Generalizations $\vec{a}\cdot\vec{b} = a_i \delta_{ij} b_j$, $\delta_{ij} = \begin{pmatrix} 1 & & \\ & 1 & \\ & & 1 \end{pmatrix}$

special relativity $a_\mu g^{\mu\nu} a_\nu$, $g_{\mu\nu} = g^{\mu\nu} = \begin{pmatrix} +1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix}$ overall sign is convention

general relativity $g_{\mu\nu} \rightarrow$ nontrivial matrix for curved space "metric tensor" $[g_{\mu\nu} \neq g^{\mu\nu}$ unless flat$]$

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7701 Lecture 2

On the board: reference $\varepsilon_{abc}\varepsilon_{aef} = \delta_{be}\delta_{cf} - \delta_{bf}\delta_{ce}$

ε-δ warm-ups (answers in lecture notes)

list in this order

② $\cdot\ \vec{\nabla}\cdot(\vec{V}\times\vec{W}) = \vec{W}\cdot(\vec{\nabla}\times\vec{V}) - \vec{V}\cdot(\vec{\nabla}\times\vec{W})$ "product rule"

③ $\cdot\ \vec{\nabla}\times(\vec{V}\times\vec{W}) = \vec{V}(\vec{\nabla}\cdot\vec{W}) - \vec{W}(\vec{\nabla}\cdot\vec{V}) + (\vec{W}\cdot\vec{\nabla})\vec{V} - (\vec{V}\cdot\vec{\nabla})\vec{W}$

① $\cdot\ \vec{\nabla}\times(\phi\vec{\nabla}\phi) = \,?\quad (=0)$

"Spot the Error" (What is wrong for each of these?)	problem
$(\vec{\nabla}\times\vec{V})\cdot(\vec{\nabla}\times\vec{W}) = \varepsilon_{abc}\frac{\partial}{\partial x_b}V_c\,\varepsilon_{def}\frac{\partial}{\partial x_e}W_f$	"d" should be "a"
$\vec{\nabla}\times(\vec{\nabla}\times\vec{V}) = \varepsilon_{abc}\frac{\partial}{\partial x_b}\varepsilon_{cde}\frac{\partial}{\partial x_d}V_e$	vector on left, "a" component on right
$(\vec{A}\times\vec{B})_a + (\vec{C}\times\vec{D})_d = \varepsilon_{abc}A_bB_c + \varepsilon_{def}C_eD_f$	"d" should be "a"
$\delta_{ab}\varepsilon_{acd}\,\varepsilon_{efa}\,\delta_{fd}$	"a" appears 3 times
$[\vec{A}\times(\vec{B}\times\vec{C})]_a = \varepsilon_{abc}A_b\,\varepsilon_{acd}B_cC_d$	"c" " "
$\vec{\nabla}\times(\vec{B}\cdot\vec{C}) = 0$	$\vec{B}\cdot\vec{C}$ is scalar

Think about programming in Mathematica using pattern matching for simplification

Recap of general algorithm with $\vec{\nabla}\times(\vec{\nabla}\times\vec{a}) = \vec{\nabla}(\vec{\nabla}\cdot\vec{a}) - \nabla^2\vec{a}$ example

1. Assign component labels (from outside in) $[\underset{j}{\vec{\nabla}}\times\underset{k}{(\vec{\nabla}\times\vec{a})}]_i$

no! i,j appear more than twice

2. Use ε_{ijk}'s for cross products: $= \varepsilon_{ijk}\frac{\partial}{\partial x_j}(\vec{\nabla}\times\vec{a})_k \overset{?}{=} \varepsilon_{ijk}\frac{\partial}{\partial x_j}\cancel{\varepsilon_{kij}\frac{\partial}{\partial x_i}a_j}$

$= \varepsilon_{ijk}\frac{\partial}{\partial x_j}\varepsilon_{klm}\frac{\partial}{\partial x_l}a_m$ correct!

3. Eliminate 2 ε's (no cross products on right side):

$\varepsilon_{ijk}\varepsilon_{klm} = \varepsilon_{kij}\varepsilon_{klm} = (\delta_{il}\delta_{jm} - \delta_{im}\delta_{jl})$

permute so k is first index (Mathematica?)

