Dimensional Analysis for Hydrogen-Like Atoms

In this example we are considering atoms with a heavy nuclear charge of $+Ze$ (Z is the number of protons) with total mass M_{nucl}, and a single electron with charge $-e$ and mass $m_e \ll M_{\mathrm{nucl}}$. We'll apply the procedure for doing systematic dimensional analysis that was presented in an earlier handout to find out about the characteristic size of the radius and energy.

General Procedure

1. Determine the relevant quantities from *physics* considerations, often based on the equation(s) that determine how the system behaves.
2. Determine the fundamental units (e.g., $[M]$, $[L]$, $[T]$) of each quantity.
3. Postulate an equation relating the quantities, with unknown exponents (a, b, ...).
4. Substitute units.
5. Equate exponents of $[M]$, $[L]$, $[T]$ and solve the resulting equations simultaneously.
6. Check your results by plugging back into the equations. Check whether the physics makes sense.

Ok, now we apply the steps ...

1. **Physics determining the typical radius and energy.**
   - The kinetic energy is $p^2/2\mu$, where μ is the reduced mass: $\mu = m_e M_{\mathrm{nucl}}/(m_e + M_{\mathrm{nucl}})$. Since $m_e \ll M_{\mathrm{nucl}}$, $\mu \approx m_e$, so the relevant quantity is m_e.
   - The Coulomb potential is

     $$V(r) = -\frac{k(Ze)e}{r} = -(Zke^2)\frac{1}{r} \ ,$$

     so the relevant quantity is Zke^2.
   - Relativity? If so, the speed of light c would be relevant. Plan: assume *not*, then check speed at end of compare the energy to $m_e c^2$.
   - Quantization $\Longrightarrow \hbar$ is relevant. (Why $\hbar$ and not $h = 2\pi\hbar$? In general, $\hbar$ is the combination that appears in the Schrödinger equation.)
2. **Units of each quantity.**

$$E \sim \frac{1}{2}mv^2 \sim [M][L]^2[T]^{-2} \qquad r \sim [L] \qquad m_e \sim [M]$$

$$Zke^2 \sim Er \sim [M][L]^3[T]^{-2} \qquad \hbar \sim \Delta x \Delta p \sim [M][L]^2[T]^{-1}$$

3. **Equations for each quantity of interest**

$$r = C_r \hbar^a (Zke^2)^b m^c \qquad\qquad E = C_E \hbar^\alpha (Zke^2)^\beta m_e^\gamma \ ,$$

where C_r and C_E are dimensionless constants that we expect to be of order unity (e.g., between 1/3 and 3 or so).

4. **Substitute units**

$$\text{radius:} \quad [L] = ([M]^a [L]^{2a} [T]^{-a})([M]^b [L]^{3b} [T]^{-2b})([M]^c)$$

$$\text{energy:} \quad ([M][L]^2[T]^{-2}) = ([M]^\alpha [L]^{2\alpha} [T]^{-\alpha})([M]^\beta [L]^{3\beta} [T]^{-2\beta})([M]^\gamma)$$

5. **Equate exponents of units and solve the simultaneous equations.**

radius: $[M] : 0 = a+b+c \quad [L] : 1 = 2a+3b \quad [T] : 0 = -a-2b \quad \Longrightarrow \quad a = 2,\ b = -1,\ c = -1$

energy: $[M] : 1 = \alpha+\beta+\gamma \quad [L] : 2 = 2\alpha+3\beta \quad [T] : 2 = -\alpha-2\beta \quad \Longrightarrow \quad \alpha = -2,\ \beta = 2,\ \gamma = -1$

6. **Do the results make sense?** Our results for the radius and energy are:

$$r = C_r \frac{\hbar^2}{(Zke^2)m_e} = C_r \frac{1}{Z} a_0 \ ,$$

where $a_0 = 0.0529\,\text{nm}$ is the Bohr radius and

$$E = C_E \frac{(Zke^2)^2 m_e}{\hbar^2} = C_E Z^2 \frac{ke^2}{a_0} \ .$$

We can verify that the trends make sense:

- If $m_e \uparrow$ (e.g. muonic atom), then $r \downarrow \surd$
- If $Z \uparrow$ (stronger Coulomb force), then $r \downarrow \surd$

Note that unless Z is large, $E \ll m_e c^2$, justifying our assumption that the atom is nonrelativistic. We can check our estimates against the exact expectation values for the ground state of the hydrogen atom: $C_r = 1$ and $C_E = 1/2 \Longrightarrow$ not bad!