1) \( \langle x | l_n \rangle^{(0)} = \begin{cases} \sqrt{\frac{2}{L}} \cos \left( \frac{n \pi x}{L} \right), & n = 1, 3, 5, \ldots \\ \sqrt{\frac{2}{L}} \sin \left( \frac{n \pi x}{L} \right), & n = 2, 4, 6, \ldots \end{cases} \)

\( E_n^{(0)} = n^2 \epsilon_1 \), \( \epsilon_1 = \frac{\hbar^2}{2mL^2} \), no degeneracies

\( H' = \mathcal{E} L \delta(x) \)

\[ \langle m | H' | n \rangle = \int_0^L dx \, \phi^*_m(x) [\mathcal{E} L \delta(x)] \phi_n(x) \]

\[ = \mathcal{E} L \phi^*_m(0) \phi_n(0) = \begin{cases} 2\mathcal{E} \text{ m+n odd} \\ 0 \text{ m+n even} \end{cases} \]

All \( | n \rangle \) with \( n \) even (odd parity) states are eigenstates, with \( E_n = E_n^{(0)} \) of \( H_0 + H' \).

All integers below are odd.

\[ E_n^{(-1)} = \langle n | H' | n \rangle = \begin{cases} 2\mathcal{E} \text{ n odd} \\ 0 \text{ n even} \end{cases} \]
(b) \[ E_n = \sum_{m>n \mod 2} \frac{(2 \delta)^2}{(m^2 - n^2)^2} \]

This is awkward to evaluate in Mathematica, so rewrite this as:

\[
\lim_{\nu \to \nu} \left( \sum_{k=0}^{\infty} \frac{1}{\nu^2 - (2k+1)^2} \right) - \frac{1}{\nu^2 - n^2}
\]

\[
= \lim_{\nu \to \nu} \left( \frac{\pi \tan \left( \frac{\nu \pi}{2} \right)}{4 \nu} - \frac{1}{\nu^2 - n^2} \right)
\]

\[
= \lim_{\delta \to 0} \left( \frac{\pi \tan \left( \frac{\nu \pi}{2} + \frac{5 \pi}{2} \right)}{4 (\nu + \delta)} - \frac{1}{(2n + 8) \delta} \right)
\]

\[
\frac{\sin \left( \frac{\nu \pi}{2} + \frac{5 \pi}{2} \right)}{\cos \left( \frac{\nu \pi}{2} + \frac{5 \pi}{2} \right)} = \frac{\sin \left( \frac{\nu \pi}{2} \right) \cos \left( \frac{5 \pi}{2} \right)}{-\sin \left( \frac{5 \pi}{2} \right) \sin \left( \frac{\nu \pi}{2} \right) = - \left( 1 - \frac{\delta^2}{8} + \cdots \right)}
\]

\[
= - \frac{\pi}{4 \delta} + O(\delta^2)
\]

\[
\lim_{\delta \to 0} \left( \frac{1}{2 (n+\delta) \delta} - \frac{1}{(2n+\delta) \delta} \right) = - \frac{1}{4n^2}
\]
\[ E_n = \frac{\varepsilon^2}{n^2 \varepsilon_1} = \frac{E_0}{E_n} \quad \text{for odd } n \]

(c) For odd \( n \), \( E_n = n^2 \varepsilon_1 + 2 \varepsilon_1 + \frac{E_0^2}{n^2 \varepsilon_1} + \ldots \)

This is an expansion in powers of \( \frac{E_0}{n^2 \varepsilon_1} \).

Should work if \( E_0 \ll \varepsilon_1 \).

This is the only type of comment I expected. Restoring some detail, we really need:

\[ \varepsilon n \ll n^2 \frac{E_0^2}{2mL} \]

We'll get convergence if \( n \) is sufficiently small (failure as \( n \to \infty \), guaranteed because of S. function bound state), convergence if \( m \) is sufficiently large or \( n \) is sufficiently large... It is much better to discuss dimensionless numbers. For \( h = c = 1 \), \( E_0 L \) and \( mL \) are dimensionless.
2) \( H_0 = \frac{p^2}{2m} - \frac{1}{r} \); \( |\Psi^{(0)} \rangle = |nlm_e; m_e, m_p \rangle \).

\( H' = \frac{3}{2} \mathbf{s} \cdot \mathbf{s} \delta^3 (\mathbf{r}) \)

Compute \( E_n^{(1)} \) for \( n = 1, 2 \).

We need to diagonalize \( H' \) in each degenerate subspace (for each \( n \)), which is accomplished by \( (\mathbf{s} = \mathbf{s}_e + \mathbf{s}_p) \) switching to the coupled spin basis.