4. Use δ_{ij}'s to eliminate indices: $= (\delta_{il}\delta_{jm} - \delta_{im}\delta_{jl})\frac{\partial}{\partial x_j}\frac{\partial}{\partial x_l}a_m = \frac{\partial}{\partial x_m}\frac{\partial}{\partial x_i}a_m - \frac{\partial}{\partial x_l}\frac{\partial}{\partial x_l}a_i$

5. Rearrange as needed and identify dot products $= \frac{\partial}{\partial x_i}\underbrace{\frac{\partial}{\partial x_m}a_m}_{\vec{\nabla}\cdot\vec{a}} - \underbrace{\frac{\partial}{\partial x_l}\frac{\partial}{\partial x_l}}_{\nabla^2}a_i = \frac{\partial}{\partial x_i}(\vec{\nabla}\cdot\vec{a}) - \nabla^2 a_i$

$= [\vec{\nabla}(\vec{\nabla}\cdot\vec{a}) - \nabla^2\vec{a}]_i$ (now drop i's)

Aside: Suppose you had a matrix $\underline{\underline{A}} \rightarrow A_{ij}$ and it happened that $a_{ij}a_{ik} = \delta_{jk}$. Write in matrix form. What kind of matrix is $\underline{A}$?

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Answers to warm-ups:

$$\left[\vec{\nabla}\times(\vec{V}\times\vec{W})\right]_a = \varepsilon_{abc}\frac{\partial}{\partial x_b}(\vec{V}\times\vec{W})_c = \varepsilon_{abc}\frac{\partial}{\partial x_b}\varepsilon_{cde}V_d W_e$$

$$= \varepsilon_{cab}\varepsilon_{cde}\frac{\partial}{\partial x_b}V_d W_e = (\delta_{ad}\delta_{be} - \delta_{ae}\delta_{bd})\frac{\partial}{\partial x_b}V_d W_e$$

or $-$

$$= \left(\frac{\partial}{\partial x_e}V_a W_e - \frac{\partial}{\partial x_b}V_b W_a\right) \qquad \frac{\partial}{\partial x_b} \text{ acts on both}$$

do derivatives

$$= \left(\frac{\partial V_a}{\partial x_e}\right)W_e + V_a\left(\frac{\partial W_e}{\partial x_e}\right) - \left(\frac{\partial V_b}{\partial x_b}\right)W_a - V_b\left(\frac{\partial W_a}{\partial x_b}\right)$$

assume $\vec{W}$ and $\vec{V}$ commute

$$= (\vec{W}\cdot\vec{\nabla})[\vec{V}]_a + [\vec{V}]_a(\vec{\nabla}\cdot\vec{W}) - (\vec{\nabla}\cdot\vec{V})[\vec{W}]_a - (\vec{V}\cdot\vec{\nabla})[\vec{W}]_a$$

remove a's

$$\vec{\nabla}\times(\vec{V}\times\vec{W}) = (\vec{W}\cdot\vec{\nabla})\vec{V} + \vec{V}(\vec{\nabla}\cdot\vec{W}) - \vec{W}(\vec{\nabla}\cdot\vec{V}) - (\vec{V}\cdot\vec{\nabla})\vec{W} \quad ✓$$

$$\vec{\nabla}\cdot(\vec{V}\times\vec{W}) = \frac{\partial}{\partial x_a}(\vec{V}\times\vec{W})_a = \frac{\partial}{\partial x_a}\varepsilon_{abc}V_b W_c = \varepsilon_{abc}\frac{\partial}{\partial x_a}V_b W_c$$

$$= \varepsilon_{abc}\left[\left(\frac{\partial V_b}{\partial x_a}\right)W_c + V_b\frac{\partial W_c}{\partial x_a}\right] \qquad \frac{\partial}{\partial x_a} \text{ acts on both}$$

assume $\vec{V}$ and $\vec{W}$ commute

$$= \underset{\uparrow}{\varepsilon_{cab}}\, W_c\frac{\partial}{\partial x_a}V_b - \underset{\uparrow}{\varepsilon_{bac}}\, V_b\frac{\partial}{\partial x_a}W_c$$

cyclic permutation $\qquad \varepsilon_{bac} = -\varepsilon_{abc}$

$$= W_c(\vec{\nabla}\times\vec{V})_c - V_b(\vec{\nabla}\times\vec{W})_b = \vec{W}\cdot(\vec{\nabla}\times\vec{V}) - \vec{V}\cdot(\vec{\nabla}\times\vec{W}) \quad ✓$$

$$\left[\vec{\nabla}\times(\phi\vec{\nabla}\phi)\right]_a = \varepsilon_{abc}\frac{\partial}{\partial x_b}\left(\phi\frac{\partial}{\partial x_c}\phi\right) = \varepsilon_{abc}\left[\left(\frac{\partial\phi}{\partial x_b}\right)\left(\frac{\partial\phi}{\partial x_c}\right) + \phi\frac{\partial}{\partial x_b}\frac{\partial}{\partial x_c}\phi\right]$$

$$= 0$$ antisymmetric × symmetric in both terms (for $b \leftrightarrow c$)

