alpha_1^r, ..., \alpha_{t-1} = \alpha_{t-1}^r, \alpha_t \geq \beta_t(a_t; a_1^r, ..., a_{t-1}^r)]$$

where

$$\beta_t(a_t; a_1^r, ..., a_{t-1}^r) = \begin{cases} \underline{\alpha}_t, & \text{if } a_t \leq q_{t-1}^L(a_1^r, ..., a_{t-1}^r) + \underline{\alpha}_t D_t(\bar{p}_t(a_1^r, ..., a_{t-1}^r)) \\ \frac{a_t - q_{t-1}^L(a_1^r, ..., a_{t-1}^r)}{D_t(\bar{p}_t(a_1^r, ..., a_{t-1}^r))}, & \text{if } a_t \geq q_{t-1}^L(a_1^r, ..., a_{t-1}^r) + \underline{\alpha}_t D_t(\bar{p}_t(a_1^r, ..., a_{t-1}^r)) \end{cases}$$

(v) For all $t = 1, ..., T$*, and all histories* $(a_1^r, ..., a_{t-1}^r)$ *we have*

$$q_{t-1}^L(a_1^r, ..., a_{t-1}^r) = \sum_{\tau=1}^{t-1} \alpha_\tau^r [D_\tau(\bar{p}_t(a_1^r, ..., a_{t-1}^r)) - D_\tau(\bar{p}_{t-1}(a_1^r, ..., a_{t-2}^r))]$$

The closed form nature of the solutions in the multiplicative demand case allow us to make qualitative predictions about the pattern of sales over time. Under suitable technical conditions, it can be shown that sales in any period t, expressed as a fraction of total sales over the demand period, are bounded above by a constant which is independent of t and approach zero as T grows arbitrarily large. The intuition for this result is that the only consumers who purchase in period t are: (i) those born before period t with valuations between $\bar{p}_{t-1}$ and $\bar{p}_t$, and these prices do not differ by much when T is large, and (ii) those born in period t with valuations above $\bar{p}_t$, and this quantity is small relative to overall sales when T is large. The derivation of this result, as well as an example with with linear per period demand and uniformly distributed uncertainty, can be found in the on-line appendix.

4.3 Empirical Implications

Our model has a rich set of predictions, regarding sales over the demand season, prices as a function of the time to the end of the season, and the variability of prices over time. Amongst these, the predictions regarding the pattern of prices appear most suited to empirical testing. Here we highlight the price predictions of our model, and compare them to some of the extant empirical literature.

It follows from the martingale characterization of prices, that for each t we have

$$E[p_t(0; a_1^r, ..., a_{t-1}^r)] = p_1(0) = c$$

Thus our model predicts that for a given market the ex ante expectation of the lowest available price posted in period t is constant. Also, because $p_t(0; a_1^r, ..., a_{t-1}^r)$ follows a martingale, measures of its ex-ante variability, such as the spread between $p_t(0; a_1^{\min}, ..., a_{t-1}^{\min})$ and $p_t(0; a_1^{\max}, ..., a_{t-1}^{\max})$, or the variance of $p_t(0; a_1^r, ..., a_{t-1}^r)$ will tend to increase with t.

Because observations on the time path of prices in a given market are rarely available, we consider a panel of related markets having an identical length of the sales season. Denote a given market by ω, and suppose the data are obtained by sampling from a prior distribution $G(\omega, \alpha_1, ..., \alpha_{T-1})$. Also assume that the marginal cost for market ω is given by $c(\omega)$. Let us compute the average lowest price in period t, over markets ω and realizations of demand uncertainty $(\alpha_1, ..., \alpha_{T-1})$. Denoting the lowest price in period t by $\underline{p}_t(\omega, \alpha_1, ..., \alpha_{T-1})$ we have

$$E(\underline{p}_t(\omega, \alpha_1, ..., \alpha_{T-1})) = E_\omega(p_1(\omega)) = E_\omega(c(\omega))$$

Under suitable assumptions on the dependence across markets, the empirical average $\frac{1}{n}\sum \underline{p}_t(\omega, \alpha_1, ..., \alpha_{T-1})$ converges in probability to $E(\underline{p}_t(\omega, \alpha_1, ..., \alpha_{T-1}))$,[27] so in large panels we should see a constant time path of average prices.[28]

Some recent empirical studies examine pricing as a function of the number of days before the end of the market period, most notably the airline industry. The airline industry fits our model well: demand is highly uncertain, the departure date defines a clear end of the market season, capacity for a given departure date must be fixed before demand is known, pricing is highly competitive (although

[27] See Holtz-Eakin, Newey and Rosen (1988), and the subsequent literature.

[28] However, conditional on the history through period $t-1$, the lowest price in period t equals the conditional expectation of the final price, which is a "shadow cost" typically different from $c(\omega)$.

admittedly not perfectly competitive), and prices for unsold seats can be adjusted frequently.

The available empirical studies on airline pricing as a function of the time to departure paint a fairly consistent picture. First, prices on a given route generally do not increase monotonically over time, and patterns vary significantly across routes (Piga and Bachis, 2006). Second, average lowest available fares, as a function of days to departure, are fairly flat until the final two weeks, at which point prices rise more and more steeply.[29] A similar pattern arises in the hotel industry: Lee et al (2009) show that normalized average prices are nearly independent of the number of days before arrival.

The empirical finding that particular realizations of lowest available fares are nonmonotonic functions of days to departure, but when averaged over many realizations remain flat up until two weeks prior to departure is consistent with our theoretical prediction that beginning-of-period prices are martingales.[30] In contrast, the static model used as the basis for the empirical work in Escobari and Gan (2007) cannot explain that lowest-available prices for particular flights are nonmonotonic, and that nearly all seats will be sold eventually.

5 Conclusion

In this paper, we developed a dynamic trading model of a "sales season" characterized by uncertainty in demand and production in advance. Information about aggregate demand is dispersed across consumers, resulting in information asymmetries amongst consumers, and between consumers and firms. Firms can adjust their prices from one period to the next, to reflect information about aggregate demand extracted from observing the aggregate quantity sold in each period. Consumers who have not yet purchased and newly born consumers visit the market once each period, and in light of their private information and information extracted from observing the past history of sales and the lowest remaining posted price, decide whether to purchase or wait to try and purchase in later periods. In equilibrium, the demand state is gradually learned over time, in a recurring pattern of increasing transaction prices within a period, reflecting increased optimism by firms regarding the

[29] See Escobari and Gan (2007), Escobari (2007, 2009), Mantin and Koo (2009), Pels and Rietveld (2004), and Bilotkach et al (2010).

[30] To explain higher fares in the last week prior to departure, one could incorporate random last minute arrivals. In equilibrium, a martingale result holds: the price following any history must equal the conditional expectation of the *lowest* price posted in period T. Pricing in period T, however, will resemble the equilibrium of the static model, with the lowest price equalling the endogenous shadow value of output rather than the marginal cost of production. Newly arriving customers in period T will therefore typically face higher prices than customers born earlier.

demand state, followed by a markdown at the start of the next period, reflecting a deterioration of expectations caused by the drying up of sales towards the end of the period. By the end of the demand season, the state of demand is fully revealed. Even more surprisingly, under some additional regularity conditions, the output produced and its distribution across consumers are efficient.

Our model assumes that firms share the same marginal cost, c. Heterogeneity in costs can easily be accomodated, as follows. Let $c(q)$ denote the cost of the q'th lowest cost firm, and suppose that $c(\cdot)$ is strictly increasing. Equilibrium is then characterized by an aggregate output q^*, such that the marginal firm to enter, whose cost equals $c(q^*)$, earns zero profits. All firms with $q < q^*$ enter and earn positive rents. In the post-entry stage, production costs are sunk, and equilibrium follows our construction for a common marginal cost $c(q^*)$.[31]

It should also be possible to relax Assumption 4, which requires that consumer types with valuation $v < \bar{p}$ have a zero delay cost. This assumption ensures that in equilibrium all output is allocated efficiently, and hence that the period T price equals $P(q^*, \alpha_1, ..., \alpha_{T-1})$. With a more general delay cost, α_t can still be revealed by the end of any period $t \leq T-1$, and the martingale property will hold, but output will typically not be allocated efficiently. As a consequence, the period T price $P_T(\alpha_1, ..., \alpha_{T-1})$ will exceed $P(q^*, \alpha_1, ..., \alpha_{T-1})$. A delicate additional fixed point argument will then be needed to establish the existence of an equilibrium function $P_T(\alpha_1, ..., \alpha_{T-1})$. Conditional upon existence, however, we expect the main qualitative properties of equilibrium to persist. In particular, equilibrium should be approximately allocatively efficient when delay costs are sufficiently low.

A more ambitious extension would relax the assumption that all production occurs prior to the demand season. Assuming that firms have an incentive to smooth production over time, they will then build inventories to guard themselves against the possibility of being unable to serve the market when demand turns out to be unusually high. To the best of our knowledge, no one has analyzed such a model when consumers can intertemporally substitute, and optimize the timing of their purchases.[32]

It would also be interesting to extend our analysis to an oligopolistic environment. The pioneering

[31] To determine aggregate output, let $q(c)$ denote the equilibrium output level in a model with common marginal cost c, and note that $q(\cdot)$ is a decreasing function. Then q^* is the unique intersection point of the curves $q(c)$ and $c(q)$.

[32] Judd (1996) develops a dynamic pricing model in which firms hold inventories. However, in his model there is neither demand uncertainty, nor intertemporal substitution in consumption.

work of Dana (1999) has shown that such an extension is possible in the static context. With oligopolistic competition, the incentive to hold fire sales once the demand state is revealed will be dampened, but will not disappear entirely. Consumers will still allocate themselves optimally over time, but the overall equilibria will fail to be efficient because of the presence of market power.

We leave these extensions for future work.

6 Appendix: Proofs

Proof of Proposition 1. We start by showing that in any sequential equilibrium $p(\cdot)$ and $\beta(\cdot)$ are continuous functions, and that (1)-(4) hold. We do this in a series of five steps.

First, we show that (1) holds. To see this, note that all firms must make zero expected profits in any sequential equilibrium. No firm's expected profits can be strictly negative, since it can always decide not to produce. Furthermore, since $M > C$, there exists a set of firms of measure no lower than $M - C$, whose equilibrium profits must be zero. Suppose now that expected profits were strictly positive for some firm m. Such a firm must necessarily be active, and charge a price $p > c$. By undercutting p by an amount $\varepsilon > 0$, an inactive firm would experience a probability of sale no lower than that of firm m. Hence for ε sufficiently small, this deviation would yield positive expected profits, contradicting equilibrium. We conclude that for any active firm equation (1) must hold, and that any such firm must offer its product for sale, i.e. that $a^{\max} = q^*$. Since $p(\cdot)$ is nondecreasing, it follows from (1) that $\beta(\cdot)$ is also nondecreasing.

Second, we argue that (2) holds. Since a firm setting a price $p < c$ would never be able to recoup its up front production cost, and since a firm pricing more than $\bar{v}$ would not sell, we must have $c \leq p(a) \leq \bar{v}$, for all $a \in [0, q^*]$. If we had $p(0) > c$, then an inactive firm could profitably deviate by producing, and selling its output at a price $p' \in (c, p(0))$, contradicting equilibrium. Thus, $p(0) = c$ holds. Define $a^c \equiv \sup\{a \leq q^* : p(a) = c\}$, and suppose that we have $a^c < a^{\min}$. For all $\varepsilon \in (0, q^* - a^c)$ define $a^\varepsilon = a^c + \varepsilon$, and note that $p(a^\varepsilon) > c$. Consider an inactive firm deviating by producing, and posting the price $p^\varepsilon \equiv \min\{c + \varepsilon, p(a^\varepsilon)\}$. This firm sells in every demand state state whenever we have $D(p^{\varepsilon}, \underline{\alpha}) \geq a^{\varepsilon}$. We know that

$$D(p^{\varepsilon}, \underline{\alpha}) \geq D(c + \varepsilon, \underline{\alpha}) \quad \text{and}$$
$$D(c, \underline{\alpha}) \geq a^{\min} > a^{c} = a^{\varepsilon} - \varepsilon$$

hold. By the continuity of demand, it follows that for sufficiently small ε, we have $D(p^{\varepsilon}, \underline{\alpha}) \geq a^{\varepsilon}$. This means the deviation is strictly profitable, contradicting equilibrium. Thus, $a^{c} = a^{\min}$, so (2) holds.

Third, we argue that (3) holds. We have already shown that $p(q^*) \leq \bar{v}$. If we had $\bar{v} > p(q^*)$, then an inactive firm choosing to produce instead, and post the price $\bar{v}$, would earn expected profits of $\bar{v}(1 - F(\beta(q^*)) - c > p(q^*)(1 - F(\beta(q^*)) - c = 0$, contradicting equilibrium. Thus $p(q^*) = \bar{v}$. Also, from (2), it follows that at least $a^{\min}$ units are sold in all states, so we have $\beta(a) = \underline{\alpha}$, for all $a \leq a^{\min}$.

Fourth, we argue that (4) holds. Recall that since consumers only have a single chance to purchase, a consumer with valuation v purchases if and only if $p(a) \leq v$. Since consumers are randomly arranged in the queue, the residual demand in state α when all units in the interval $[0, a)$ have been sold equals

$$D(p, \alpha) - \int_0^a \frac{D(p, \alpha)}{D(p(z), \alpha)} dz, \text{ for } p \geq p(a)$$

By definition of $\beta(\cdot)$, this residual demand must be zero at $\alpha = \beta(a)$ and $p = p(a)$. It follows that (4) holds whenever $D(p(a), \beta(a)) > 0$, which is the case when $p(a) < \bar{v}$. We now claim that $p(a) < \bar{v}$ for all $a < q^*$, establishing that (4) is valid for all $a < q^*$. Suppose that contrary to the claim we have $p(a) = \bar{v}$ for some $a < q^*$. Then of positive measure of output, at least $q^* - a$, is priced at $\bar{v}$, but $D(\bar{v}, \alpha) = 0$ holds for all α. Therefore, the probability that a firm posting $\bar{v}$ sells is zero, contradicting equilibrium.

Let us now show that (4) also holds at $a = q^*$. Since $\beta(\cdot)$ is nondecreasing, for any $a < q^*$ we have

$$1 = \int_0^a \frac{dz}{D(p(z), \beta(a))} \geq \int_0^a \frac{dz}{D(p(z), \beta(q^*))}$$

Taking limits as a approaches q^* from below we therefore have

$$\int_0^{q^*} \frac{dz}{D(p(z),\beta(q^*))} \leq 1. \tag{12}$$

We shall now establish that (12) holds with equality. To this effect, define

$$R(p,\beta) = \begin{cases} \frac{D(p,\beta)}{D(p,\beta(q^*))}, & \text{if } p < \bar{v} \\ \frac{D_p(\bar{v},\beta)}{D_p(\bar{v},\beta(q^*))}, & \text{if } p = \bar{v} \end{cases}$$

and let $r(\beta) = \min_p R(p,\beta)$. Since the function R is jointly continuous in (p,β), it follows from the theorem of the maximum that $r(\cdot)$ is a continuous function. If (12) held with strict inequality, there would exist $K > 0$ such that

$$\begin{aligned} 1-K = \int_0^{q^*} \frac{dz}{D(p(z),\beta(q^*))} &= \int_0^{a} \frac{dz}{D(p(z),\beta(q^*))} + \int_a^{q^*} \frac{dz}{D(p(z),\beta(q^*))} \\ &\geq r(\beta(a)) \int_0^{a} \frac{dz}{D(p(z),\beta(q^*))} + \int_a^{q^*} \frac{dz}{D(p(z),\beta(q^*))} \end{aligned} \tag{13}$$

We shall establish below that (12) implies that $\beta(\cdot)$ is continuous at q^*. It then follows from the Lebesgue dominated convergence theorem and (12) that the right side of (13) converges to $r(\beta(q^*)) = 1$ as a approaches q^* from below, establishing a contradiction. Thus (4) holds at q^*.