\( H' = \frac{3}{2} (s^2 - \frac{3}{2}) \delta^3 (\mathbf{r}) \)

\( |\Psi^{(0)} \rangle = |nlm_e; s, m_s \rangle \); \( s = 0, 1 \)

There will be off-diagonal matrix elements between states with different \( n \) (which we don't need for \( E_n^{(1)} \)) but

\[
\langle nlm_e; s, m_s | H' | n'l'm'_e; s', m'_s \rangle
\]

\[
= \text{constant} \times \delta_{nl} \delta_{lm_e} \delta_{s s'_e} \delta_{m_s m'_s}
\]
\[ \langle \text{nlm}_x, s, m_s | H | \text{nlm}_x, s, m_s \rangle \]

\[ = \frac{\hbar}{2} \left( s(s+1) - \frac{1}{2} \right) \int d^3r \, \psi_{\text{nlm}_x}(\vec{r}) \, \delta^3(\vec{r}) \cdot \psi_{\text{nlm}_x}(\vec{r}) \]

\[ = \frac{\hbar}{2} \left( s(s+1) - \frac{1}{2} \right) / \psi_{\text{nlm}_x}(0) \]

All wave functions vanish at the origin except for \( l = 0 \), so:

\[ E^{(1)}_{2s} = 0 \]

12 - states

\[ | \psi_{1s}(0) |^2 = \frac{1}{4\pi} \cdot 4 \left( \frac{1}{a_0} \right)^3 = \frac{1}{\pi a_0^3} \]

\[ | \psi_{2s}(0) |^2 = \frac{1}{4\pi} \cdot 4 \left( \frac{1}{2a_0} \right)^3 = \frac{1}{8\pi a_0^3} \]

\[ E^{(1)}_{1s} = \begin{cases} \frac{3}{4\pi a_0^3} & s = 1 - 3 \text{ states} \\ -\frac{3\hbar}{4\pi a_0^3} & s = 0 - 1 \text{ state} \end{cases} \]

\[ E^{(1)}_{2s} = \begin{cases} \frac{5}{32\pi a_0^3} & s = 1 - 3 \text{ states} \\ -\frac{3\hbar}{32\pi a_0^3} & s = 0 - 1 \text{ state} \end{cases} \]
3) Spin-1 bosons on a sphere

To characterize any state, we will need 6 quantum numbers (not counting $s = s_1 - s_2$).
A basis is $| l_1, m_1, l_2, m_2, s, m_s \rangle$

but we will need to couple angular momenta.

$$H = \frac{1}{2I} (l_1^2 + l_2^2 + l_1 \cdot l_2 + s_1 \cdot s_2)$$

$$\vec{L} = \vec{l}_1 + \vec{l}_2, \quad \vec{S} = \vec{s}_1 \cdot \vec{s}_2$$

$$l_1 \cdot l_2 = \frac{1}{2} (l_1^2 - l_1 \cdot l_2)$$

$$s_1 \cdot s_2 = \frac{1}{2} (s_1^2 - s_1 \cdot s_2) = \frac{1}{2} (s^2 - 4)$$

$$\Rightarrow \quad H = \frac{1}{4I} (l_1^2 + l_2^2 + l_1 \cdot l_2 + s^2 - 4)$$

Eigenstates (before symmetrization)

$$| l_1, l_2, l, m, s, m_s \rangle$$

6 labels.

$$E = \frac{1}{4I} (l_1(l_1 + 1) + l_2(l_2 + 1) + l(l+1) + s(s+1) - 4)$$

$$l_1, l_2 = 0, 1, 2, \ldots; \quad l = | l_1 - l_2 |, | l_1 - l_2 |+1, \ldots, l_1 + l_2$$

$$m_s = -l, l+1, \ldots; \quad s = 0, 1, 2; \quad m_s = -s, -s+1, \ldots, s$$
Only symmetric states are allowed, so:

\[ |4\rangle = \begin{cases} \\
|4_{\text{sym.}}\rangle \otimes |4_{\text{spin}}\rangle \\
|4_{\text{A}}\rangle \otimes |4_{\text{spin}}\rangle \\
\end{cases} \]

The \( s=0,2 \) states are symmetric.

The \( s=1 \) states are anti-symmetric.

For \( l = l_1 = l_2 \) :
\[ |l, l, l, M_e\rangle \]
includes both symmetric and antisymmetric states:
\[ l = 2l_2 \text{ in } S, \ l = 2l_2 - 1 \text{ in } A, \]
\[ l = 2l_2 - 2 \text{ in } S, \text{ and so on}. \]

For \( (l_1, l_2) = (l, l) \) , we must use:
\[ \frac{1}{\sqrt{2}} (|l_1, l_2, l, M_e\rangle + |l_2, l_1, l, M_e\rangle) \]
to form symmetric (+) or antisymmetric (-) states.
6) Ground state: \[ l_1 = l_2 = l = s = 0 \]
\[ E = -\frac{1}{4I}, \text{ degeneracy } = 1 \]

1st excited state:
\[ (l_1, l_2) = (0, 1), l = 1, s = 0 \quad E = 0, d = 3 \]
\[ (l_1, l_2) = (1, 1), l = 0, s = 0 \quad E = 0, d = 1 \]

