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7701 Introductory Comments

① Prof. Dick Furnstahl, Nuclear Theory Group
- low-energy nuclear physics
  - cf. relativistic heavy-ion collisions like Prof. Yuri Kovchegov (next semester)
  - nuclear structure and reactions from microscopic calculations (e.g. energies, decay rates → astrophysics
  - connect to underlying quantum chromodynamics
- effective field theory, renormalization group are tools ⇒ math methods
- many computational aspects → numerical math methods (cf. computational physics course I teach spring semester)

② 7701 logistics — see Web pages
- 7701 home page is communication center for course
  - always check "recent changes" section at top for new things
- Course description and info is linked
  - texts, prerequisites, material, schedule, grading, office hours
  - some is still tentative → check back
- Assignments, Handouts, Lecture Notes (PS#1 handed out only today ⇒ otherwise online
  - username: physics password: 7701
  - it is essential to do problems to learn
  - goal: useful but not excessive homework
    - bonus problems not required but recommended for theorists or math-inclined experimentalists
  - interact with classmates but don't ask for collective help too soon ⇒ everyone needs to struggle some to really learn
- Text and references
  - see comments on webpages and in emails
  - no fixed text — standard topics
    - Cahill, Physical Mathematics → interesting examples and topics like singular value decomposition, Feynman propagator for fourier transforms, cosmic microwave background (CMB) radiation for spherical harmonics. Topics for future reference: integral equations, path integrals, Monte Carlo, group theory

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- To learn the basics, Cahill (or Arfken/Weber) is not as good as Lea (or Boas)
  - ⇒ I will provide appropriate excerpts for certain topics

③ Overview of 7701 Analytic and Numeric Methods of Physics

- originated two years ago as a replacement of the first quarter of Jackson electrodynamics.
- last year in transition with switch to semester — more boundary value / Green's function methods but not tied to E/M in second semester.
- This year: Add equivalence of 1st three chapters of Jackson ⇒ electrostatic boundary value problems as "lead in" to "first" semester of E/M
  - same math methods, particular physics context
  - requires rearranging topics somewhat
- Goals:
  - establish mathematical core competencies for E+M, serves as foundation for other physics (but not everything — e.g. linear algebra in QM mostly)
  - learn how and where to look things up (including Mathematica)
  - gain <u>some</u> background and appreciation of numerical issues.

⇒ math as a <u>tool</u>, rather than a distraction from physics

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- What topics should 7701 cover?
- Consider the first equations in the introduction to Jackson's "Classical Electrodynamics" text: Maxwell's Equations

$$\vec{\nabla}\cdot\vec{D} = \rho$$
$$\vec{\nabla}\times\vec{H} - \frac{\partial\vec{D}}{\partial t} = \vec{J} \qquad \text{all functions of } (\vec{x}, t)$$
$$\vec{\nabla}\times\vec{E} + \frac{\partial\vec{B}}{\partial t} = 0 \qquad \text{local equations (at a point and nearby from derivatives)}$$
$$\vec{\nabla}\cdot\vec{B} = 0 \qquad \Rightarrow \text{find global consequences}$$

To deal with these we need: (see also Jackson covers)
- vector calculus
  - intrinsically 3-dimensional
  - use different coordinate systems to simplify (geometry)
- differential equations → special functions, eigenfunction expansions (Sturm-Liouville)
  → linear ⇒ superposition → Green's functions
- point sources → delta (generalized) functions → Green's functions
- Fourier series and integrals (solving diff. eqs, momentum space, ...)
- complex analysis - contour integrals, dispersion relations
  ⇒ see Wigner essay on relevance of complex numbers
- numerical solutions!

- List of topics is covered in any good math methods book
  - Choose according to personal preference (likely more than one!)

- Many topics and limited time (in and out of class)
  - examples to provide foundation
  - practice looking things up (eg. special functions) in books, Mathematica, ...

- Issues:
  - vocabulary - many terms may be new (or different from past use) ⇒ ask if unclear
  - notation - become exposed to full variety. Notation matters!
    eg, $\hat{x}$ or $\hat{i}$ or $\hat{e}_x$ or ... Use multiple notations on purpose

$\boxed{\text{* If you know these topics well, consider placing out of 7701}}$

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How is 770 different from a math course?