Fifth, we show that $\beta(\cdot) : [0,q^*] \to [\underline{\alpha}, F^{-1}(1-\frac{\varepsilon}{\bar{v}})]$ is a continuous function. The argument proceeds in two steps; let us begin by demonstrating that $\beta(\cdot)$ is continuous at q^*. Suppose to the contrary that $\beta(\cdot)$ has a discontinuity at q^*. Let $\alpha' = \lim_{a\uparrow q^*} \beta(a)$ and $p' = \lim_{a\uparrow q^*} p(a)$. We then have $\alpha' < \beta(q^*)$; it then follows from (1) that $p' < \bar{v}$. Now select $\widetilde{a} < q^*$ such that

$$\frac{1-F(\alpha')}{1-F(\beta(\widetilde{a}))}\bar{v} > p'.$$

By construction, the expected revenues of the firm posting $p(\widetilde{a})$ is no more than $(1-F(\beta(\widetilde{a}))p'$. Suppose now that this firm deviated, and posted the price $\bar{v}-\varepsilon$ instead, where $\varepsilon \in (0, \beta(q^*) - \alpha')$. Since the measure of sold output in state α' equals q^*, there must be a positive measure of consumers remaining in the queue when unit q^* is sold in state $\alpha' + \varepsilon$. Thus the firm posting $\bar{v}-\varepsilon$ would sell in all states $\alpha \geq \alpha' + \varepsilon$. Its expected revenues would be no less than $(1 - F(\alpha' + \varepsilon))(\bar{v} - \varepsilon)$, which exceeds $(1 - F(\beta(\widetilde{a}))p'$ for ε sufficiently small, contradicting the hypothesis that $\beta(\cdot)$ has a discontinuity at q^*.

Next, suppose that $\beta(\cdot)$ had a discontinuity at some $a \in [0, q^*)$. Since (1) implies that $\beta(\cdot)$ must be nondecreasing, it follows that either there exists $a_n \uparrow a$ such that $\widetilde{\beta} = \lim_{n\to\infty} \beta(a_n) < \beta(a)$, or there exists $a_n \downarrow a$ such that $\beta(a) > \widetilde{\beta} = \lim_{n\to\infty} \beta(a_n)$. We shall treat the former case here; the argument for the latter case is analogous. Since $D(p, \beta)$ is strictly increasing in β, and since α_n is an increasing sequence, there exists an increasing sequence δ_n with $\delta_n > 0$ such that for all n,

$$\int_0^{a_n} \frac{dz}{D(p(z), \beta(a))} = \int_0^{a_n} \frac{dz}{D(p(z), \widetilde{\beta})} - \delta_n \leq \int_0^{a_n} \frac{dz}{D(p(z), \beta_n(a))} - \delta_n \leq 1 - \delta_1 \tag{14}$$

At the same time, we have

$$\int_0^{a_n} \frac{dz}{D(p(z), \beta(a))} = \int_0^{a} \frac{dz}{D(p(z), \beta(a))} - \int_{a_n}^{a} \frac{dz}{D(p(z), \beta(a))} \tag{15}$$

It follows from (4) that the first integral on the right side of (15) equals 1. Furthermore, the Lebesgue dominated convergence theorem and (4) imply that the second integral converges to 0 as $n \to \infty$. Thus the left side of (15), and hence of (14), converges to 1 as as $n \to \infty$, yielding a contradiction. This establishes that $\beta(\cdot)$ is continuous on $[0, q^*)$. We conclude that $\beta(\cdot)$ is continuous on all of $[0, q^*]$, and hence from (1) that $p(\cdot)$ is continuous as well.

Finally, we show that any triple $\{q^*, p(\cdot), \beta(\cdot)\}$ that solves (1)-(4), is a sequential equilibrium. Given the consumer strategy and the price function, the minimum state such that unit a is sold is given by $\beta(a)$, from (4). This equation implies that $\beta(\cdot)$ is nondecreasing; equation (1) then shows that $p(\cdot)$ is nondecreasing. It follows from the argument in the previous paragraph that in any solution to (1)-(4) the function $p(\cdot)$ is continuous. Therefore, from (1)-(3) we see that active firms are making zero profits, and that no profitable entry is possible. Thus, the corresponding strategy profile is a sequential equilibrium. ■

We now turn to the question of existence and uniqueness of a number q^* and function $\beta(a)$ satisfying (1) and (4) on the interval $[0, q^*]$. It follows from (3) and (1) that $\bar{v}(1 - F(\beta(q^*)) = c$, or equivalently that

$$\beta(q^*) = F^{-1}(1 - \frac{c}{\bar{v}}).$$

The relations (1) and (4) define a pair functional equations in the unknown functions $p(\cdot)$ and $\beta(\cdot)$. Solving (1) for $p(a)$ and substituting the resulting expression into (4) produces

$$\int_0^a \frac{dz}{D(\frac{c}{1-F(\beta(z))}, \beta(a))} = 1. \tag{16}$$

A solution to the system (1) and (4) is thus a non-decreasing function $\beta(\cdot)$ such that (i) $\beta(a) = \underline{\alpha}$ for $\alpha \in [0, D(c, \underline{\alpha})]$; (ii) β satisfies (16) on the domain $[0, q^*]$, where q^* is the solution in a to

$$\int_0^a \frac{dz}{D(\frac{c}{1-F(\beta(z))}, F^{-1}(1 - \frac{c}{\bar{v}}))} = 1; \tag{17}$$

and (iii) $\beta(q^*) = F^{-1}(1 - \frac{c}{\bar{v}})$.

Let $C([0, D(\underline{v}, \bar{\alpha})])$ denote the set of continuous functions on the domain $[0, D(\underline{v}, \bar{\alpha})]$. We make $C([0, D(\underline{v}, \bar{\alpha})])$ into a metric space, by endowing it with the sup norm. Thus if $\beta \in C([0, D(\underline{v}, \bar{\alpha})])$, its norm is defined as $||\beta|| = \max_{a \in [0, D(\underline{v}, \bar{\alpha})]} |\beta(a)|$.

Let $M \subset C([0, D(\underline{v}, \bar{\alpha})])$ denote the set of non-decreasing continuous functions $\beta : [0, D(\underline{v}, \bar{\alpha})]$ such that $\beta(a) = \underline{\alpha}$ for $\alpha \in [0, D(c, \underline{\alpha})]$ and such that $\beta(D(\underline{v}, \bar{\alpha})) = F^{-1}(1 - \frac{c}{\bar{v}})$. We now define an operator T on M, as follows. Given $\beta \in M$, define q^* to be the solution in a to (17). For every $a \in [0, D(c, \underline{\alpha})]$ define $T\beta(a) = \underline{\alpha}$. For every $a \in [D(c, \underline{\alpha}), q^*]$ define $T\beta(a)$ to be the solution in β to the equation

$$\int_0^a \frac{dz}{D(\frac{c}{1-F(\beta(z))}, \beta)} = 1.$$

Finally, for every $a \in [q^*, D(\underline{v}, \bar{\alpha})])$ define $T\beta(a) = F^{-1}(1 - \frac{c}{\bar{v}})$.

Let let $\underline{M} = T(M)$. It is evident from this definition that $T : \underline{M} \to \underline{M}$, and that a fixed point of T on $\underline{M}$ is a solution to the system (1) and (4), and vice versa. We first prove that any such fixed point must be unique.

Lemma 2 *(Uniqueness) Let $\beta_0 \in \underline{M}$ and $\beta_1 \in \underline{M}$ be such that $T\beta_0 = \beta_0$ and $T\beta_1 = \beta_1$. Then $\beta_0 = \beta_1$.*

Proof. Without loss of generality, let us assume that $q_0^* \leq q_1^*$. Let $a < q_0^*$. Then from (4) we have:

$$
\begin{aligned}
0 &= \int_0^a \frac{dz}{D(\frac{c}{1-F(\beta_0(z))},\beta_0(a))} - \int_0^a \frac{dz}{D(\frac{c}{1-F(\beta_1(z))},\beta_1(a))} \\
&= \int_0^a \frac{D(\frac{c}{1-F(\beta_1(z))},\beta_1(a)) - D(\frac{c}{1-F(\beta_0(z))},\beta_0(a))}{D(\frac{c}{1-F(\beta_0(z))},\beta_0(a))D(\frac{c}{1-F(\beta_1(z))},\beta_1(a))}dz \\
&= \int_0^a \frac{D(\frac{c}{1-F(\beta_1(z))},\beta_1(a)) - D(\frac{c}{1-F(\beta_1(z))},\beta_0(a)) + D(\frac{c}{1-F(\beta_1(z))},\beta_0(a)) - D(\frac{c}{1-F(\beta_0(z))},\beta_0(a))}{D(\frac{c}{1-F(\beta_0(z))},\beta_0(a))D(\frac{c}{1-F(\beta_1(z))},\beta_1(a))}dz \\
&= \int_0^a \frac{D_\alpha(\frac{c}{1-F(\beta_1(z))},\widetilde{\beta}(a))(\beta_1(a)-\beta_0(a)) + D_p(\widetilde{p},\beta_0(a))(\frac{c}{1-F(\beta_1(z))} - \frac{c}{1-F(\beta_0(z))})}{D(\frac{c}{1-F(\beta_0(z))},\beta_0(a))D(\frac{c}{1-F(\beta_1(z))},\beta_1(a))}dz \\
&= (\beta_1(a)-\beta_0(a))\int_0^a \frac{D_\alpha(\frac{c}{1-F(\beta_1(z))},\widetilde{\beta})}{D(\frac{c}{1-F(\beta_0(z))},\beta_0(a))D(\frac{c}{1-F(\beta_1(z))},\beta_1(a))}dz + c\int_0^a \frac{D_p(\widetilde{p}(z),\beta_0(a))(\frac{F(\beta_1(z))-F(\beta_0(z))}{(1-F(\beta_1(z)))(1-F(\beta_0(z)))})}{D(\frac{c}{1-F(\beta_0(z))},\beta_0(a))D(\frac{c}{1-F(\beta_1(z))},\beta_1(a))}dz
\end{aligned}
$$

where the fourth equality follows from the Mean Value Theorem, and where $\widetilde{\beta}(a) = \eta\beta_0(a) + (1-\eta)\beta_1(a)$ for some $\eta \in (0,1)$ and $\widetilde{p}(z) = \zeta p_0(z) + (1-\zeta)p_1(z)$ for some $\zeta \in (0,1)$. Thus we have

$$
|\beta_1(a)-\beta_0(a)| \; \leq \frac{c\int_0^a \frac{-D_p(\widetilde{p},\beta_0(a))\frac{|F(\beta_1(z))-F(\beta_0(z))|}{(1-F(\beta_1(z)))(1-F(\beta_0(z)))})}{D(\frac{c}{1-F(\beta_0(z))},\beta_0(a))D(\frac{c}{1-F(\beta_1(z))},\beta_1(a))}dz}{\int_0^a \frac{D_\alpha(\frac{c}{1-F(\beta_1(z))},\widetilde{\beta})}{D(\frac{c}{1-F(\beta_0(z))},\beta_0(a))D(\frac{c}{1-F(\beta_1(z))},\beta_1(a))}dz} \tag{18}
$$

Let $\bar{f} = \max f(\alpha)$, $\underline{D}_\alpha = \min D_\alpha(p,\alpha)$, $\underline{D}_p = \min D_p(p,\alpha)$. Note that $1-F(\beta_i(z)) \geq 1-F(\beta_i(q^*)) = \frac{c}{\bar{v}}$. Since for $z \leq a^{\min}$, we have $\beta_1(z) = \beta_0(z) = \underline{\alpha}$, it follows from (18) that

$$
|\beta_1(a)-\beta_0(a)| \; \leq \frac{\bar{v}^2(-\underline{D}_p)\bar{f}}{c\underline{D}_\alpha}\frac{\int_{a^{\min}}^a \frac{|\beta_1(z)-\beta_0(z)|}{D(\frac{c}{1-F(\beta_0(z))},\beta_0(a))D(\frac{c}{1-F(\beta_1(z))},\beta_1(a))}dz}{\int_0^a \frac{1}{D(\frac{c}{1-F(\beta_0(z))},\beta_0(a))D(\frac{c}{1-F(\beta_1(z))},\beta_1(a))}dz} \tag{19}
$$

Since

$$
|(\beta_1(z)) - \beta_0(z))| \; \leq \max_{z\in[0,a]} |(\beta_1(z)) - \beta_0(z))|
$$

we conclude from (19) that

$$
|\beta_1(a)-\beta_0(a)| \; \leq \frac{\bar{v}^2(-\underline{D}_p)\bar{f}}{c\underline{D}_\alpha}L_0(a)\max_{z\in[0,a]} |(\beta_1(z)) - \beta_0(z))|
$$

where

$$
L_0(a) = \frac{\int_{a^{\min}}^a \frac{dz}{D(\frac{c}{1-F(\beta_0(z))},\beta_0(a))D(\frac{c}{1-F(\beta_1(z))},\beta_1(a))}}{\int_0^a \frac{dz}{D(\frac{c}{1-F(\beta_0(z))},\beta_0(a))D(\frac{c}{1-F(\beta_1(z))},\beta_1(a))}}
$$

Since $L_0(a^{\min}) = 0$, and since L is continuous in a, we may select $\widetilde{a}_1 > a^{\min}$ such that

$$L_0(a) \leq \frac{1}{2} \frac{c\underline{D}_\alpha}{\bar{v}^2(-\underline{D}_p)\bar{f}}$$

for all $a \in [0, \widetilde{a}_1]$. It follows that

$$|\beta_1(a) - \beta_0(a)| \; \leq \frac{1}{2} \max_{z \in [0,a]} |(\beta_1(z)) - \beta_0(z))| \, ,$$

for all $a \in [a^{\min}, \widetilde{a}_1]$, and hence that $\beta_1(a) = \beta_0(a)$ for all $a \in [0, \widetilde{a}_1]$.

Given that $\beta_1(a) = \beta_0(a)$ for all $a \in [0, \widetilde{a}_n]$, we may then inductively define

$$L_{n+1}(a) = \frac{\int_{\widetilde{a}_n}^a \frac{dz}{D(\frac{c}{1-F(\beta_0(z))}, \beta_0(a)) D(\frac{c}{1-F(\beta_1(z))}, \beta_1(a))}}{\int_0^a \frac{dz}{D(\frac{c}{1-F(\beta_0(z))}, \beta_0(a)) D(\frac{c}{1-F(\beta_1(z))}, \beta_1(a))}}$$

and select $\widetilde{a}_{n+1} > \widetilde{a}_n$ such that

$$L_{n+1}(a) \leq \frac{1}{2} \frac{c\underline{D}_\alpha}{\bar{v}^2(-\underline{D}_p)\bar{f}}$$

for all $a \in [\widetilde{a}_n, \widetilde{a}_{n+1}]$ to conclude that $\beta_1(a) = \beta_0(a)$ for all $a \in [0, \widetilde{a}_{n+1}]$.

This process yields an increasing sequence $\widetilde{a}_n$, which satisfies $\lim_{n\to\infty} \widetilde{a}_n = q_1^*$. We conclude that $q_0^* = q_1^*$ and $\beta_0(a) = \beta_1(a)$ for all $a \in [0, q_1^*]$, as was to be demonstrated. ■

It remains to be shown that T has a fixed point. We provide one such proof below.

Lemma 3 *(Existence) There exists a $\beta \in \underline{M}$ such that $T\beta = \beta$.*

Proof. Select $p_0(a) = c$ for all $a \in [0, D(\underline{v}, \bar{\alpha})]$. Define $q_0^* = D(c, F^{-1}(1 - \frac{c}{\bar{v}}))$, and for $a \in [a^{\min}, q_0^*]$ define $\beta_0(a)$ to be the solution to

$$\int_0^a \frac{dz}{D(p_0(z), \beta_0(a))} = 1$$

i.e. $\beta_0(a)$ then solves

$$D(c, \beta_0(a)) = a.$$

For $a \in [q_0^*, D(\underline{v}, \bar{\alpha})]$ define $\beta_0(a) = \beta_0(q_0^*)$.