That's it for δ ε's for now. But we'll keep coming back...

motivational question: consider $f(x) = \frac{1}{1+x^2}$ and its Taylor expansion about $x=0 \Rightarrow 1-x^2+x^4+\ldots$

- For what range of x does this series converge?
- One way: Consider $f(z)$ for complex z and find nearest singularities!

(10)

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Switching gears $\Rightarrow$ Complex Analysis

Introductory remarks:

Complex analysis is a big subject, generally a semester course by itself in a math department

- We'll spend about 4-5 lectures!
- So we have to focus our attention and concentrate on results, not proofs

Partial list of motivations for complex analysis (see Arfken's discussion)

- real physical quantities can become complex as physical theory is made more general
  - add absorption and real index of refraction of light becomes complex
  - allow decay and real energies become complex (optical potential)
- $k \to ik$ relates Helmholtz to diffusion equation
- $t \to \tau = it \Rightarrow$ imaginary time equations ("Euclidean") are key to numerical solutions of field theories (and other Monte Carlo methods like Green's Function Monte Carlo for S-eqn)
- insight into and tools for solving differential equations
- integrals in complex plane have many applications

Core competencies ("what you need to know")

1. comfortable with manipulations of complex variables and functions
   - both Cartesian and polar form
2. evaluation of integrals. Requires:
   - understanding of analyticity (Cauchy-Riemann relations)
   - singularities (poles, branch points, essential singularities)
   - Laurent series expansion
   - specific applications of contour integration (residues et al)
3. dispersion relations (but just a preview — details next semester)

probably omitted: conformal mapping, proofs of any of this!

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Reminders of complex number representation...

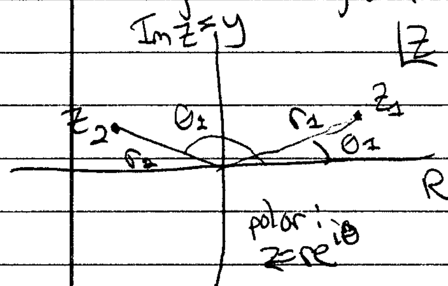
<u>notation</u>: $z = x + iy \overset{\text{one to one}}{\Longleftrightarrow} (x, y)$ must know without thinking!

real

$\boxed{i^2 = -1,\ i^3 = -i,\ i^4 = 1}$

⇒ "Argand diagram" or "complex z plane"

$\mathbb{Z}$ ← notation



usually choose $0 \le \theta < 2\pi$ Mathematica!

or $-\pi < \theta \le \pi$ so θ unique

but clearly some freedom:

$$3e^{i\pi/3} = 3e^{i(\pi/3 + 2\pi)} = 3e^{i(\pi/3 + 4\pi)} = 3e^{-5\pi/3}$$

Complex conjugate notation $z^* = x - iy$ (or $\bar{z}$ sometimes) $\quad zz^* = z^* z = |z|^2 = x^2 + y^2 \ge 0$

Recall how simple manipulations work through examples,

Lea, problem 2.2: Find an expression for $\cos 3\theta$ and for $\sin 3\theta$ in terms of $\cos\theta$, $\sin\theta$

Connection with complex numbers $\boxed{e^{i\theta} = \cos\theta + i\sin\theta}$ ← know results for $\theta = \frac{\pi}{2}, \pi, \frac{3\pi}{2}, 2\pi, \ldots$ (read from Argand diagram)

How do I know this? Taylor series probably best:

$$e^{i\theta} = \sum_{n=0}^{\infty} \frac{(i\theta)^n}{n!} = \sum_{n\ \text{even}} + \sum_{n\ \text{odd}} = \underbrace{1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \ldots}_{\cos\theta} \ \underbrace{+ \frac{i\theta}{1!} - \frac{i\theta^3}{3!} + \frac{i\theta^5}{5!}}_{i\sin\theta} \quad ✓$$