① Emphasis on physics intuition rather than mathematical rigor.
⇒ derivations and motivations but seldom rigorous proofs (in class)
"physics proofs"

② Using math as a tool for physics (and E+M in particular)
⇒ selective coverage and only as general/abstract as needed (for working "by hand")

③ Mathematica as a tool and substitute, when appropriate
• but be careful to recognize limitations

④ Numerical solutions (as opposed to symbolic "analytic")
• also conceptual simplifications, eg. operators → matrices
Hilbert spaces → finite dimensional matrices and vectors

• Random sampling of Mathematica capabilities (without explaining)
• many math tasks formerly done by hand (or book) are best done by Mathematica (eg. series expansions, sum, ...)
?SphericalBesselJ (and follow help) cf. Abramovich + Stegun
• plots
• Series[SphericalBesselJ[3,x], {x,0,5}] small x
{x, Infinity, 3}] large x (asymptotic)
//FullSimplify ["O" means "order of"]

• SphericalBesselJ[2,x]
//FunctionExpand Notation: brackets matter!
• LegendreP[100,x] [≠ (≠ {

Integrate[Cos[x]/x, {x,-1,2}]
//TraditionalForm but undefined!
add PrincipalValue -> True (more soon!)

Jackson covers; we'll return later for vector calculus theorems,
• for now, vector manipulations and identities with Kronecker delta function and Levi-Civita (Epsilon) symbol

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Kronecker Delta function and Levi-Cinta (Epsilon) Symbol

• first go through annotated sheet sections 1, 2, 3.

• first proof: ε can move anywhere ← be careful of moving A, B, C if derivatives

$$\vec A\cdot(\vec B\times\vec C) = A_i(\vec B\times\vec C)_i = A_i\varepsilon_{ijk}B_jC_k = \varepsilon_{ijk}A_iB_jC_k$$

works from outside in

✓ cyclic or even permutation

$$= \varepsilon_{kij}A_iB_jC_k = \varepsilon_{kij}C_kA_iB_j = C_k(\vec A\times\vec B)_k = \vec C\cdot(\vec A\times\vec B)$$

if C_k, A_i, B_j commute **

• class vote $\stackrel{?}{=} \vec B\cdot(\vec C\times\vec A)$ ✓ cyclic!

or $\stackrel{?}{=} \vec B\cdot(\vec A\times\vec C)$ ×

• Two quick examples (more from Jackson front cover to try!)

relabel dummy indices $j\leftrightarrow k$; now switch k,j

(1) $\vec a\times\vec a \Rightarrow (\vec a\times\vec a)_i = \varepsilon_{ijk}a_ja_k = \varepsilon_{ikj}a_ka_j = -\varepsilon_{ijk}a_ka_j$

antisymmetric × symmetric $= 0$

if commuting $= -\varepsilon_{ijk}a_ja_k = 0$

↙ can be interchanged

cf., $[\nabla\times\vec\nabla\psi(\vec x)]_i = \varepsilon_{ijk}\frac{\partial}{\partial x_j}\frac{\partial}{\partial x_k}\psi(\vec x) = 0$ ↖ but $\frac{\partial}{\partial x_i}x_j f(\vec x) \neq x_j\frac{\partial}{\partial x_i}f(\vec x)$

↖ ε_{ijk} changes sign under $j\leftrightarrow k$ while $\frac{\partial}{\partial x_j}\frac{\partial}{\partial x_k}$ doesn't ⇒ antisym × sym = 0

notation! $(\vec\nabla)_i \equiv \frac{\partial}{\partial x_i} \equiv \partial_i \equiv \nabla_i$

(2) $[\underset{j}{\vec\nabla}\times\underbrace{(\vec\nabla\times\vec a)}_{k}]_i = \varepsilon_{ijk}\frac{\partial}{\partial x_j}(\vec\nabla\times\vec a)_k \stackrel{?}{=} \varepsilon_{ijk}\cancel{\frac{\partial}{\partial x_j}}\varepsilon_{kij}\frac{\partial}{\partial x_i}a_j$ no! i, j appear more than twice

$= \varepsilon_{ijk}\frac{\partial}{\partial x_j}\varepsilon_{klm}\frac{\partial}{\partial x_l}a_m$ correct

outside in

permute repeated index to first place

• simplify: eliminate 2 ε's with $\varepsilon_{ijk}\varepsilon_{klm} = \varepsilon_{kij}\varepsilon_{klm} = (\delta_{il}\delta_{jm} - \delta_{im}\delta_{jl})$

eliminate δ's

$$= (\delta_{il}\delta_{jm} - \delta_{im}\delta_{jl})\frac{\partial}{\partial x_j}\frac{\partial}{\partial x_l}a_m = \frac{\partial}{\partial x_m}\frac{\partial}{\partial x_i}a_m - \frac{\partial}{\partial x_l}\frac{\partial}{\partial x_l}a_i$$

commute ∂ $= \frac{\partial}{\partial x_i}\frac{\partial}{\partial x_m}a_m - \frac{\partial}{\partial x_l}\frac{\partial}{\partial x_l}a_i$ identify $= \frac{\partial}{\partial x_i}\vec\nabla\cdot\vec a - \nabla^2 a_i$

If time, try more. $= [\vec\nabla(\vec\nabla\cdot\vec a) - \nabla^2\vec a]_i$ (we can drop the i's now)

8/21/13 • good prototype method!