Given β_n and q_n^*, inductively define p_{n+1} by

$$p_{n+1}(a) = \frac{c}{1 - F\left(\beta_n(a)\right)}$$

for all $a \in [0, D(\underline{v}, \bar{\alpha})]$. Given p_{n+1}, define q_{n+1}^* as the solution in a to

$$\int_0^a \frac{dz}{D(p_{n+1}(z), F^{-1}(1 - \frac{c}{\bar{v}}))} = 1$$

Finally, define β_{n+1} as follows. For $a \in [a^{\min}, q_{n+1}^*]$ the demand state $\beta_{n+1}(a)$ is the solution in β to the equation

$$\int_0^a \frac{dz}{D(p_{n+1}(z), \beta)} = 1, \tag{20}$$

and for $a \in [q_{n+1}^*, D(\underline{v}, \bar{\alpha})]$ we have $\beta_{n+1}(a) = \beta_{n+1}(q_{n+1}^*)$.

Observe that we necessarily have $p_1 \geq p_0$. Next, we claim that $p_{n+1} \geq p_n$ implies $\beta_{n+1} \geq \beta_n$. To see this, observe that

$$\int_0^a \frac{dz}{D(p_n(z), \beta_n(a))} = 1$$

and $p_{n+1} \geq p_n$ imply

$$\int_0^a \frac{dz}{D(p_{n+1}(z), \beta_n(a))} \geq 1$$

It then follows from the definition of $\beta_{n+1}(a)$ and the fact that $D(p, \beta)$ is strictly increasing in β that $\beta_{n+1}(a) \geq \beta_n(a)$ for all a. Observe that $\beta_{n+1} \geq \beta_n$ implies

$$p_{n+2} = \frac{c}{1 - F\left(\beta_{n+1}(a)\right)} \geq \frac{c}{1 - F\left(\beta_n(a)\right)} = p_{n+1}$$

Hence we have shown that the sequences β_n and p_n are nondecreasing. The definition of q_n^* and the fact that p_n is nondecreasing then imply that the sequence q_n^* is nonincreasing.

Observe that for each a, the monotone sequence $p_n(a)$ belongs to the compact interval $[c, \bar{v}]$, and hence has a limit, which we shall denote by $p(a)$. Let $q^* = \lim_n q_n^*$. Since for each $a \leq q^*$ and each $\beta \leq F^{-1}(1 - \frac{c}{\bar{v}})$ we have

$$\int_0^a \frac{dz}{D(p_n(z), \beta)} \leq \int_0^{q^*} \frac{dz}{D(p_n(z), \beta)} \leq \int_0^{q_n^*} \frac{dz}{D(p_n(z), \beta)} \leq \int_0^{q_n^*} \frac{dz}{D(p_n(z), F^{-1}(1 - \frac{c}{\bar{v}}))} = 1,$$

it follows from the Lebesgue dominated convergence theorem that

$$\int_0^a \frac{dz}{D(p_n(z),\beta)} \to \int_0^a \frac{dz}{D(p(z),\beta)} \tag{21}$$

Let $\beta(a)$ denote the solution in β to

$$\int_0^a \frac{dz}{D(p(z),\beta)} = 1 \tag{22}$$

The strict monotonicity of $D(p,\beta)$ in β then implies that $\beta_n(a) \to \beta(a) \leq q^*$. Therefore for all $a \leq q^*$we have

$$p(a) = \lim_{n\to\infty} p_{n+1}(a) = \lim_{n\to\infty} \frac{c}{1-F(\beta_n(a))} = \frac{c}{1-F(\beta(a))}$$

We conclude that the pair of functions $p(\cdot), \beta(\cdot)$ solves the system (1) and (4), i.e. is a fixed point of T. ■

Proof of Proposition 2. From (1) and (3), we have

$$1 - F(\beta(q^*)) = 1 - \frac{c}{\bar{v}} > 0.$$

For sufficiently small positive ε, and for all α such that $\bar{a}(\alpha) \in (q^* - \varepsilon, q^*)$, it is the case that: (i) there is unsold output in state α; and (ii) the highest price at which output is sold is near $\bar{v}$ and therefore greater than $\underline{v}$. As transactions prices rise above $\underline{v}$, there are type $\underline{v}$ consumers who arrive to the head of the queue and choose not to purchase. If a planner were to allocate the unsold output to consumers who did not purchase in these states, surplus would be higher. ■

Proof of Proposition 3. From the martingale condition (9) we have:

$$p_t(a_t; a_1^r, ..., a_{t-1}^r) = \frac{\int_{\beta_t(a_t;a_1^r,...,a_{t-1}^r)}^{\bar{\alpha}_t} \int_{\underline{\alpha}_{t+1}}^{\bar{\alpha}_{t+1}} \cdots \int_{\underline{\alpha}_{T-1}}^{\bar{\alpha}_{T-1}} P(q^e, \alpha_1^r, ..., \alpha_{t-1}^r, \alpha_t, ..., \alpha_{T-1}) f(\alpha_1^r, ..., \alpha_{t-1}^r, \alpha_t, ..., \alpha_{T-1}) d\alpha_{T-1}...d\alpha_t}{\int_{\beta_t(a_t;a_1^r,...,a_{t-1}^r)}^{\bar{\alpha}_t} \int_{\underline{\alpha}_{t+1}}^{\bar{\alpha}_{t+1}} \cdots \int_{\underline{\alpha}_{T-1}}^{\bar{\alpha}_{T-1}} f(\alpha_1^r, ..., \alpha_{t-1}^r, \alpha_t, ..., \alpha_{T-1}) d\alpha_{T-1}...d\alpha_t}$$

Since $\beta_t(a_t; a_1^r, ..., a_{t-1}^r)$ is constant in a_t for $a_t \leq a_t^{\min}$, it follows that $p_t(a_t; a_1^r, ..., a_{t-1}^r)$ is constant over the interval $a_t \leq a_t^{\min}$. Also, since $\beta_t(a_t; a_1^r, ..., a_{t-1}^r)$ is strictly increasing in a_t for $a_t \geq a_t^{\min}$, it follows that over this interval the sign of $\frac{\partial p_t(a_t;a_1^r,...,a_{t-1}^r)}{\partial a_t}$ equals the sign of

$$
\begin{aligned}
&-\left(\int_{\underline{\alpha}_{t+1}}^{\bar{\alpha}_{t+1}} \cdots \int_{\underline{\alpha}_{T-1}}^{\bar{\alpha}_{T-1}} P(q^e,\alpha_1^r,...,\alpha_{t-1}^r,\beta_t(a_t;a_1^r,...,a_{t-1}^r),...,\alpha_{T-1}) f(\alpha_1^r,...,\alpha_{t-1}^r,\beta_t(a_t;a_1^r,...,a_{t-1}^r),...,\alpha_{T-1}) d\alpha_{T-1}...d\alpha\right. \\
&\left(\int_{\beta_t(a_t;a_1^r,...,a_{t-1}^r)}^{\bar{\alpha}_t} \int_{\underline{\alpha}_{t+1}}^{\bar{\alpha}_{t+1}} \cdots \int_{\underline{\alpha}_{T-1}}^{\bar{\alpha}_{T-1}} f(\alpha_1^r,...,\alpha_{t-1}^r,\alpha_t,...,\alpha_{T-1}) d\alpha_{T-1}...d\alpha_t\right) \\
&+\left(\int_{\underline{\alpha}_{t+1}}^{\bar{\alpha}_{t+1}} \cdots \int_{\underline{\alpha}_{T-1}}^{\bar{\alpha}_{T-1}} f(\alpha_1^r,...,\alpha_{t-1}^r,\beta_t(a_t;a_1^r,...,a_{t-1}^r),...,\alpha_{T-1}) d\alpha_{T-1}...d\alpha_{t+1}\right) \\
&\left(\int_{\beta_t(a_t;a_1^r,...,a_{t-1}^r)}^{\bar{\alpha}_t} \int_{\underline{\alpha}_{t+1}}^{\bar{\alpha}_{t+1}} \cdots \int_{\underline{\alpha}_{T-1}}^{\bar{\alpha}_{T-1}} P(q^e,\alpha_1^r,...,\alpha_{t-1}^r,\alpha_t,...,\alpha_{T-1}) f(\alpha_1^r,...,\alpha_{t-1}^r,\alpha_t,...,\alpha_{T-1}) d\alpha_{T-1}...d\alpha_t\right)
\end{aligned}
$$

which is just the integral from $\beta_t(a_t;a_1^r,...,a_{t-1}^r)$ to $\bar{\alpha}_t$ of the following expression

$$
\begin{aligned}
&-\left(\int_{\underline{\alpha}_{t+1}}^{\bar{\alpha}_{t+1}} \cdots \int_{\underline{\alpha}_{T-1}}^{\bar{\alpha}_{T-1}} P(q^e,\alpha_1^r,...,\alpha_{t-1}^r,\beta_t(a_t;a_1^r,...,a_{t-1}^r),...,\alpha_{T-1}) f(\alpha_1^r,...,\alpha_{t-1}^r,\beta_t(a_t;a_1^r,...,a_{t-1}^r),...,\alpha_{T-1}) d\alpha_{T-1}...d\alpha\right. \\
&\left(\int_{\underline{\alpha}_{t+1}}^{\bar{\alpha}_{t+1}} \cdots \int_{\underline{\alpha}_{T-1}}^{\bar{\alpha}_{T-1}} f(\alpha_1^r,...,\alpha_{t-1}^r,\alpha_t,...,\alpha_{T-1}) d\alpha_{T-1}...d\alpha_{t+1}\right) \\
&+\left(\int_{\underline{\alpha}_{t+1}}^{\bar{\alpha}_{t+1}} \cdots \int_{\underline{\alpha}_{T-1}}^{\bar{\alpha}_{T-1}} f(\alpha_1^r,...,\alpha_{t-1}^r,\beta_t(a_t;a_1^r,...,a_{t-1}^r),...,\alpha_{T-1}) d\alpha_{T-1}...d\alpha_{t+1}\right) \\
&\int_{\underline{\alpha}_{t+1}}^{\bar{\alpha}_{t+1}} \cdots \int_{\underline{\alpha}_{T-1}}^{\bar{\alpha}_{T-1}} P(q^e,\alpha_1^r,...,\alpha_{t-1}^r,\alpha_t,...,\alpha_{T-1}) f(\alpha_1^r,...,\alpha_{t-1}^r,\alpha_t,...,\alpha_{T-1}) d\alpha_{T-1}...d\alpha_{t+1}
\end{aligned}
$$

Thus we will have $\frac{\partial p_t(a_t;a_1^r,...,a_{t-1}^r)}{\partial a_t} > 0$ if

$$
\begin{aligned}
&\frac{\int_{\underline{\alpha}_{t+1}}^{\bar{\alpha}_{t+1}} \cdots \int_{\underline{\alpha}_{T-1}}^{\bar{\alpha}_{T-1}} P(q^e,\alpha_1^r,...,\alpha_{t-1}^r,\alpha_t,...,\alpha_{T-1}) f(\alpha_1^r,...,\alpha_{t-1}^r,\alpha_t,...,\alpha_{T-1}) d\alpha_{T-1}...d\alpha_{t+1}}{\int_{\underline{\alpha}_{t+1}}^{\bar{\alpha}_{t+1}} \cdots \int_{\underline{\alpha}_{T-1}}^{\bar{\alpha}_{T-1}} f(\alpha_1^r,...,\alpha_{t-1}^r,\alpha_t,...,\alpha_{T-1}) d\alpha_{T-1}...d\alpha_{t+1}} \\
&\geq \frac{\int_{\underline{\alpha}_{t+1}}^{\bar{\alpha}_{t+1}} \cdots \int_{\underline{\alpha}_{T-1}}^{\bar{\alpha}_{T-1}} P(q^e,\alpha_1^r,...,\alpha_{t-1}^r,\beta_t(a_t;a_1^r,...,a_{t-1}^r),...,\alpha_{T-1}) f(\alpha_1^r,...,\alpha_{t-1}^r,\beta_t(a_t;a_1^r,...,a_{t-1}^r),...,\alpha_{T-1}) d\alpha_{T-1}...d\alpha_{t+1}}{\int_{\underline{\alpha}_{t+1}}^{\bar{\alpha}_{t+1}} \cdots \int_{\underline{\alpha}_{T-1}}^{\bar{\alpha}_{T-1}} f(\alpha_1^r,...,\alpha_{t-1}^r,\beta_t(a_t;a_1^r,...,a_{t-1}^r),...,\alpha_{T-1}) d\alpha_{T-1}...d\alpha_{t+1}}
\end{aligned}
$$

holds for all $\alpha_t \geq \beta_t(a_t;a_1^r,...,a_{t-1}^r)$. Let

$$
r_{f,t}^i(\alpha_1^r,...,\alpha_{t-1}^r,\alpha_t,...,\alpha_{T-1}) = -\frac{\partial}{\partial \alpha_i}(\ln f(\alpha_1^r,...,\alpha_{t-1}^r,\alpha_t,...,\alpha_{T-1}))
$$

It follows from Assumption 2 that

$$
r_{f,t}^i(\alpha_1^r,...,\alpha_{t-1}^r,\alpha_t,...,\alpha_{T-1}) < r_{f,t}^i(\alpha_1^r,...,\alpha_{t-1}^r,\beta_t(a_t;a_1^r,...,a_{t-1}^r),...,\alpha_{T-1})
$$

for all $\alpha_t > \beta_t(a_t; a_1^r, ..., a_{t-1}^r)$. Thus the distribution associated with $f(\alpha_1^r, ..., \alpha_{t-1}^r, \alpha_t, ..., \alpha_{T-1})$ strictly dominates the distribution associated with $f(\alpha_1^r, ..., \alpha_{t-1}^r, \beta_t(a_t; a_1^r, ..., a_{t-1}^r), ..., \alpha_{T-1})$ in the strong monotone likelihood ratio order, and hence in the order of first order stochastic dominance (Whitt (1982)). Since $P(q^e, \alpha_1^r, ..., \alpha_{t-1}^r, \alpha_t, ..., \alpha_{T-1})$ is strictly increasing in α_t, it follows that $p_t(a_t; a_1^r, ..., a_{t-1}^r)$ is strictly increasing in a_t for $a_t \geq a_t^{\min}$. ■

Proof of Theorem 1.

We prove the theorem in a sequence of Lemmata.

Lemma 4 *Suppose consumer behavior is given by (iv), (v) and (vii). Then (vi) holds, and* $\bar{a}_t(\alpha_t; a_1^r, ..., a_{t-1}^r)$ *is strictly increasing in* α_t. *Thus given the revealed history* $(a_1^r, ..., a_{t-1}^r)$ *the minimum amount sold in period* t *equals* $a_t^{\min}$, *and the maximum amount sold in period* t *equals* $a_t^{\max}$, *where*

$$1 = \int_{q_{t-1}^L(a_1^r,...,a_{t-1}^r)}^{a_t^{\min}} \frac{dz}{D_t(v_t^*(z; a_1^r, ..., a_{t-1}^r), \underline{\alpha}_t)}, \text{ and}$$
$$1 = \int_{q_{t-1}^L(a_1^r,...,a_{t-1}^r)}^{a_t^{\max}} \frac{dz}{D_t(v_t^*(z; a_1^r, ..., a_{t-1}^r), \bar{\alpha}_t)}. \tag{23}$$

Proof. If (iv) and (v) hold, then given the history $(a_1^r, ..., a_{t-1}^r)$ and implied revealed states $(\alpha_1^r, ..., \alpha_{t-1}^r)$ the measure of consumers that were born in some periods $\tau \leq t-1$, and that purchase no later than in period t equals $D_\tau(\bar{p}_t(a_1^r, ..., a_{t-1}^r), \alpha_\tau^r)$. Since the measure of consumers that purchase prior to period t is given by $\sum_{\tau=1}^{t-1} a_\tau^r$, it follows that (7) holds.