Question: is $e^{iz} = \cos z + i\sin z$ for complex z? (cf. real θ)
(ans: yes, the Taylor expansion holds)

remember z is complex here ⇒ $\cos z = \dfrac{e^{iz} + e^{-iz}}{2}$, $\cosh(z) = \dfrac{e^{z} + e^{-z}}{2}$, $\sin z = \dfrac{e^{iz} - e^{-iz}}{2i}$, $\sinh z = \dfrac{e^{z} - e^{-z}}{2}$

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Back to our problem ....

Take $r=1$, $z=e^{i\theta}$

$$z^3 = (e^{i\theta})^3 = e^{i3\theta}$$

$$(\cos\theta + i\sin\theta)^3 = \cos 3\theta + i\sin 3\theta$$

expand

$$= \cos^3\theta + 3\cos^2\theta(i\sin\theta) + 3\cos(i\sin\theta)^2 + (i\sin\theta)^3 = \cos 3\theta + i\sin 3\theta$$

In an equation, real and imaginary parts must be separately equal.

[How do we know? Put all terms on one side!

follows from complex conjugation preserving equality

(Real part) + i(Imaginary part) = 0

complex conjugate (Real part) − i (Imaginary part) = 0

⇒ add and subtract: 2 Real part = 0, $2i$ Imag part = 0

so each separately zero]

equal: real $\cos^3\theta - 3\cos\sin^2\theta = \cos 3\theta$ ✓

$3\cos^2\theta\sin\theta - \sin^3\theta = \sin 3\theta$

Is this a particularly powerful trick?

example notebook on Mathematica page

- For example, can Mathematica tell us this?
- Try these on computer

  Cos[3t] → nothing

  Cos[3t] // Simplify or // FullSimplify (do both postfix // and functional)

  ?Trig* to see choices

  Cos[3t] // TrigExpand or TrigFactor

  ⇒ works! Lesson: Mathematica can be very helpful, but not always automatically!

- Try Cos[5t] or Cos[50t] you wouldn't want to do by hand!
- Try TrigToExp and ExpToTrig on

  Cos[z] + I Sin[z] and Exp[iz]
- Why does cos[3t] // TrigExpand do nothing? Need Cos, not cos (uppercase)

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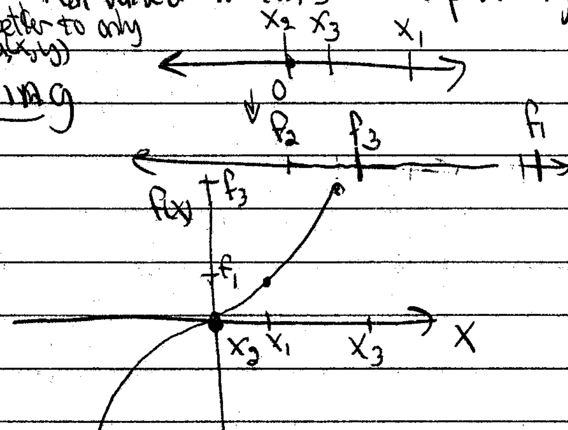
Functions of a Complex Variable

$z = x + iy \rightarrow w = f(z) = u(z) + i v(z)$

real valued functions of complex arguments

(maybe better to only use $u(x,y)$)

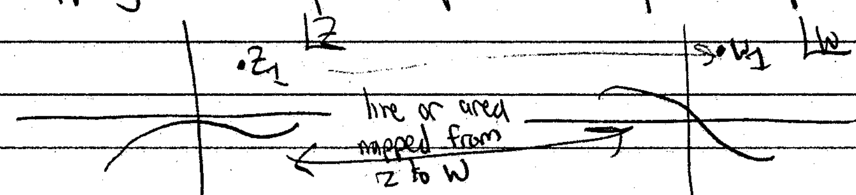


ordinary function $f(x)$ is a mapping from $x \in \mathbb{R}$ to $f(x) \in \mathbb{R}$

"element of" real numbers

$\Rightarrow$ represent as graph

Complex $z = x + iy$ to $w = u + iv$ is mapping from complex z plane to complex w plane



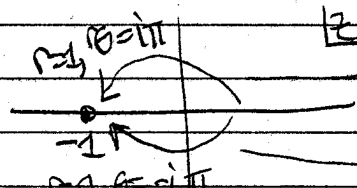
- Hard to draw an analog to $f(x)$ vs. x (need 4-dimensional representation)
- Instead, look how a line or a region maps. (more later)

Strange behavior can happen. Consider $w = z^{1/2}$

$$w = z^{1/2} = (re^{i\theta})^{1/2} = r^{1/2} e^{i\theta/2}$$

$r^{1/2}$ ← well defined positive square root

$\sqrt{-1}$?