Kronecker Delta Function δ_{ij} and Levi-Civita (Epsilon) Symbol ε_{ijk}.

• There are multiple ways to deal with vectors. This one generalizes to higher-rank tensors and numerical applications.

1. **Definitions**

$$\delta_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{otherwise} \end{cases} \qquad \varepsilon_{ijk} = \begin{cases} +1 & \text{if } \{ijk\} = 123,\ 312, \text{ or } 231 \\ -1 & \text{if } \{ijk\} = 213,\ 321, \text{ or } 132 \\ 0 & \text{all other cases (i.e., any two equal)} \end{cases}$$

don't confuse with other εs (dielectric tensor ε_{ij})

← dense a, mnemonic to remember

notation: ε_{ijk} or ε_{abc} or ε_{lmn}

- So, for example, $\varepsilon_{112} = \varepsilon_{313} = \varepsilon_{222} = 0$.
- The +1 (or *even*) permutations are related by rotating the numbers around; think of starting with 123 and moving (in your mind) the 3 to the front of the line, to get 312. Do it again with the 2 and you get 231. The −1 (or *odd*) permutations starting with 213 are related to each other the same way; they are related to 123 by interchanging just two of the numbers (e.g., switch the 1 and 3 to get 321). ← need to do this in your head

2. **Applying δ_{ij} and ε_{ijk} to Vectors in Cartesian coordinates** ← (can this work with other coordinates?!)

- ✱ Instead of using x, y, and z to label the components of a vector, we use 1, 2, 3.
- ✱ Then the letters i, j, k, ... can be used as (dummy) summation variables, running from 1 to 3. (We could use any other letters, like a, b, ...; it is merely a convention.)

↑ later: basis unit vectors are not constant as $\hat{x}, \hat{y}, \hat{z}$ are

- Don't confuse the use of the dummy summation variables i, j, k, each of which can be 1, 2, or 3, with the unit vectors $\hat{\mathbf{i}}$, $\hat{\mathbf{j}}$, $\hat{\mathbf{k}}$. These are two independent notations!
- The dot product of two vectors $\mathbf{A}\cdot\mathbf{B} = A_xB_x + A_yB_y + A_zB_z$ in this notation is

$$\mathbf{A}\cdot\mathbf{B} = A_1B_1 + A_2B_2 + A_3B_3 = \sum_{i=1}^{3} A_iB_i = \sum_{i=1}^{3}\sum_{j=1}^{3} A_iB_j\delta_{ij} = \sum_{i=1}^{3}\sum_{j=1}^{3} A_i\delta_{ij}B_j \ .$$

could be more than 3 (numerical applications)

$(A)\begin{pmatrix} 1 & \\ & 1 \end{pmatrix}(B)$

Note that there are nine terms in the final sums, but only three terms are non-zero, and that the final sum is in the form of a matrix multiplication.

- The i^{th} component of the cross product of two vectors $\mathbf{A}\times\mathbf{B}$ becomes

$$(\mathbf{A}\times\mathbf{B})_i = \sum_{j=1}^{3}\sum_{k=1}^{3} \varepsilon_{ijk}A_jB_k \ . = \sum_{l=1}^{3}\sum_{m=1}^{3} \varepsilon_{ilm}A_lB_m$$

← dummy indices

Again, there are nine terms in the sum, but this time only two of them are non-zero. Note also that this expression summarizes three equations, namely for $i = 1, 2, 3$.

$$(A\times B)_1 = \underbrace{\varepsilon_{123}}_{+1} A_2B_3 + \underbrace{\varepsilon_{132}}_{-1} A_3B_2 = A_2B_3 - A_3B_2$$

3. **Einstein Summation Convention**

- We might notice that the summations in the expressions for $\mathbf{A}\cdot\mathbf{B}$ and $\mathbf{A}\times\mathbf{B}$ are redundant, because they only appear when an index like i or j appears twice on one side of an equation. So we can omit them. Thus

$$\sum_{i=1}^{3}\sum_{j=1}^{3} A_iB_j\delta_{ij} \longrightarrow A_iB_j\delta_{ij} = A_iB_i \qquad \text{and} \qquad \sum_{j=1}^{3}\sum_{k=1}^{3} \varepsilon_{ijk}A_jB_k \longrightarrow \varepsilon_{ijk}A_jB_k \ .$$