When the demand state in period t equals α_t, the length of the queue of newly arriving consumers (after $q_{t-1}^L(a_1^r, ..., a_{t-1}^r)$ units of output have been sold) is $D_t(\underline{v}, \alpha_t)$. Conditional on being in the queue of newly arriving consumers, the probability density that a consumer is type (v, t) equals

$$-\frac{\frac{\partial}{\partial p} D_t(v, \alpha_t)}{D_t(\underline{v}, \alpha_t)}.$$

Let q denote the position of a consumer in the queue of newly arriving consumers, and let $a_t(q)$ denote the measure of sales made to newly arriving consumers when the consumer in position q is released from the queue. Since only newly arriving consumer types with valuations above $v_t^*(a_t; a_1^r, ..., a_{t-1}^r)$

purchase, we must have

$$\frac{da_t}{dq}=\int_{v_t^*(a_t;a_1^r,....,a_{t-1}^r)}^{\bar{v}}-\frac{\frac{\partial}{\partial p}D_t(v,\alpha_2)}{D_t(\underline{v},\alpha_2)}dv=\frac{D_t(v_t^*(a_t;a_1^r,....,a_{t-1}^r),\alpha_t)}{D_t(\underline{v},\alpha_t)}.$$

Separating by variables, and integrating this equation from $q_{t-1}^L(a_1^r,...,a_{t-1}^r)$ to $\bar{a}_t$ then yields (6). Since D_t is strictly increasing in α_t, it follows that $\bar{a}_t(\alpha_t;a_1^r,...,a_{t-1}^r)$ is strictly increasing in α_t. ■

Lemma 4 implies that the belief of firms when the measure of sales in period t equals a_t is a truncation of the prior to the interval $[\beta_t(a_t;a_1^r,...,a_{t-1}^r),\bar{\alpha}_t]$, where for $a_t\leq a_t^{\min}$ we have $\beta_t(a_t;a_1^r,...,a_{t-1}^r)=\underline{\alpha}_t$, and $\beta_t(a_t;a_1^r,...,a_{t-1}^r)$ equals the solution in β of the equation

$$1=\int_{q_{t-1}^L(a_1^r,...,a_{t-1}^r)}^{a_t}\frac{dz}{D_t(v_t^*(z;a_1^r,....,a_{t-1}^r),\beta)}$$

for any $a_t\in[a_t^{\min},a_t^{\max}]$. Because $\beta_t(a_t;a_1^r,...,a_{t-1}^r)=\underline{\alpha}_t$ for $a_t\leq a_t^{\min}$, it then follows from the condition $\Delta_t(v_t^*,a_t;a_1^r,....,a_{t-1}^r))=0$ that $v_t^*(a_t;a_1^r,....,a_{t-1}^r)$ is constant on the interval $[q_{t-1}^L(a_1^r,...,a_{t-1}^r),a_t^{\min}]$. Lemma 4 therefore implies that a_t^r reveals the period t demand state to equal $\beta_t(a_t;a_1^r,...,a_{t-1}^r)$. Lemma 4 also has another important implication: in every period $t\leq T-1$, and for every history $(a_1^r,...,a_{t-1}^r)$ firms always reserve some of the produced output for sale in future periods. Thus stockouts never occur:

Lemma 5 *For every $t\leq T-1$ and every history $(a_1^r,...,a_{t-1}^r)$ we have*

$$a_t^{\max}+\sum_{\tau=1}^{t-1}a_\tau^r<q^e.$$

Proof. In any period t, given the prescribed history, $a_t^{\max}$ is defined by (23). Since $v_t^*(a_t;a_1^r,....,a_{t-1}^r)>\widehat{v}\geq\bar{p}$, we have

$$1=\int_{q_{t-1}^L(a_1^r,...,a_{t-1}^r)}^{a_t^{\max}}\frac{dz}{D_t(v_t^*(z;a_1^r,....,a_{t-1}^r),\bar{\alpha}_t)}>\int_{q_{t-1}^L(a_1^r,...,a_{t-1}^r)}^{a_t^{\max}}\frac{dz}{D_t(\bar{p},\bar{\alpha}_t)}$$

and so

$$D_t(\bar{p},\bar{\alpha}_t)>a_t^{\max}-q_{t-1}^L(a_1^r,...,a_{t-1}^r)$$

Using (7) we thus obtain

$$a_t^{\max} + \sum_{\tau=1}^{t-1} a_\tau^r < D_t(\bar{p}, \bar{\alpha}_t) + \sum_{\tau=1}^{t-1} D_\tau(P(q^e, \alpha_1^r, ..., \alpha_{t-1}^r, \bar{\alpha}_t, ..., \bar{\alpha}_{T-1}), \alpha_\tau^r)$$

$$\leq D_t(P(q^e, \alpha_1^r, ..., \alpha_{t-1}^r, \bar{\alpha}_t, ..., \bar{\alpha}_{T-1}), \bar{\alpha}_t) + \sum_{\tau=1}^{t-1} D_\tau(P(q^e, \alpha_1^r, ..., \alpha_{t-1}^r, \bar{\alpha}_t, ..., \bar{\alpha}_{T-1}), \alpha_\tau^r)$$

$$\leq \sum_{\tau=t}^{T-1} D_\tau(P(q^e, \alpha_1^r, ..., \alpha_{t-1}^r, \bar{\alpha}_t, ..., \bar{\alpha}_{T-1}), \bar{\alpha}_\tau) + \sum_{\tau=1}^{t-1} D_\tau(P(q^e, \alpha_1^r, ..., \alpha_{t-1}^r, \bar{\alpha}_t, ..., \bar{\alpha}_{T-1}), \alpha_\tau^r) = q^e$$

■

We have already argued that consumer behavior implies that in state $(\alpha_1, ..., \alpha_{T-1})$ the final price equals the market clearing price $P(q^e, \alpha_1, ..., \alpha_{T-1})$. Our next Lemma links $p_t(a_t; a_1^r, ..., a_{t-1}^r)$ to $P(q^e, \alpha_1, ..., \alpha_{T-1})$:

Lemma 6 *Given the assumed consumer behavior, sequential rationality of firms pricing behavior is satisfied if and only if condition (iii) of Theorem 1 holds.*

Proof. First, we prove sequential rationality implies condition (iii) holds. Consider a firm that posts the price $p_t(a_t; a_1^r, ..., a_{t-1}^r)$ following history $(a_1^r, ..., a_{t-1}^r)$. This firm sells its unit in period t if and only if $\alpha_t \geq \beta_t(a_t; a_1^r, ..., a_{t-1}^r)$. Hence its probability of selling the unit in period t equals

$$\pi_t(a_t; a_1^r, ..., a_{t-1}^r) = \frac{\int_{\beta_t(a_t; a_1^r, ..., a_{t-1}^r)}^{\bar{\alpha}_t} \int_{\underline{\alpha}_{t+1}}^{\bar{\alpha}_{t+1}} \cdots \int_{\underline{\alpha}_{T-1}}^{\bar{\alpha}_{T-1}} f(\alpha_1^r, ..., \alpha_{t-1}^r, \alpha_t, ..., \alpha_{T-1}) d\alpha_{T-1} ... d\alpha_t}{\int_{\underline{\alpha}_t}^{\bar{\alpha}_t} \int_{\underline{\alpha}_{t+1}}^{\bar{\alpha}_{t+1}} \cdots \int_{\underline{\alpha}_{T-1}}^{\bar{\alpha}_{T-1}} f(\alpha_1^r, ..., \alpha_{t-1}^r, \alpha_t, ..., \alpha_{T-1}) d\alpha_{T-1} ... d\alpha_t}$$

In the event the unit does not sell in period t, it must be an optimal continuation strategy to withhold output until the last period, and then sell at the price $P(q^e, \alpha_1, ..., \alpha_{T-1})$. Thus the expected profit of the this firm equals:

$$p_t(a_t; a_1^r, ..., a_{t-1}^r)\pi_t(a_t; a_1^r, ..., a_{t-1}^r) + E[P(q^e, \alpha_1, ..., \alpha_{T-1}) | \alpha_1 = \alpha_1^r, ..., \alpha_{t-1} = \alpha_{t-1}^r, \alpha_t < \beta_t(a_t; a_1^r, ..., a_{t-1}^r)]$$
$$(1 - \pi_t(a_t; a_1^r, ..., a_{t-1}^r)]$$

Meanwhile, a firm that following the history $(a_1^r, ..., a_{t-1}^r)$ selects to always withhold output until the last period has an expected profit of $E[P(q^e, \alpha_1, ..., \alpha_{t-1}, \alpha_t, ..., \alpha_{T-1}) | \alpha_1 = \alpha_1^r, ..., \alpha_{t-1} = \alpha_{t-1}^r]$.

The latter profit can be rewritten as

$$E[P(q^e, \alpha_1, ..., \alpha_{T-1})|\alpha_1 = \alpha_1^r, ..., \alpha_{t-1} = \alpha_{t-1}^r, \alpha_t \geq \beta_t(a_t; a_1^r, ..., a_{t-1}^r)]\pi_t(a_t; a_1^r, ..., a_{t-1}^r) +$$
$$E[P(q^e, \alpha_1, ..., \alpha_{T-1})|\alpha_1 = \alpha_1^r, ..., \alpha_{t-1} = \alpha_{t-1}^r, \alpha_t < \beta_t(a_t; a_1^r, ..., a_{t-1}^r)](1 - \pi_t(a_t; a_1^r, ..., a_{t-1}^r)]$$

Since expected revenues following the history $(a_1^r, ..., a_{t-1}^r)$ must be equated across firms, we must have

$$p_t(a_t; a_1^r, ..., a_{t-1}^r) = E[P(q^e, \alpha_1^r, ..., \alpha_{t-1}^r, \alpha_t, ..., \alpha_{T-1})|\alpha_1 = \alpha_1^r, ..., \alpha_{t-1} = \alpha_{t-1}^r, \alpha_t \geq \beta_t(a_t; a_1^r, ..., a_{t-1}^r)]$$

i.e. condition (iii) of Theorem 1 holds.

Conversely, if condition (iii) holds, the expected profit of any firm that has unsold output following the history $(a_1^r, ..., a_{t-1}^r)$ equals

$$E[P(q^e, \alpha_1, ..., \alpha_{t-1}, \alpha_t, ..., \alpha_{T-1})|\alpha_1 = \alpha_1^r, ..., \alpha_{t-1} = \alpha_{t-1}^r] \, .$$

A firm that deviates by posting a price below $p_t(0; a_1^r, ..., a_{t-1}^r)$ would be sure to sell, but earn strictly less than $p_t(0; a_1^r, ..., a_{t-1}^r) = E[P(q^e, \alpha_1^r, ..., \alpha_{t-1}^r, \alpha_t, ..., \alpha_{T-1})]$. A firm that posts a price above $p_t(a_t^{\max}; a_1^r, ..., a_{t-1}^r)$ would never sell in period t, and hence would still earn the same expected profit as any other firm. ■

We now turn to the optimality of consumer behavior. Consider the purchasing behavior of newly arriving customers. We first need an auxiliary Lemma.

Lemma 7 *Suppose that Assumption 5 holds. Then*

$$\ln\left(-\frac{\partial D_t}{\partial p}(p, \alpha_t)\right)$$

is supermodular in (p, α_t).

Proof. We will treat the differentiable case here. Observe that

$$\frac{\partial^2}{\partial p \partial \alpha} \ln\left(-\frac{\frac{\partial D_t}{\partial p}(p, \alpha_t)}{D_t(p, \alpha_t)}\right) = \frac{\partial^2}{\partial p \partial \alpha} \ln\left(-\frac{\partial D_t}{\partial p}(p, \alpha_t)\right) - \frac{\partial^2}{\partial \alpha \partial p} \ln(D_t(p, \alpha_t))$$

It follows from Assumption 5 (ii) that

$$\frac{\partial^2}{\partial p \partial \alpha} \ln \left(-\frac{\frac{\partial D_t}{\partial p}(p, \alpha_t)}{D_t(p, \alpha_t)} \right) > 0$$

and from Assumption 5 (i) that

$$\frac{\partial}{\partial \alpha} \left(-\frac{\frac{\partial D_t}{\partial p}(v, \alpha_t)}{D_t(v, \alpha_t)} \right) = -\frac{\partial}{\partial \alpha} \frac{\partial}{\partial p} (\ln(D_t(p, \alpha_t)) < 0,$$

implying the desired result. ■

We may now state

Lemma 8 *Suppose there exists a function* $v_t^*(a_t; a_1^r, ..., a_{t-1}^r) : [q_{t-1}^L(a_1^r, ..., a_{t-1}^r), a_t^{\max}] \to (\widehat{v}, \bar{v})$ *such that conditions (v) and (vii) of Theorem 1 hold, so that* $v_t^*(a_t; a_1^r, ..., a_{t-1}^r)$ *solves* $\Delta_t(v, a_t; a_1^r, ..., a_{t-1}^r) = 0$*. Then if a consumer of type* (v, t) *is released from the queue in period* t*, when the measure of output sold in that period equals* $a_t \in [q_{t-1}^L(a_1^r, ..., a_{t-1}^r), a_t^{\max}]$*, it is optimal to purchase whenever* $v > v_t^*(a_t; a_1^r, ..., a_{t-1}^r)$ *, and to delay trade whenever* $v < v_t^*(a_t; a_1^r, ..., a_{t-1}^r)$*.*

Proof. First, we claim that Δ_t is strictly increasing in v. Since $\delta(v)$ is nondecreasing in v, and since $p_t(a_t; a_1^r, ..., a_{t-1}^r)$ is independent of v, it suffices to show that

$$\frac{\int_{\beta_t(a_t;a_1^r,...,a_{t-1}^r)}^{\bar{\alpha}_t} \int_{\underline{\alpha}_{t+1}}^{\bar{\alpha}_{t+1}} \cdots \int_{\underline{\alpha}_{T-1}}^{\bar{\alpha}_{T-1}} P(q^e, \alpha_1^r, ..., \alpha_{t-1}^r, \alpha_t, ..., \alpha_{T-1}) \frac{-\frac{\partial}{\partial p} D_t(v,\alpha_t)}{D_t(v_t^*(a_t;a_1^r,...,a_{t-1}^r),\alpha_t)} f(\alpha_1^r, ..., \alpha_{t-1}^r, \alpha_t, ..., \alpha_{T-1}) d\alpha_{T-1}...d\alpha_t}{\int_{\beta_t(a_t;a_1^r,...,a_{t-1}^r)}^{\bar{\alpha}_t} \int_{\underline{\alpha}_{t+1}}^{\bar{\alpha}_{t+1}} \cdots \int_{\underline{\alpha}_{T-1}}^{\bar{\alpha}_{T-1}} \frac{-\frac{\partial}{\partial p} D_t(v,\alpha_t)}{D_t(v_t^*(a_t;a_1^r,...,a_{t-1}^r),\alpha_t)} f(\alpha_1^r, ..., \alpha_{t-1}^r, \alpha_t, ..., \alpha_{T-1}) d\alpha_{T-1}...d\alpha_t}$$

is strictly increasing in v. For $i = t, ..., T-1$ define

$$r_t^i(v, \alpha_t, ..., \alpha_{T-1}; \alpha_1^r, ..., \alpha_{t-1}^r) = -\frac{\partial}{\partial \alpha_i} \ln f(\alpha_1^r, ..., \alpha_{t-1}^r, \alpha_t, ..., \alpha_{T-1}) - \frac{\partial}{\partial \alpha_i} \ln \left(\frac{-\frac{\partial}{\partial p} D_t(v, \alpha_t)}{D_t(v_t^*(a_t; a_1^r, ..., a_{t-1}^r), \alpha_t)} \right)$$

It follows from Assumption 2 that for any $i \neq t$, any $(\alpha_t', ..., \alpha_{T-1}') > (\alpha_t, ..., \alpha_{T-1})$ with $\alpha_i' = \alpha_i$ and any $v' > v$ we have

$$r_t^i(v', \alpha_t', ..., \alpha_{T-1}'; \alpha_1^r, ..., \alpha_{t-1}^r) < r_t^i(v, \alpha_t, ..., \alpha_{T-1}; \alpha_1^r, ..., \alpha_{t-1}^r) \quad (24)$$

Furthermore, it follows from Assumption 2 and Lemma 7 that if $(\alpha_t', ..., \alpha_{T-1}') > (\alpha_t, ..., \alpha_{T-1})$ is such that $\alpha'_t = \alpha_t$, and if $v' > v$, then

$$r_t^t(v', \alpha'_t, ..., \alpha'_{T-1}; \alpha_1^r, ..., \alpha_{t-1}^r) < r_t^t(v, \alpha_t, ..., \alpha_{T-1}; \alpha_1^r, ..., \alpha_{t-1}^r) \tag{25}$$

It follows from (24) and (25) that the distribution over states $(\alpha_t, ..., \alpha_{T-1})$ of type (v', t) strictly dominates the distribution over states $(\alpha_t, ..., \alpha_{T-1})$ of type (v, t) in the strong monotone likelihood ratio order, and hence in the order of first order stochastic dominance. Since $P(q^e, \alpha_1^r, ..., \alpha_{t-1}^r, \alpha_t, ..., \alpha_{T-1})$ is strictly increasing in $(\alpha_t, ..., \alpha_{T-1})$, this proves the the claim.