If we try \( s = 1 \), we have to also increase \( l \)'s to make a symmetric state.

c) \[ H' = \frac{\mathbf{L} \cdot \mathbf{S}}{I} \quad \mathbf{J} = \mathbf{L} + \mathbf{S} \Rightarrow H' = \frac{1}{2I}(J^2 - L^2 - S^2) \]
\[ H + H' = \frac{1}{4I} \left( L_1^2 + L_2^2 - L^2 - S^2 + 2J^2 - 4 \right) \]

To keep our symmetries, we must build these coupled states only from the symmetric states above (e.g., still can't have \( l_1 = l_2 = l = 0, \ s = 1 \)).
Define:

\[ (l_a, l_b)^s \cdot l_s \cdot s \cdot j \cdot m_j \equiv \frac{1}{2} (l_a, b, l, s, j, m_j) = |l_a, l_b, l, s, j, m_j \rangle \]

\((l_a, l_b)^s\) goes with \(s = 0, 2\)

\((l_a, l_b)^p\) goes with \(s = 1\)

If \(l_a = l_b\), \(l^2 = 2l_a + s\) (goes with \(s = 0, 2\)), \(l = \frac{l_a - 1}{2}\) is \(A\) (goes with \(s = 1\)), etc.

For any \(l_a, l_b\), \(l = l_a - l_b, \ldots, l_a + l_b\), \(s = 1, 2, \ldots\), \(m_j = -j, -j + 1, \ldots, j\).

To find the ground state and first excited state energies, we must consider all possible set of \(l_a, l_b, l, s, j\), where:

\[ E = \frac{1}{4l} \left[ l(l+1) + b(b+1) - l(l+1) - s(s+1) + 2j(j+1) - 4 \right] \]

I will start by simply taking some low-lying levels, then turn to an analysis. This part graded liberally.
<table>
<thead>
<tr>
<th>$(l_1, l_2)$</th>
<th>$l$</th>
<th>$s$</th>
<th>$j$</th>
<th>$AI \times F$</th>
<th>notes</th>
</tr>
</thead>
<tbody>
<tr>
<td>(0, 0)</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>-4</td>
<td></td>
</tr>
<tr>
<td>(0, 1)</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>want large $s$ + small $j$ (ignore $j = 3$) need small $j$</td>
</tr>
<tr>
<td></td>
<td>2</td>
<td>1</td>
<td>2</td>
<td>-6</td>
<td>balance of $s \times j$</td>
</tr>
<tr>
<td>(0, 1)</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>-2</td>
<td></td>
</tr>
<tr>
<td>(1, 1)</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td></td>
</tr>
<tr>
<td></td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>-4</td>
<td>$l = l_1 + l_2$ $s = 2$ $j =</td>
</tr>
<tr>
<td></td>
<td>2</td>
<td>2</td>
<td>0</td>
<td>-12</td>
<td></td>
</tr>
<tr>
<td>(0, 2)</td>
<td>2</td>
<td>2</td>
<td>0</td>
<td>-10</td>
<td>$= 5 + 4$ for $j = 1$</td>
</tr>
<tr>
<td></td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>-6</td>
<td></td>
</tr>
<tr>
<td>(1, 2)</td>
<td>2</td>
<td>2</td>
<td>0</td>
<td>-8</td>
<td>balance of $l + j$</td>
</tr>
<tr>
<td></td>
<td>3</td>
<td>2</td>
<td>1</td>
<td>-10</td>
<td></td>
</tr>
<tr>
<td>(2, 2)</td>
<td>4</td>
<td>2</td>
<td>2</td>
<td>-6</td>
<td></td>
</tr>
</tbody>
</table>

Searching a large table makes it clear that finding the 1st excited state requires looking at several possible excitations, but for ground state we want:

\[ l = l_1 + l_2 \] , because of \(-l(l+1)\)

\[ s = 2 \] , because of \(-s(s+1)\)

\[ j = |l-s| \] , because of \(2j(j+1)\)
Minimizing:

\[ E = \frac{1}{7} \left[ l_1(l_1+1) + l_2(l_2+1) - (l_1+l_2)(l_1+l_2+1) \right. \\
+ 2 l_1 l_2 - 21 (l_1^2 + l_2^2 - 2l_1 l_2 + 1) - 10 \left. \right] \]

leads to \( l_1 = l_2 = 1 \).

**Ground state** \( l_1 = l_2 = 1 \); \( L = 2 \); \( S = 2 \); \( J = 0 \)

\[ E = -\frac{3}{2} \]

\( \text{degeneracy} = 1 \)

It is very difficult to find the 1st excited state without simply searching this vicinity.

1st excited states \( E = -\frac{5}{21} \)

\( \begin{align*}
(l_1,l_2) &= (0,2) \quad l = 2, S = 2, J = 0 \quad d = 1 \\
(l_1,l_2) &= (1,2) \quad l = 3, S = 2, J = 1 \quad d = 3 \\
\end{align*} \)

\( d_{tot} = 4 \)