- Rules: If an index appears (exactly) twice, then it is summed over and appears only on one side of an equation. A single index (called a *free index*) appears once on *each side* of the equation. So

Careful: it's easy to reuse an index by mistake

$$\textit{Valid:}\ A_i = A_j\delta_{ij}\,, \quad B_k = \varepsilon_{ikl}A_iC_l \qquad \textit{Invalid:}\ A_i = B_iC_i\,, \quad A_i = \varepsilon_{ijk}B_iC_j\ .$$

Simplification rules (implement in Mathematica!)
① replace doubled ε's
② use δ's to eliminate indices

- When you have a Kronecker delta δ_{ij} and one of the indices is repeated (say i), then you simplify by replacing the other i index on that side of the equation by j and removing the δ_{ij}. For example: eliminate index ✓ look at δ, find one of the indices, and substitute the other.

$$A_j\delta_{ij} = A_i\,, \qquad B_{ij}C_{jk}\delta_{ik} = B_{kj}C_{jk} = B_{ij}C_{ji}$$

Note that in the second case we had two choices of how to simplify the equation; use either one!

If B, C are matrices, what is this expression? Tr(BC)

- The triple or box product $\mathbf{A}\cdot(\mathbf{B}\times\mathbf{C})$ can be written

$$\mathbf{A}\cdot(\mathbf{B}\times\mathbf{C}) = \varepsilon_{ijk}A_iB_jC_k = \varepsilon_{kij}A_iB_jC_k = \varepsilon_{kij}C_kA_iB_j = \mathbf{C}\cdot(\mathbf{A}\times\mathbf{B})\ ,$$

where we've used the properties of ε_{ijk} in the middle equations to prove a relation among triple products with the vectors in a different order.

- A very useful identity (if the repeated index is not first in both ε's, permute until it is):

If repeated ε's with common index, use this.

$$\varepsilon_{ijk}\varepsilon_{ilm} = \delta_{jl}\delta_{km} - \delta_{jm}\delta_{kl}\ .$$

eg $\varepsilon_{123}\varepsilon_{113} = 0$, only $\varepsilon_{123}\varepsilon_{123}$ or $\varepsilon_{123}\varepsilon_{132}$

j and l same or k,m same — with sign flip

(be able to write this without looking it up)

4. **Example: Proving a Vector Identity**

- We'll write the i^{th} Cartesian component of the gradient operator $\boldsymbol{\nabla}$ as ∂_i (cf. $\frac{\partial}{\partial x_i}$).
- Let's simplify $\boldsymbol{\nabla}\times(\boldsymbol{\nabla}\times\mathbf{A}(\mathbf{x}))$. We start by considering the i^{th} component and then we use our expression for the cross product (working from the outside in):

$$(\boldsymbol{\nabla}\times(\boldsymbol{\nabla}\times\mathbf{A}))_i = \varepsilon_{ijk}\partial_j(\boldsymbol{\nabla}\times\mathbf{A})_k\ .$$

Next we replace the remaining cross product, making sure to introduce new dummy summation variables l and m:

$$(\boldsymbol{\nabla}\times(\boldsymbol{\nabla}\times\mathbf{A}))_i = \varepsilon_{ijk}\partial_j\varepsilon_{klm}\partial_lA_m = \varepsilon_{kij}\varepsilon_{klm}\partial_j\partial_lA_m\ .$$

(The partial derivatives act only on the components of $\mathbf{A}$, so we can pull out the ε's.) We rotated the indices in one of the ε's in the last step so that we can now directly apply our very useful identity (and simplify):

$$(\boldsymbol{\nabla}\times(\boldsymbol{\nabla}\times\mathbf{A}))_i = (\delta_{il}\delta_{jm} - \delta_{im}\delta_{lj})\partial_j\partial_lA_m = \partial_m\partial_iA_m - \partial_l\partial_lA_i = \partial_i(\partial_mA_m) - (\partial_l\partial_lA)_i$$

or, finally,

$$\boldsymbol{\nabla}\times(\boldsymbol{\nabla}\times\mathbf{A}) = \boldsymbol{\nabla}(\boldsymbol{\nabla}\cdot\mathbf{A}) - \nabla^2\mathbf{A}\ .$$