Given the fact that α_t is revealed by period t behavior, the private history provides no additional information about future states, so all expectations of future prices conditional on the public history and the private history equal the expectation conditional on the public history alone. Thus, sequentially rational behavior does not depend on the private history. Consider first any type (v, t) with $v < v_t^*(a_t; a_1^r, ..., a_{t-1}^r)$. By the claim, we have $\Delta_t(v, a_t; a_1^r, ..., a_{t-1}^r) < 0$, i.e. waiting until period $t+1$ and then purchasing at the price $p_{t+1}(0; a_1^r, ..., a_{t-1}^r, a_t^r)$ yields higher expected utility than purchasing at the price $p_t(a_t; a_1^r, ..., a_{t-1}^r)$. Hence (v, t) does not purchase at the latter price.

Next, consider a type (v, t) with $v > v_t^*(a_t; a_1^r, ..., a_{t-1}^r)$. Since $\bar{p}_{t+1}(a_1^r, ..., a_{t-1}^r, a_t) \leq \bar{p} \leq \widehat{v} < v_t^*(a_t; a_1^r, ..., a_{t-1}^r)$, it follows from condition (v) of Theorem 1 that if such a type does not purchase in period t, she will purchase for sure in period $(t+1)$. Because we have $\Delta_t(v, a_t; a_1^r, ..., a_{t-1}^r) > 0$, purchasing in period t at the price $p_t(a_t; a_1^r, ..., a_{t-1}^r)$ is a better option, so (v, t) purchases at this price. ■

Next, we consider the period t purchasing strategy of prior generation consumers, i.e. consumers of type (v, t') with $t' < t$.

Lemma 9 *Given the history $(a_1^r, ..., a_{t-1}^r)$ it is optimal for a type (v, t') with $t' < t$ to purchase in period t if and only if $v \geq \bar{p}_t(a_1^r, ..., a_{t-1}^r)$.*

Proof. Given the history $(a_1^r, ..., a_{t-1}^r)$, it has been revealed that $(\alpha_1, ..., \alpha_{t-1}) = (\alpha_1^r, ..., \alpha_{t-1}^r)$. Since a consumer of type (v, t') only has private information about $\alpha_{t'}$, and since it has been revealed that $\alpha_{t'} = \alpha_{t'}^r$, such a consumer holds the same beliefs as firms about the distribution of future states $(\alpha_t, ..., \alpha_{T-1})$. Thus if $v \geq \bar{p}_t(a_1^r, ..., a_{t-1}^r)$ such a consumer is indifferent between purchasing in period t at the price $p_t(0; a_1^r, ..., a_{t-1}^r)$, and delaying trade to any future period. By assumption 6, such consumers must purchase in period t, provided their expected surplus from doing so is nonnegative. Nonnegative expected surplus is obvious, since $p_t(0; a_1^r, ..., a_{t-1}^r) \leq \bar{p}_t(a_1^r, ..., a_{t-1}^r) \leq v$.

If $v < \bar{p}_t(a_1^r, ..., a_{t-1}^r)$, then there is positive probability that $P(q^e, \alpha_1^r, ..., \alpha_{t-1}^r, \alpha_t, ..., \alpha_{T-1})$ will exceed v. Hence waiting to purchase until period T, and then purchasing only if $P(q^e, \alpha_1^r, ..., \alpha_{t-1}^r, \alpha_t, ..., \alpha_{T-1}) \leq v$ yields a higher expected surplus than purchasing at the price $p_t(0; a_1^r, ..., a_{t-1}^r)$. Such consumers therefore strictly prefer not to trade at $p_t(0; a_1^r, ..., a_{t-1}^r)$. ■

It remains to be shown that after every history $(a_1^r, ..., a_{t-1}^r)$ there exist a unique pair of cutoff functions $\beta_t(a_t; a_1^r, ..., a_{t-1}^r)$ and $v_t^*(a_t; a_1^r, ..., a_{t-1}^r)$, such that (i) $\beta_t(a_t; a_1^r, ..., a_{t-1}^r) = \underline{\alpha}_t$ for all $a_t \leq a_t^{\min}(a_1^r, ..., a_{t-1}^r)$, and for all $a_t \in [a_t^{\min}(a_1^r, ..., a_{t-1}^r), a_t^{\max}(a_1^r, ..., a_{t-1}^r)]$ equals the solution in β to the equation

$$1 = \int_{q_{t-1}^L(a_1^r, ..., a_{t-1}^r)}^{a_t} \frac{da_t}{D_t(v_t^*(a_t; a_1^r, ..., a_{t-1}^r), \beta)}; \tag{26}$$

(ii) $v_t^*(a_t; a_1^r, ..., a_{t-1}^r)$ is the solution in v to the equation

$$0 = \int_{\beta_t(a_t)}^{\bar{\alpha}_t} \int_{\underline{\alpha}_{t+1}}^{\bar{\alpha}_{t+1}} \cdots \int_{\underline{\alpha}_{T-1}}^{\bar{\alpha}_{T-1}} P(q^e, \alpha_1^r, ..., \alpha_{t-1}^r, \alpha_t, ..., \alpha_{T-1}) h(v, \beta_t(a_t), \alpha_t, ..., \alpha_{T-1} | \alpha_1^r, ..., \alpha_{t-1}^r) d\alpha_{T-1} ... d\alpha_t$$
$$- p_t(a_t; a_1^r, ..., a_{t-1}^r)) + \delta(v), \tag{27}$$

for all $a_t \in [0, a_t^{\max}(a_1^r, ..., a_{t-1}^r)]$, where

$$h(v, \beta_t(a_t), \alpha_t, ..., \alpha_{T-1} | \alpha_1^r, ..., \alpha_{t-1}^r) = \frac{\frac{-\frac{\partial D_t}{\partial p}(v, \alpha_t)}{D_t(v, \alpha_t)} f(\alpha_1^r, ..., \alpha_{t-1}^r, \alpha_t, ..., \alpha_{T-1})}{\int_{\beta_t(a_t)}^{\bar{\alpha}_t} \int_{\underline{\alpha}_{t+1}}^{\bar{\alpha}_{t+1}} \cdots \int_{\underline{\alpha}_{T-1}}^{\bar{\alpha}_{T-1}} \frac{-\frac{\partial D_t}{\partial p}(v, \alpha_t')}{D_t(v, \alpha_t')} f(\alpha_1^r, ..., \alpha_{t-1}^r, \alpha_t', ..., \alpha_{T-1}') d\alpha_{T-1}' ... d\alpha_t'};$$

and (iii) we have $\hat{v} < v_t^*(a_t; a_1^r, ..., a_{t-1}^r) < \bar{v}$ for all $a_t \in [0, a_t^{\max}(a_1^r, ..., a_{t-1}^r)]$.

Note that equation (27) differs from the condition $\Delta_t(v, a_t; a_1^r, ..., a_{t-1}^r) = 0$ because in the formula for h, v appears in $D_t(v, \alpha_t)$, but in the formula for Δ_t the analogous expression is $D_t(v_t^*(a_t; a_1^r, ..., a_{t-1}^r), \alpha_t)$. However, for the type satisfying (27), we have $v = v_t^*(a_t; a_1^r, ..., a_{t-1}^r)$, and therefore the condition $\Delta_t(v, a_t; a_1^r, ..., a_{t-1}^r) = 0$ is satisfied at $v = v_t^*(a_t; a_1^r, ..., a_{t-1}^r)$.

Our strategy will be to reduce the coupled system of functional equations (26) and (27) to a single functional equation in the function $\beta_t(a_t; a_1^r, ..., a_{t-1}^r)$. For this purpose, we first study solutions to (27). Define the following following analogue to (27):

$$F_t(v,\beta;a_1^r,...,a_{t-1}^r) \equiv \delta(v) - \frac{\int_\beta^{\bar{\alpha}_t}\int_{\underline{\alpha}_{t+1}}^{\bar{\alpha}_{t+1}}\cdots\int_{\underline{\alpha}_{T-1}}^{\bar{\alpha}_{T-1}} P(q^e,\alpha_1^r,...,\alpha_{t-1}^r,\alpha_t,...,\alpha_{T-1})f(\alpha_1^r,...,\alpha_{t-1}^r,\alpha_t,...,\alpha_{T-1})d\alpha_{T-1}...d\alpha_t}{\int_\beta^{\bar{\alpha}_t}\int_{\underline{\alpha}_{t+1}}^{\bar{\alpha}_{t+1}}\cdots\int_{\underline{\alpha}_{T-1}}^{\bar{\alpha}_{T-1}} f(\alpha_1^r,...,\alpha_{t-1}^r,\alpha_t,...,\alpha_{T-1})d\alpha_{T-1}...d\alpha_t}$$
$$+\frac{\int_\beta^{\bar{\alpha}_t}\int_{\underline{\alpha}_{t+1}}^{\bar{\alpha}_{t+1}}\cdots\int_{\underline{\alpha}_{T-1}}^{\bar{\alpha}_{T-1}} P(q^e,\alpha_1^r,...,\alpha_{t-1}^r,\alpha_t,...,\alpha_{T-1})\frac{\frac{\partial}{\partial p}D_t(v,\alpha_t)}{D_t(v,\alpha_t)}f(\alpha_1^r,...,\alpha_{t-1}^r,\alpha_t,...,\alpha_{T-1})d\alpha_{T-1}...d\alpha_t}{\int_\beta^{\bar{\alpha}_t}\int_{\underline{\alpha}_{t+1}}^{\bar{\alpha}_{t+1}}\cdots\int_{\underline{\alpha}_{T-1}}^{\bar{\alpha}_{T-1}} \frac{\frac{\partial}{\partial p}D_t(v,\alpha_t)}{D_t(v,\alpha_t)}f(\alpha_1^r,...,\alpha_{t-1}^r,\alpha_t,...,\alpha_{T-1})d\alpha_{T-1}...d\alpha_t} \tag{28}$$

whenever $\beta < \bar{\alpha}_t$, and

$$F_t(v,\bar{\alpha}_t;a_1^r,...,a_{t-1}^r) = \delta(v),$$

otherwise.[33]

We start by characterizing the function $F_t(v,\beta;a_1^r,...,a_{t-1}^r)$:

Lemma 10 *For each $(a_1^r,...,a_{t-1}^r)$, the function $F_t(v,\beta;a_1^r,...,a_{t-1}^r)$ is a C^1 function of (v,β).*

Proof. First, let us demonstrate that F_t is C^0 in (v,β). Consider any sequence $(v^n,\beta^n)\to(v,\beta)$. If $\beta<\bar{\alpha}_t$ then $\lim_{n\to\infty}F_t(v^n,\beta^n)=F_t(v,\beta)$ follows because the denominator in each of the fractions appearing on the right side of (28) is bounded away from zero, and the continuity of all terms in (v,β). Consider therefore $\beta=\bar{\alpha}_t$. It follows from the intermediate value theorem that there exist η^n and $\gamma^n\in(\beta^n,\bar{\alpha}_t)$ such that

$$\frac{\int_{\beta^n}^{\bar{\alpha}_t}\int_{\underline{\alpha}_{t+1}}^{\bar{\alpha}_{t+1}}\cdots\int_{\underline{\alpha}_{T-1}}^{\bar{\alpha}_{T-1}} P(q^e,\alpha_1^r,...,\alpha_{t-1}^r,\alpha_t,...,\alpha_{T-1})\frac{\frac{\partial}{\partial p}D_t(v^n,\alpha_t)}{D_t(v^n,\alpha_t)}f(\alpha_1^r,...,\alpha_{t-1}^r,\alpha_t,...,\alpha_{T-1})d\alpha_{T-1}...d\alpha_t}{\int_{\beta^n}^{\bar{\alpha}_t}\int_{\underline{\alpha}_{t+1}}^{\bar{\alpha}_{t+1}}\cdots\int_{\underline{\alpha}_{T-1}}^{\bar{\alpha}_{T-1}} \frac{\frac{\partial}{\partial p}D_t(v^n,\alpha_t)}{D_t(v^n,\alpha_t)}f(\alpha_1^r,...,\alpha_{t-1}^r,\alpha_t,...,\alpha_{T-1})d\alpha_{T-1}...d\alpha_t}$$
$$=\frac{(\bar{\alpha}_t-\beta^n)\int_{\underline{\alpha}_{t+1}}^{\bar{\alpha}_{t+1}}\cdots\int_{\underline{\alpha}_{T-1}}^{\bar{\alpha}_{T-1}} P(q^e,\alpha_1^r,...,\alpha_{t-1}^r,\eta^n,...,\alpha_{T-1})f(\alpha_1^r,...,\alpha_{t-1}^r,\eta^n,\alpha_{t+1},...,\alpha_{T-1})d\alpha_{T-1}...d\alpha_{t+1}}{(\bar{\alpha}_t-\beta^n)\int_{\underline{\alpha}_{t+1}}^{\bar{\alpha}_{t+1}}\cdots\int_{\underline{\alpha}_{T-1}}^{\bar{\alpha}_{T-1}} f(\alpha_1^r,...,\alpha_{t-1}^r,\gamma^n,\alpha_{t+1},...,\alpha_{T-1})d\alpha_{T-1}...d\alpha_{t+1}}$$
$$=\frac{\int_{\underline{\alpha}_{t+1}}^{\bar{\alpha}_{t+1}}\cdots\int_{\underline{\alpha}_{T-1}}^{\bar{\alpha}_{T-1}} P(q^e,\alpha_1^r,...,\alpha_{t-1}^r,\eta^n,...,\alpha_{T-1})f(\alpha_1^r,...,\alpha_{t-1}^r,\eta^n,\alpha_{t+1},...,\alpha_{T-1})d\alpha_{T-1}...d\alpha_{t+1}}{\int_{\underline{\alpha}_{t+1}}^{\bar{\alpha}_{t+1}}\cdots\int_{\underline{\alpha}_{T-1}}^{\bar{\alpha}_{T-1}} f(\alpha_1^r,...,\alpha_{t-1}^r,\gamma^n,\alpha_{t+1},...,\alpha_{T-1})d\alpha_{T-1}...d\alpha_{t+1}}$$
$$\to\frac{\int_{\underline{\alpha}_{t+1}}^{\bar{\alpha}_{t+1}}\cdots\int_{\underline{\alpha}_{T-1}}^{\bar{\alpha}_{T-1}} P(q^e,\alpha_1^r,...,\alpha_{t-1}^r,\bar{\alpha}_t,...,\alpha_{T-1})f(\alpha_1^r,...,\alpha_{t-1}^r,\bar{\alpha}_t,\alpha_{t+1},...,\alpha_{T-1})d\alpha_{T-1}...d\alpha_{t+1}}{\int_{\underline{\alpha}_{t+1}}^{\bar{\alpha}_{t+1}}\cdots\int_{\underline{\alpha}_{T-1}}^{\bar{\alpha}_{T-1}} f(\alpha_1^r,...,\alpha_{t-1}^r,\bar{\alpha}_t,\alpha_{t+1},...,\alpha_{T-1})d\alpha_{T-1}...d\alpha_{t+1}}$$

An entirely similar argument shows that the second term in (28) converges to the same limit. Hence we have $F_t(v,\bar{\alpha}_t)-\lim_{n\to\infty}F_t(v^n,\beta^n)=\lim_{n\to\infty}(\delta(v)-\delta(v^n))=0$, establishing continuity of F_t in

[33] To economize on notation, in the remainder of the proof of Theorem 1, we will often suppress the dependence of the functions of $a_t^{\min}$, $a_t^{\max}$, β_t , $v_t^*(a_t)$ and F_t on the history $(a_1^r,...,a_{t-1}^r)$.

(v, β).

Next, let us compute $\frac{\partial F_t}{\partial \beta}$. For $\beta < \bar{\alpha}_t$ we have:

$$\frac{\partial F_t}{\partial \beta} = \frac{A}{B} - \frac{A'}{B'},$$

where

$$A = \left(\int_{\beta}^{\bar{\alpha}_t} \int_{\underline{\alpha}_{t+1}}^{\bar{\alpha}_{t+1}} \cdots \int_{\underline{\alpha}_{T-1}}^{\bar{\alpha}_{T-1}} P(q^e, \alpha_1^r, ..., \alpha_{t-1}^r, \alpha_t, ..., \alpha_{T-1}) f(\alpha_1^r, ..., \alpha_{t-1}^r, \alpha_t, ..., \alpha_{T-1}) d\alpha_{T-1} ... d\alpha_t \right)$$
$$\left(\int_{\underline{\alpha}_{t+1}}^{\bar{\alpha}_{t+1}} \cdots \int_{\underline{\alpha}_{T-1}}^{\bar{\alpha}_{T-1}} f(\alpha_1^r, ..., \alpha_{t-1}^r, \beta, \alpha_{t+1}, ..., \alpha_{T-1}) d\alpha_{T-1} ... d\alpha_{t+1} \right)$$
$$- \left(\int_{\underline{\alpha}_{t+1}}^{\bar{\alpha}_{t+1}} \cdots \int_{\underline{\alpha}_{T-1}}^{\bar{\alpha}_{T-1}} P(q^e, \alpha_1^r, ..., \alpha_{t-1}^r, \beta, \alpha_{t+1}, ..., \alpha_{T-1}) f(\alpha_1^r, ..., \alpha_{t-1}^r, \beta, \alpha_{t+1}, ..., \alpha_{T-1}) d\alpha_{T-1} ... d\alpha_{t+1} \right)$$
$$\left(\int_{\beta}^{\bar{\alpha}_t} \int_{\underline{\alpha}_{t+1}}^{\bar{\alpha}_{t+1}} \cdots \int_{\underline{\alpha}_{T-1}}^{\bar{\alpha}_{T-1}} f(\alpha_1^r, ..., \alpha_{t-1}^r, \alpha_t, ..., \alpha_{T-1}) d\alpha_{T-1} ... d\alpha_t \right),$$
$$B = \left(\int_{\beta}^{\bar{\alpha}_t} \int_{\underline{\alpha}_{t+1}}^{\bar{\alpha}_{t+1}} \cdots \int_{\underline{\alpha}_{T-1}}^{\bar{\alpha}_{T-1}} f(\alpha_1^r, ..., \alpha_{t-1}^r, \alpha_t, ..., \alpha_{T-1}) d\alpha_{T-1} ... d\alpha_t \right)^2$$

$$A' = \left(\int_{\beta}^{\bar{\alpha}_t} \int_{\underline{\alpha}_{t+1}}^{\bar{\alpha}_{t+1}} \cdots \int_{\underline{\alpha}_{T-1}}^{\bar{\alpha}_{T-1}} P(q^e, \alpha_1^r, ..., \alpha_{t-1}^r, \alpha_t, ..., \alpha_{T-1}) \frac{\frac{\partial}{\partial p} D_t(v, \alpha_t)}{D_t(v, \alpha_t)} f(\alpha_1^r, ..., \alpha_{t-1}^r, \alpha_t, ..., \alpha_{T-1}) d\alpha_{T-1} ... d\alpha_t \right)$$
$$\frac{\frac{\partial}{\partial p} D_t(v, \beta)}{D_t(v, \beta)} \left(\int_{\underline{\alpha}_{t+1}}^{\bar{\alpha}_{t+1}} \cdots \int_{\underline{\alpha}_{T-1}}^{\bar{\alpha}_{T-1}} f(\alpha_1^r, ..., \alpha_{t-1}^r, \beta, \alpha_{t+1}, ..., \alpha_{T-1}) d\alpha_{T-1} ... d\alpha_{t+1} \right)$$
$$- \left(\int_{\underline{\alpha}_{t+1}}^{\bar{\alpha}_{t+1}} \cdots \int_{\underline{\alpha}_{T-1}}^{\bar{\alpha}_{T-1}} P(q^e, \alpha_1^r, ..., \alpha_{t-1}^r, \beta, \alpha_{t+1}, ..., \alpha_{T-1}) f(\alpha_1^r, ..., \alpha_{t-1}^r, \beta, \alpha_{t+1}, ..., \alpha_{T-1}) d\alpha_{T-1} ... d\alpha_{t+1} \right)$$
$$\frac{\frac{\partial}{\partial p} D_t(v, \beta)}{D_t(v, \beta)} \left(\int_{\beta}^{\bar{\alpha}_t} \int_{\underline{\alpha}_{t+1}}^{\bar{\alpha}_{t+1}} \cdots \int_{\underline{\alpha}_{T-1}}^{\bar{\alpha}_{T-1}} f(\alpha_1^r, ..., \alpha_{t-1}^r, \alpha_t, ..., \alpha_{T-1}) \frac{\frac{\partial}{\partial p} D_t(v, \alpha_t)}{D_t(v, \alpha_t)} d\alpha_{T-1} ... d\alpha_t \right),$$

and

$$B' = \left(\int_{\beta}^{\bar{\alpha}_t} \int_{\underline{\alpha}_{t+1}}^{\bar{\alpha}_{t+1}} \cdots \int_{\underline{\alpha}_{T-1}}^{\bar{\alpha}_{T-1}} f(\alpha_1^r, ..., \alpha_{t-1}^r, \alpha_t, ..., \alpha_{T-1}) \frac{\frac{\partial}{\partial p} D_t(v, \alpha_t)}{D_t(v, \alpha_t)} d\alpha_{T-1} ... d\alpha_t \right)^2$$

For $\beta = \bar{\alpha}_t$, note that by the mean value theorem, for every $\beta < \bar{\alpha}_t$ there exists $\mu \in (\beta, \bar{\alpha}_t)$ such that:

$$F_t(v, \beta) = F_t(v, \bar{\alpha}_t) - \frac{\partial F_t}{\partial \beta}(v, \mu)(\bar{\alpha}_t - \beta)$$

Hence it follows from the definition of a partial derivative that

$$\begin{aligned}\frac{\partial F_t}{\partial \beta}(v, \bar{\alpha}_t) &= \lim_{\beta \to \bar{\alpha}_t} \frac{F_t(v, \bar{\alpha}_t) - F_t(v, \beta)}{\bar{\alpha}_t - \beta} \\ &= \lim_{\beta \to \bar{\alpha}_t} \frac{\partial F_t}{\partial \beta}(v, \mu)\end{aligned}$$

provided the latter limit exists. We shall establish this fact below.

Next, we wish to argue that $\frac{\partial F_t}{\partial \beta}(v, \beta)$ is continuous in (v, β). Consider any sequence $(v^n, \beta^n) \to (v, \beta)$. If $\beta < \bar{\alpha}_t$, it follows from the fact that the denominator in every expression appearing in the formula for $\frac{\partial F_t}{\partial \beta}$ is bounded away from zero, that $\lim_{n \to \infty} \frac{\partial F_t}{\partial \beta}(v^n, \beta^n) = \frac{\partial F_t}{\partial \beta}(v, \beta)$. To show continuity at $\beta = \bar{\alpha}_t$, expanding A' into a second order Taylor series yields:

$$\begin{aligned}A'(v^n, \beta^n) = A'(v, \bar{\alpha}_t) &+ \frac{\partial A'}{\partial \beta}(v, \bar{\alpha}_t)(\bar{\alpha}_t - \beta^n) + \frac{\partial A'}{\partial v}(v, \bar{\alpha}_t)(v - v^n) + \frac{1}{2}\frac{\partial^2 A'}{\partial \beta^2}(v, \bar{\alpha}_t)(\bar{\alpha}_t - \beta^n)^2 \\ &+ \frac{1}{2}\frac{\partial^2 A'}{\partial v^2}(v, \bar{\alpha}_t)(v - v^n)^2 + \frac{1}{2}\frac{\partial^2 A'}{\partial v \partial \beta}(v, \bar{\alpha}_t)(v - v^n)(\bar{\alpha}_t - \beta^n) + o(||(v, \bar{\alpha}_t) - (v^n, \beta^n)||^2).\end{aligned}$$

It is straightforward but tedious to calculate

$$A'(v, \bar{\alpha}_t) = \frac{\partial A'}{\partial \beta}(v, \bar{\alpha}_t) = \frac{\partial A'}{\partial v}(v, \bar{\alpha}_t) = \frac{\partial^2 A'}{\partial v^2}(v, \bar{\alpha}_t) = \frac{\partial^2 A'}{\partial v \partial \beta}(v, \bar{\alpha}_t) = 0.$$

Hence we have

$$A'(v^n, \beta^n) = \frac{1}{2}\frac{\partial^2 A'}{\partial \beta^2}(v, \bar{\alpha}_t)(\bar{\alpha}_t - \beta^n)^2 + o(||(v, \bar{\alpha}_t) - (v^n, \beta^n)||^2).$$

Similarly, we may derive

$$B'(v, \bar{\alpha}_t) = \frac{\partial B'}{\partial \beta}(v, \bar{\alpha}_t) = \frac{\partial B'}{\partial v}(v, \bar{\alpha}_t) = \frac{\partial^2 B'}{\partial v^2}(v, \bar{\alpha}_t) = \frac{\partial^2 B'}{\partial v \partial \beta}(v, \bar{\alpha}_t) = 0,$$

so that we have

$$B'(v^n, \beta^n) = \frac{1}{2}\frac{\partial^2 B'}{\partial \beta^2}(v, \bar{\alpha}_t)(\bar{\alpha}_t - \beta^n)^2 + o(||(v, \bar{\alpha}_t) - (v^n, \beta^n)||^2).$$

It follows that

$$\lim_{n\to\infty}\frac{A'(v^n,\beta^n)}{B'(v^n,\beta^n)}=\lim_{n\to\infty}\frac{\frac{1}{2}\frac{\partial^2 A'}{\partial\beta^2}(v,\bar{\alpha}_t)(\bar{\alpha}_t-\beta^n)^2+o(||(v,\bar{\alpha}_t)-(v^n,\beta^n)||^2)}{\frac{1}{2}\frac{\partial^2 B'}{\partial\beta^2}(v,\bar{\alpha}_t)(\bar{\alpha}_t-\beta^n)^2+o(||(v,\bar{\alpha}_t)-(v^n,\beta^n)||^2)}=\frac{\frac{\partial^2 A'}{\partial\beta^2}(v,\bar{\alpha}_t)}{\frac{\partial^2 B'}{\partial\beta^2}(v,\bar{\alpha}_t)}.$$

An entirely parallel argument establishes that

$$\lim_{n\to\infty}\frac{A(\beta^n)}{B(\beta^n)}=\lim_{n\to\infty}\frac{\frac{1}{2}\frac{\partial^2 A}{\partial\beta^2}(\bar{\alpha}_t)(\bar{\alpha}_t-\beta^n)^2+o((\bar{\alpha}_t-\beta^n)^2)}{\frac{1}{2}\frac{\partial^2 B}{\partial\beta^2}(\bar{\alpha}_t)(\bar{\alpha}_t-\beta^n)^2+o((\bar{\alpha}_t-\beta^n)^2)}=\frac{\frac{\partial^2 A}{\partial\beta^2}(\bar{\alpha}_t)}{\frac{\partial^2 B}{\partial\beta^2}(\bar{\alpha}_t)}.$$

It follows that

$$\lim_{n\to\infty}\frac{\partial F_t}{\partial\beta}(v^n,\beta^n)=\lim_{n\to\infty}\frac{A(\beta^n)}{B(\beta^n)}-\lim_{n\to\infty}\frac{A'(v^n,\beta^n)}{B'(v^n,\beta^n)}=\frac{\frac{\partial^2 A}{\partial\beta^2}(\bar{\alpha}_t)}{\frac{\partial^2 B}{\partial\beta^2}(\bar{\alpha}_t)}-\frac{\frac{\partial^2 A'}{\partial\beta^2}(v,\bar{\alpha}_t)}{\frac{\partial^2 B'}{\partial\beta^2}(v,\bar{\alpha}_t)}=0,$$

where the final equality follows from straightforward but tedious calculations. We conclude that

$$\frac{\partial F_t}{\partial\beta}(v,\bar{\alpha}_t)=0$$

and that

$$\lim_{n\to\infty}\frac{\partial F_t}{\partial\beta}(v^n,\beta^n)=\frac{\partial F_t}{\partial\beta}(v,\bar{\alpha}_t)$$

holds for all sequences (v^n,β^n) converging to $(v,\bar{\alpha}_t)$. Hence, $\frac{\partial F_t}{\partial\beta}(v,\bar{\alpha}_t)$ exists and is a continuous function. Similar arguments also establish that $\frac{\partial F_t}{\partial\beta}(v,\beta)$ exists for all (v,β), and that $\frac{\partial F_t}{\partial\beta}(v,\beta)$ is continuous in (v,β). If follows that $F_t(v,\beta)$ is a C^1 function in (v,β). ■

We are now in a position to describe the solution to the equation $F_t(v,\beta;a_1^r,...,a_{t-1}^r)=0$:

Lemma 11 *For every history $(a_1^r,...,a_{t-1}^r)$ there exists a C^1 function $\nu_t:[\underline{\alpha}_t,\bar{\alpha}_t]\to[\underline{v},\bar{v}]$ such that $F_t(v,\beta)=0$ if and only if $v=\nu_t(\beta)$. This solution satisfies: (i) $\nu_t(\bar{\alpha}_t)=\widehat{v}$; (ii) $\nu_t(\beta)>\widehat{v}$ for all $\beta<\bar{\alpha}_t$; and (iii) there exists $\varepsilon>0$ such that $\nu_t(\beta)\leq\bar{v}-\varepsilon$, for all β.*

Proof. First, we claim that $\lim_{v\uparrow\bar{v}}F_t(v,\beta)=\delta(\bar{v})>0$. To see this, observe that

$$\lim_{v \uparrow \bar{v}} \frac{\int_{\beta}^{\bar{\alpha}_t} \int_{\underline{\alpha}_{t+1}}^{\bar{\alpha}_{t+1}} \cdots \int_{\underline{\alpha}_{T-1}}^{\bar{\alpha}_{T-1}} P(q^e, \alpha_1^r, ..., \alpha_{t-1}^r, \alpha_t, ..., \alpha_{T-1}) \frac{\frac{\partial}{\partial p} D_t(v,\alpha_t)}{D_t(v,\alpha_t)} f(\alpha_1^r, ..., \alpha_{t-1}^r, \alpha_t, ..., \alpha_{T-1}) d\alpha_{T-1} ... d\alpha_t}{\int_{\beta}^{\bar{\alpha}_t} \int_{\underline{\alpha}_{t+1}}^{\bar{\alpha}_{t+1}} \cdots \int_{\underline{\alpha}_{T-1}}^{\bar{\alpha}_{T-1}} \frac{\frac{\partial}{\partial p} D_t(v,\alpha_t)}{D_t(v,\alpha_t)} f(\alpha_1^r, ..., \alpha_{t-1}^r, \alpha_t, ..., \alpha_{T-1}) d\alpha_{T-1} ... d\alpha_t}$$
$$= \int_{\beta}^{\bar{\alpha}_t} \int_{\underline{\alpha}_{t+1}}^{\bar{\alpha}_{t+1}} \cdots \int_{\underline{\alpha}_{T-1}}^{\bar{\alpha}_{T-1}} P(q^e, \alpha_1^r, ..., \alpha_{t-1}^r, \alpha_t, ..., \alpha_{T-1}) L(\bar{v}) d\alpha_{T-1} ... d\alpha_t,$$

where

$$L(\bar{v}) \equiv \lim_{v \uparrow \bar{v}} \frac{\frac{D_t(v,\bar{\alpha}_t) \frac{\partial}{\partial p} D_t(v,\alpha_t)}{D_t(v,\alpha_t)} f(\alpha_1^r, ..., \alpha_{t-1}^r, \alpha_t, ..., \alpha_{T-1})}{\int_{\beta}^{\bar{\alpha}_t} \int_{\underline{\alpha}_{t+1}}^{\bar{\alpha}_{t+1}} \cdots \int_{\underline{\alpha}_{T-1}}^{\bar{\alpha}_{T-1}} \frac{D_t(v,\bar{\alpha}_t) \frac{\partial}{\partial p} D_t(v,\alpha_t)}{D_t(v,\alpha_t)} f(\alpha_1^r, ..., \alpha_{t-1}^r, \alpha_t, ..., \alpha_{T-1}) d\alpha_{T-1} ... d\alpha_t}$$
$$= \frac{\frac{\frac{\partial}{\partial p} D_t(v,\bar{\alpha}_t) \frac{\partial}{\partial p} D_t(\bar{v},\alpha_t)}{\frac{\partial}{\partial p} D_t(\bar{v},\alpha_t)} f(\alpha_1^r, ..., \alpha_{t-1}^r, \alpha_t, ..., \alpha_{T-1})}{\int_{\beta}^{\bar{\alpha}_t} \int_{\underline{\alpha}_{t+1}}^{\bar{\alpha}_{t+1}} \cdots \int_{\underline{\alpha}_{T-1}}^{\bar{\alpha}_{T-1}} \frac{\frac{\partial}{\partial p} D_t(\bar{v},\bar{\alpha}_t) \frac{\partial}{\partial p} D_t(\bar{v},\alpha_t)}{\frac{\partial}{\partial p} D_t(\bar{v},\alpha_t)} f(\alpha_1^r, ..., \alpha_{t-1}^r, \alpha_t, ..., \alpha_{T-1}) d\alpha_{T-1} ... d\alpha_t}$$
$$= \frac{f(\alpha_1^r, ..., \alpha_{t-1}^r, \alpha_t, ..., \alpha_{T-1}) d\alpha_{T-1} ... d\alpha_t}{\int_{\beta}^{\bar{\alpha}_t} \int_{\underline{\alpha}_{t+1}}^{\bar{\alpha}_{t+1}} \cdots \int_{\underline{\alpha}_{T-1}}^{\bar{\alpha}_{T-1}} f(\alpha_1^r, ..., \alpha_{t-1}^r, \alpha_t, ..., \alpha_{T-1}) d\alpha_{T-1} ... d\alpha_t}.$$

Thus we have $\lim_{v \uparrow \bar{v}} F_t(v, \beta) = \delta(\bar{v}) > 0$.

Second, we claim that $F_t(\widehat{v}, \beta) < 0$ for all $\beta < \bar{\alpha}_t$. Our strategy of proof will be to show that the distribution associated with $f(\alpha_1^r, ..., \alpha_{t-1}^r, \alpha_t, ..., \alpha_{T-1})$ dominates the distribution associated with $h(v, \beta, \alpha_t, ..., \alpha_{T-1} | \alpha_1^r, ..., \alpha_{t-1}^r)$ in the strong monotone likelihood ratio order, and hence in the order of first order stochastic dominance (Whitt, 1982). Recall that

$$r_{f,t}^i(\alpha_1^r, ..., \alpha_{t-1}^r, \alpha_t, ..., \alpha_{T-1}) = -\frac{\partial}{\partial \alpha_i} \ln f(\alpha_1^r, ..., \alpha_{t-1}^r, \alpha_t, ..., \alpha_{T-1})$$

and similarly, let

$$\begin{aligned} r_{h,t}^i(\alpha_1^r, ..., \alpha_{t-1}^r, \alpha_t, ..., \alpha_{T-1} | v, \beta) &= -\frac{\partial}{\partial \alpha_i} \ln h(v, \beta, \alpha_t, ..., \alpha_{T-1} | \alpha_1^r, ..., \alpha_{t-1}^r) \\ &= -\frac{\partial}{\partial \alpha_i} \ln f(\alpha_1^r, ..., \alpha_{t-1}^r, \alpha_t, ..., \alpha_{T-1}) - \frac{\partial}{\partial \alpha_i} \ln \left(\frac{-\frac{\partial}{\partial p} D_t(v, \alpha_t)}{D_t(v, \alpha_t)} \right) \\ &= r_{f,t}^i(\alpha_1^r, ..., \alpha_{t-1}^r, \alpha_t, ..., \alpha_{T-1}) - \frac{\partial}{\partial \alpha_i} \ln \left(\frac{-\frac{\partial}{\partial p} D_t(v, \alpha_t)}{D_t(v, \alpha_t)} \right). \end{aligned}$$

We need to show that for any (v, β) and any $(\alpha_t', ..., \alpha_{T-1}') > (\alpha_t, ..., \alpha_{T-1})$ with $\alpha_i' = \alpha_i$, we have

$$r_{f,t}^i(\alpha_1^r, ..., \alpha_{t-1}^r, \alpha_t', ..., \alpha_{T-1}') \leq r_{h,t}^i(\alpha_1^r, ..., \alpha_{t-1}^r, \alpha_t, ..., \alpha_{T-1} | v, \beta)$$

It follows from Assumption 2 that

$$\frac{\partial}{\partial \alpha_i} \ln f(\alpha_1^r, ..., \alpha_{t-1}^r, \alpha_t', ..., \alpha_{T-1}') > \frac{\partial}{\partial \alpha_i} \ln f(\alpha_1^r, ..., \alpha_{t-1}^r, \alpha_t, ..., \alpha_{T-1})$$

and hence that

$$r_{f,t}^i(\alpha_1^r, ..., \alpha_{t-1}^r, \alpha_t', ..., \alpha_{T-1}') < r_{f,t}^i(\alpha_1^r, ..., \alpha_{t-1}^r, \alpha_t, ..., \alpha_{T-1}).$$

Assumption 5(ii) then guarantees that $r_{f,t}^i(\alpha_1^r, ..., \alpha_{t-1}^r, \alpha_t', ..., \alpha_{T-1}') < r_{h,t}^i(\alpha_1^r, ..., \alpha_{t-1}^r, \alpha_t, ..., \alpha_{T-1}|v, \beta)$. Hence we have $F_t(\widehat{v}, \beta) < \delta(\widehat{v}) = 0$. Furthermore, at $\beta = \bar{\alpha}_t$, we have $F_t(\widehat{v}, \bar{\alpha}_t) = \delta(\widehat{v}) = 0$. It follows from the previous two claims that for every β there exists a solution in v to the equation $F_t(v, \beta) = 0$, satisfying $v > \widehat{v}$ for all $\beta < \bar{\alpha}_t$, and $v = \widehat{v}$ for $\beta = \bar{\alpha}_t$.

Third, we claim that the solution to (27) in the interval $[\widehat{v}, \bar{v}]$ is unique. It will suffice to prove that F_t is strictly increasing in v over the interval $[\widehat{v}, \bar{v}]$. To see why, observe that if $v' > v$ then by Assumption 5 (ii) the distribution function associated with $h(v', \alpha_t, ..., \alpha_{T-1}|\alpha_1^r, ..., \alpha_{t-1}^r)$ strictly dominates the distribution associated with $h(v, \alpha_t, ..., \alpha_{T-1}|\alpha_1^r, ..., \alpha_{t-1}^r)$ in the strong monotone likelihood ratio order, and hence in the order of multivariate first order stochastic dominance. Since δ is strictly increasing in v for $v > \widehat{v}$, it follows that we have $F_t(v', \beta) > F_t(v, \beta)$, establishing the claim.

Fourth, let $\nu_t(\beta)$ denote the unique solution to $F_t(v, \beta) = 0$ in the interval $[\widehat{v}, \bar{v}]$. We claim that there exists $\varepsilon > 0$ s.t. $\nu_t(\beta) \leq \bar{v} - \varepsilon$, for all $\beta \in [\underline{\alpha}_t, \bar{\alpha}_t]$. To see this, define $\kappa(v) = \min F_t(v, \beta)$. By the Theorem of the Maximum, κ is a continuous function of v. Since $\kappa(\bar{v}) = F_t(\bar{v}, \beta)$ for some $\beta \in [\underline{\alpha}_t, \bar{\alpha}_t]$, we have $\kappa(\bar{v}) = \delta(\bar{v}) > 0$. It follows that there exists $\varepsilon > 0$ such that $\kappa(v) > 0$ for all $v > \bar{v} - \varepsilon$. Since $F_t(v, \beta)$ is strictly increasing in v, this in turn shows that $\nu_t(\beta) \leq \bar{v} - \varepsilon$, for all $\beta \in [\underline{\alpha}_t, \bar{\alpha}_t]$, establishing the claim.

Finally, since $F_t(v, \beta)$ is a C^1 function of (v, β), and since $\frac{\partial F_t}{\partial v} > 0$, it follows from the implicit function theorem that $\nu_t(\beta)$ is a C^1 function of β, satisfying

$$\frac{\partial \nu_t}{\partial \beta} = -\frac{\frac{\partial F_t}{\partial \beta}}{\frac{\partial F_t}{\partial v}}.$$

■

It follows from Lemma 11 that $v_t^*(a_t) = \nu_t(\beta_t(a_t))$. Substituting this into (26) yields a single functional equation in $\beta_t(\cdot)$:

$$1 = \int_{q_{t-1}^L}^{a_t} \frac{dz}{D_t(\nu_t(\beta_t(z)), \beta_t(a_t))} \tag{29}$$

where $\beta_t(a_t) = \underline{\alpha}_t$ for all $a_t \in [0, a_t^{\min}]$ and $\beta_t(a_t)$ is strictly increasing in a_t for $a_t \in [a_t^{\min}, a_t^{\max}]$. Notice that $a_t^{\min}$ does not depend on the function $\beta_t(a_t)$. Furthermore, Lemma 11 (iii) implies that any solution to (29) is continuous. We next establish that it must also be unique.

Lemma 12 *(Uniqueness) If β_t^0 and β_t^1 both solve (29) then $\beta_t^0 = \beta_t^1$.*

Proof. We have already shown that $a_{t,0}^{\min} = a_{t,1}^{\min} = a_t^{\min}$, and $\beta_t^0(a_t) = \beta_t^1(a_t)$ for $a_t \leq a_t^{\min}$. Without loss of generality, we may assume that $a_{t,0}^{\max} \leq a_{t,1}^{\max}$. Let $a_t^{\min} < a_t < a_{t,0}^{\max}$ hold. Then from (29) we have

$$\begin{aligned}
0 &= \int_{q_{t-1}^L}^{a_t} \frac{dz}{D_t(\nu_t(\beta_t^0(z)), \beta_t^0(a_t))} - \int_{q_{t-1}^L}^{a_t} \frac{dz}{D_t(\nu_t(\beta_t^1(z)), \beta_t^1(a_t))} \\
&= \int_{q_{t-1}^L}^{a_t} \frac{D_t(\nu_t(\beta_t^1(z)), \beta_t^1(a_t)) - D_t(\nu_t(\beta_t^0(z)), \beta_t^0(a_t))}{D_t(\nu_t(\beta_t^0(z)), \beta_t^0(a_t)) D_t(\nu_t(\beta_t^1(z)), \beta_t^1(a_t))} dz \\
&= \int_{q_{t-1}^L}^{a_t} \frac{D_t(\nu_t(\beta_t^1(z)), \beta_t^1(a_t)) - D_t(\nu_t(\beta_t^1(z)), \beta_t^0(a_t)) + D_t(\nu_t(\beta_t^1(z)), \beta_t^0(a_t)) - D_t(\nu_t(\beta_t^0(z)), \beta_t^0(a_t))}{D_t(\nu_t(\beta_t^0(z)), \beta_t^0(a_t)) D_t(\nu_t(\beta_t^1(z)), \beta_t^1(a_t))} dz \\
&= \int_{q_{t-1}^L}^{a_t} \frac{\frac{\partial D_t}{\partial \alpha_t}(\nu_t(\beta_t^1(z)), \widetilde{\beta}(a_t))(\beta_t^1(a_t) - \beta_t^0(a_t)) + \frac{\partial D_t}{\partial p}(\nu_t(\widehat{\beta}(z)), \beta_t^0(a_t)) \frac{d\nu_t}{d\beta}(\widehat{\beta}(z))(\beta_t^1(z) - \beta_t^0(z))}{D_t(\nu_t(\beta_t^0(z)), \beta_t^0(a_t)) D_t(\nu_t(\beta_t^1(z)), \beta_t^1(a_t))} dz
\end{aligned}$$

where the final equality follows from the mean value theorem, $\widetilde{\beta}(a_t) = \mu \beta_t^1(a_t) + (1-\mu)\beta_t^0(a_t)$ for some $\mu \in (0,1)$, and $\widehat{\beta}(z) = \eta(z)\beta_t^1(z) + (1-\eta(z))\beta_t^0(z)$ for some $\eta(z) \in (0,1)$. Therefore we have

$$\begin{aligned}
&|\beta_t^1(a_t) - \beta_t^0(a_t)| \int_{q_{t-1}^L}^{a_t} \frac{\frac{\partial D_t}{\partial \alpha_t}(\nu_t(\beta_t^1(z)), \widetilde{\beta}(a_t))}{D_t(\nu_t(\beta_t^0(z)), \beta_t^0(a_t)) D_t(\nu_t(\beta_t^1(z)), \beta_t^1(a_t))} dz \\
&\leq \int_{q_{t-1}^L}^{a_t} \frac{|\frac{\partial D_t}{\partial p}(\nu_t(\widehat{\beta}(z)), \beta_t^0(a_t))| \, |\frac{d\nu_t}{d\beta}(\widehat{\beta}(z))| \, |\beta_t^1(z) - \beta_t^0(z)|}{D_t(\nu_t(\beta_t^0(z)), \beta_t^0(a_t)) D_t(\nu_t(\beta_t^1(z)), \beta_t^1(a_t))} dz \\
&\leq \int_{a_t^{\min}}^{a_t} \frac{|\frac{\partial D_t}{\partial p}(\nu_t(\widehat{\beta}(z)), \beta_t^0(a_t))| \, |\frac{d\nu_t}{d\beta}(\widehat{\beta}(z))| \, |\beta_t^1(z) - \beta_t^0(z)|}{D_t(\nu_t(\beta_t^0(z)), \beta_t^0(a_t)) D_t(\nu_t(\beta_t^1(z)), \beta_t^1(a_t))} dz
\end{aligned}$$

Let $\underline{D}_t^\alpha = \min \frac{\partial D_t}{\partial \alpha_t}(p, \alpha_t)$, $\bar{D}_t^p = \max |\frac{\partial D_t}{\partial p}|$, and $K = \max |\frac{d\nu_t}{d\beta}|$. Then we have

$$|\beta_t^1(a_t) - \beta_t^0(a_t)| \ \leq K \frac{\bar{D}_t^p}{\underline{D}_t^\alpha} \max_{z \in [0, a_t]} \ |\beta_t^1(z) - \beta_t^0(z)| \ L_t^0(a_t) \tag{30}$$

where

$$L_t^0(a_t) = \frac{\int_{a_t^{\min}}^{a_t} \frac{dz}{D_t(\nu_t(\beta_t^0(z)),\beta_t^0(a_t))D_t(\nu_t(\beta_t^1(z)),\beta_t^1(a_t))}}{\int_{q_{t-1}^L}^{a_t} \frac{dz}{D_t(\nu_t(\beta_t^0(z)),\beta_t^0(a_t))D_t(\nu_t(\beta_t^1(z)),\beta_t^1(a_t))}}$$

Since $L_t^0(a_t^{\min}) = 0$, and since L_t^0 is continuous in a_t, we may select $a_t^0 > a_t^{\min}$ such that

$$L_t^0(a_t^0) = \frac{1}{2} \frac{\underline{D}_t^\alpha}{\left(\bar{D}_t^p\right) K}.$$

if such a solution exists, and $a_t^0 = a_t^{\max}$, otherwise. It then follows from (30) that for all $a_t \in [0, a_t^0]$ we have

$$|\beta_t^1(a_t) - \beta_t^0(a_t)| \leq \frac{1}{2} \max_{z \in [0,a_t]} |\beta_t^1(z) - \beta_t^0(z)|$$

implying that $\beta_t^1(a_t) = \beta_t^0(a_t)$ for all $a_t \in [0, a_t^0]$.

Given that $\beta_t^1(a_t) = \beta_t^0(a_t)$ for all $a_t \in [0, a_t^n]$, we may the inductively define

$$L_t^{n+1}(a_t) = \frac{\int_{a_t^n}^{a_t} \frac{dz}{D_t(\nu_t(\beta_t^0(z)),\beta_t^0(a_t))D_t(\nu_t(\beta_t^1(z)),\beta_t^1(a_t))}}{\int_{q_{t-1}^L}^{a_t} \frac{dz}{D_t(\nu_t(\beta_t^0(z)),\beta_t^0(a_t))D_t(\nu_t(\beta_t^1(z)),\beta_t^1(a_t))}}$$

and select $a_t^{n+1} > a_t^n$ such that

$$L_t^{n+1}(a_t^{n+1}) = \frac{1}{2} \frac{\underline{D}_t^\alpha}{\left(\bar{D}_t^p\right) K}$$

if such a solution exists, and $a_t^{n+1} = a_t^{\max}$, otherwise, to conclude that $\beta_t^1(a_t) = \beta_t^0(a_t)$ for all $a_t \in [0, a_t^{n+1}]$. This process will end in a finite number of steps, since $D_t(\nu_t(\beta_t^0(z)), \beta_t^0(a_t))$ and $D_t(\nu_t(\beta_t^1(z)), \beta_t^1(a_t))$ are uniformly bounded away from zero. Indeed, Lemma 11 implies that $D_t(\nu_t(\beta_t^0(z)), \beta_t^0(a_t)) \geq D_t(\bar{v} - \varepsilon, \beta_t^0(a_t)) \geq D_t(\bar{v} - \varepsilon, \underline{\alpha}_t)$. We conclude that $a_{t,0}^{\max} = a_{t,1}^{\max}$ and $\beta_t^1(a_t) = \beta_t^0(a_t)$ for all $a_t \in [0, a_t^{\max}]$, as was to be demonstrated. ■

Finally, we establish that there exists a solution to the functional equation (29).

Lemma 13 *(Existence) There exists a solution β_t to (29) satisfying $\beta_t(a_t) = \underline{\alpha}_t$ for all $a_t \leq a_t^{\min}$.*

Proof. Let a_t^0 be defined as in the proof of Lemma 12, and let $M_t^0 = \{\beta \in C([0, a_t^0]) \mid \beta$ is nondecreasing and $\beta(a_t) = \underline{\alpha}_t$ for all $a_t \in [0, a_t^{\min}]\}$. Next, define the operator T_t as follows. For $a_t \in [0, a_t^{\min}]$ let $T_t\beta(a_t) = \underline{\alpha}_t$, and for $a_t \in [a_t^{\min}, a_t^0]$ let $T_t\beta(a_t)$ be the solution in α_t to the equation

$$\int_{q_{t-1}^L}^{a_t} \frac{dz}{D_t(\nu_t(\beta_t(z)), \alpha_t)} = 1 \tag{31}$$

Let $W_t^0 = T_t(M_t^0)$. It is evident that $T_t : W_t^0 \to W_t^0$. Furthermore, it follows from the proof of Lemma 12 that T_t is a contraction with modulus $\frac{1}{2}$.

We endow the set $C([0, a_t^0])$ with the l_∞ metric, that is for any $f \in C([0, a_t^0])$, we let $||f|| = \max\{ |f(x)| : x \in [0, a_t^0]\}$, and define $d(f, g) = ||f - g||$. Note that the Weierstrass theorem guarantees that this metric is well defined. It is well known that under this metric the space $C([0, a_t^0])$ is complete.

We shall now argue that W_t^0 is a closed subset of $C([0, a_t^0])$. It follows from (31) that for any $\beta \in W^0$ and any pair (a_t, a_t') satisfying $a_t^{\min} \le a_t < a_t' \le a_t^0$ we have

$$
\begin{aligned}
0 &= \int_{q_{t-1}^L}^{a_t'} \frac{dz}{D_t(\nu_t(\beta(z)), T_t\beta(a_t'))} - \int_{q_{t-1}^L}^{a_t} \frac{dz}{D_t(\nu_t(\beta(z)), T_t\beta(a_t))} \\
&= \int_{a_t}^{a_t'} \frac{dz}{D_t(\nu_t(\beta(z)), T_t\beta(a_t'))} + \int_{q_{t-1}^L}^{a_t} \frac{dz}{D_t(\nu_t(\beta(z)), T_t\beta(a_t'))} - \int_{q_{t-1}^L}^{a_t} \frac{dz}{D_t(\nu_t(\beta(z)), T_t\beta(a_t))} \\
&= \int_{a_t}^{a_t'} \frac{dz}{D_t(\nu_t(\beta(z)), T_t\beta(a_t'))} + \int_{q_{t-1}^L}^{a_t} \left(\frac{D_t(\nu_t(\beta(z)), T_t\beta(a_t)) - D_t(\nu_t(\beta(z)), T_t\beta(a_t'))}{D_t(\nu_t(\beta(z)), T_t\beta(a_t'))D_t(\nu_t(\beta(z)), T_t\beta(a_t))} \right) dz \\
&= \int_{a_t}^{a_t'} \frac{dz}{D_t(\nu_t(\beta(z)), T_t\beta(a_t'))} + \int_{q_{t-1}^L}^{a_t} \left(\frac{\frac{\partial D_t}{\partial \alpha_t}(\nu_t(\beta(z)), \alpha_t)(T_t\beta(a_t) - T_t\beta(a_t'))}{D_t(\nu_t(\beta(z)), T_t\beta(a_t'))D_t(\nu_t(\beta(z)), T_t\beta(a_t))} \right) dz
\end{aligned}
$$

where the final equality follows from an application of the mean value theorem. Therefore we have

$$
\begin{aligned}
&(T_t\beta(a_t') - T_t\beta(a_t)) \int_{q_{t-1}^L}^{a_t} \left(\frac{\frac{\partial D_t}{\partial \alpha_t}(\nu_t(\beta(z)), \alpha_t)}{D_t(\nu_t(\beta(z)), T_t\beta(a_t'))D_t(\nu_t(\beta(z)), T_t\beta(a_t))} \right) dz \\
&= \int_{a_t}^{a_t'} \frac{dz}{D_t(\nu_t(\beta(z)), T_t\beta(a_t'))}.
\end{aligned}
$$

Let $\underline{D}_t = D_t(\bar{v} - \varepsilon, \underline{\alpha}_t)$, and $\bar{D}_t = D_t(\underline{v}, \bar{\alpha}_t)$. It follows from the previous inequality that

$$
|T_t\beta(a_t') - T_t\beta(a_t)| \le S(a_t' - a_t)
$$

where

$$
S = \frac{(\bar{D}_t)^2}{\underline{D}_t^\alpha \underline{D}_t a_t^{\min}}
$$

It follows that all members of W_t^0 are Lipschitz continuous with Lipschitz constant S. Therefore M_t^0 is an equicontinuous subset of $C([0, a_t^0])$. The Arzela-Ascoli Theorem then implies that W_t^0 is a closed subset of $C([0, a_t^0])$.

We conclude that all the conditions of the Banach fixed point theorem are satisfied, and hence that there exists a unique $\beta_t^* \in C([0, a_1^0])$ such that $T_t \beta_t^* = \beta_t^*$.

We may now proceed inductively, by defining a_t^n as in the proof of Lemma 12, and by defining $M_t^n = \{\beta \in C([0, a_t^n]) \mid \beta \text{ is nondecreasing and } \beta(a_t) = \beta_t^*(a_t) \text{ for all } a_t \in [0, a_t^n]\}$, and $W_t^n = T_t M_t^n$, to conclude that for each n there exists a unique $\beta_t^* \in C([0, a_t^n])$ such that $T_t \beta_t^* = \beta_t^*$. Since the process completes in a finite number of steps, this completes the proof of existence of a unique $\beta_t^* \in C([0, a_t^{\max}])$ such that $T_t \beta_t^* = \beta_t^*$. ■

Proof of Lemma 1. First we show that

$$D_t(v, \alpha_t) = H(\alpha_t) D_t(v) \tag{32}$$

implies $\Delta_t(v, a_t; a_1^r, ..., a_{t-1}^r) - \delta(v) = 0$ for all $t, v, a_t, a_1^r, ..., a_{t-1}^r$. First note that (32) implies

$$\frac{-\frac{\partial}{\partial p} D_t(v, \alpha_t)}{D_t(v_t^*(a_t; a_1^r, ..., a_{t-1}^r), \alpha_t)} = \frac{-\frac{\partial}{\partial p} D_t(v)}{D_t(v_t^*(a_t; a_1^r, ..., a_{t-1}^r))},$$

which does not depend on α_t. Therefore, we can factor this term out of the integrals in the expression for Δ_t, yielding the desired result. Hence the information effect is absent for all t, all v, and all $(a_1^r, ..., a_{t-1}^r, a_t)$.

Now we show that $\Delta_t(v, a_t; a_1^r, ..., a_{t-1}^r) - \delta(v) = 0$ for all $t, v, a_t, a_1^r, ..., a_{t-1}^r$ implies that demand can be written as (32). Using the logic of the proof of Lemma 7, we can show that

$$\frac{\partial}{\partial \alpha} \frac{\partial}{\partial p} \ln D_t(v, \alpha_t) \geq 0 \tag{33}$$

holds for all v, α_t. Suppose that inequality (33) held strictly for some v, α_t. Then the argument behind Lemma 8 can be modified to show that, for v' and v'' in a neighborhood of v where $v'' > v'$, the distribution over states $(\alpha_t, ..., \alpha_{T-1})$ of type (v'', t) strictly dominates the distribution over states $(\alpha_t, ..., \alpha_{T-1})$ of type (v', t) in the strong monotone likelihood ratio order, and hence in the order of first order stochastic dominance. Since $P(q^e, \alpha_1^r, ..., \alpha_{t-1}^r, \alpha_t, ..., \alpha_{T-1})$ is strictly increasing in $(\alpha_t, ..., \alpha_{T-1})$, this implies that $\Delta_t(v, a_t; a_1^r, ..., a_{t-1}^r) - \delta(v)$ is locally increasing in v, a contradiction. Therefore, (33) must hold as an equality. It follows that $\frac{\partial}{\partial p} \ln D_t(v, \alpha_t)$ is independent of α_t. Hence there exists a function $\hat{D}_t(v)$ such that

$$\frac{\partial}{\partial p} \ln D_t(v, \alpha_t) = \hat{D}_t(v). \tag{34}$$

Integrating (34) then yields

$$D_t(v, \alpha_t) = e^{\int \hat{D}_t(v)dv} e^{\hat{H}(\alpha_t)}$$

for some function $\hat{H}(\alpha_t)$. Therefore, we can express demand in the form $D_t(v, \alpha_t) = H(\alpha_t)D_t(v)$ ■

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Department of Economics, University of Wisconsin, Madison, WI 53706 and Department of Economics, The Ohio State University, Columbus, OH 43210.

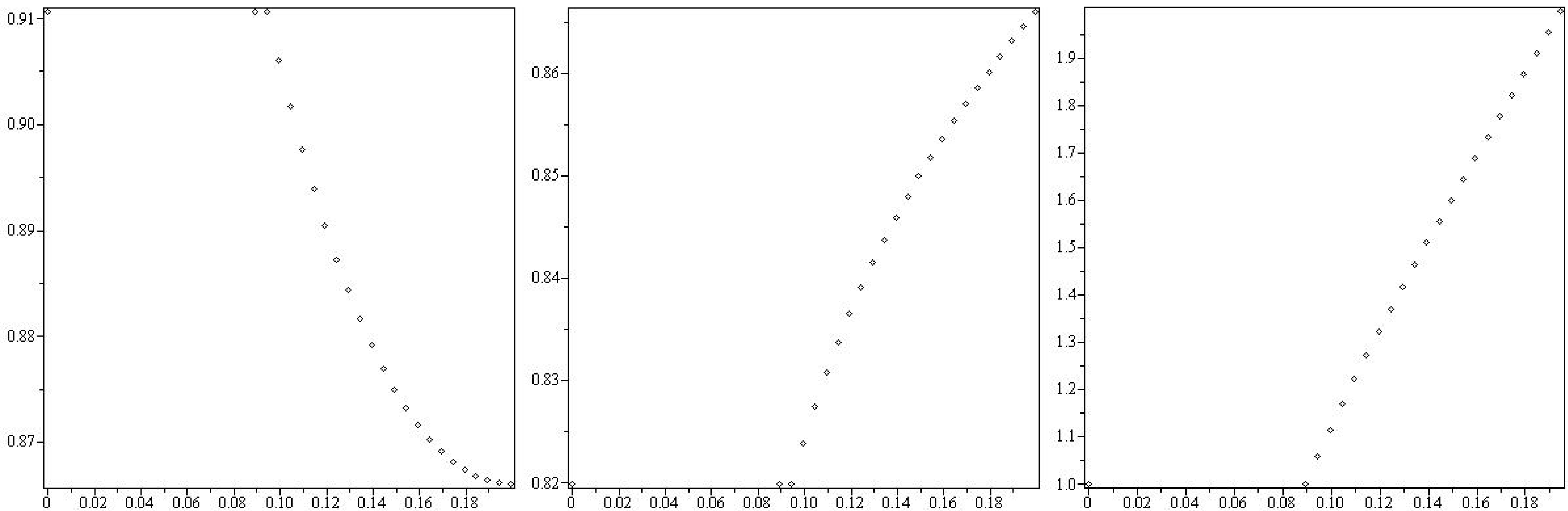

Figure 1: Graphs of $v^*(a_1)$, $p_1(a_1)$ and $\beta(a_1)$