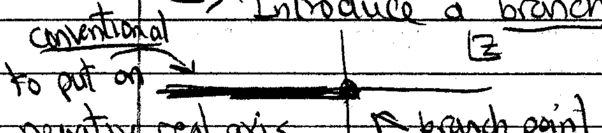
$\rightarrow 1^{1/2} e^{i\pi/2} = +i$

$\rightarrow 1^{1/2} e^{-i\pi/2} = -i$

} different answers depending on how z is specified!

- Looks like same point in $\lfloor z$ plane maps to two different values.

$\Rightarrow$ Introduce a branch cut to define $z^{1/2}$ as single valued.



More later! for now we've simply cut the plane so bottom and top not connected. (Can't reach same point, so single-valued)

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- Other functions with branch points: $(z-z_0)^{1/n}$, $\ln(z-z_0)$ (z_0 is branch point)

- Two example problems (cf. PS#1)

a) Solve $f(z) = z^m = a$ with m integer and a complex
$\Rightarrow$ find all distinct z

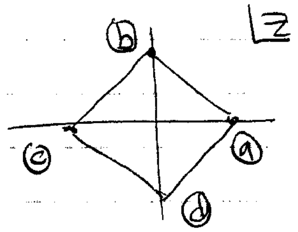
Simpler first $\Rightarrow$ "a" real

e.g. $z^4 = 2$, Key step $\Rightarrow$ write $2 = 2e^{2\pi i n}$ where n is any integer

z^4:	$2e^{i0}$	$2e^{2\pi i}$	$2e^{4\pi i}$	$2e^{6\pi i}$	$2e^{8\pi i}$	$2e^{10\pi i}$
z:	$2^{1/4}$	$2^{1/4}e^{2\pi i/4}$	$2^{1/4}e^{i\pi}$	$2^{1/4}e^{i3\pi/2}$	$2^{1/4}e^{i2\pi}$	$2^{1/4}e^{i5\pi/2}$
	$\times 1$	$\times e^{i\pi/2} = i$	$\times -1$	$\times -i$	$\times 1$	$\times i$
	(a)	(b)	(c)	(d)	(a)	(b)

repeats

$\Rightarrow$ four distinct solutions



constant angle increase until repeats $\Rightarrow$ vertices of a square.

different branch cut $\Rightarrow$ $-\pi < \theta \le \pi$ or $0 \le \theta < 2\pi$

Generalize: $a = re^{i\theta}e^{i2\pi n} \Rightarrow \underline{z = a^{1/m}} = r^{1/m}e^{i\theta/m}e^{2\pi i n/m}$ $\leftarrow$ m distinct solutions

b) Find solutions in $\sin z = 6$. First question: Which is most useful: z, $x+iy$, or $re^{i\theta}$?

Here: use $\sin(x+iy) = (\sin x)(\cos iy) + (\cos x)(\sin iy) = 6$ $\leftarrow$ prove using exponentials $e^{i(z_1+z_2)} = e^{iz_1}e^{iz_2}$ with $e^{iz} = \cos z + i\sin z$

$= \underbrace{(\sin x)(\cosh y)}_{u} + i\underbrace{(\cos x)(\sinh y)}_{v} = u + iv$

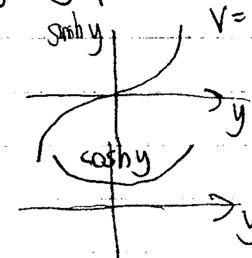
$\Rightarrow$ Equate real and imaginary parts: $u = (\sin x)(\cosh y) = 6$

$v = (\cos x)(\sinh y) = 0$ $\leftarrow$ start here, because it is zero

use plots to guide and check your solution

$\sinh y = \dfrac{e^y - e^{-y}}{2} \Rightarrow$

$\cosh y = \dfrac{e^y + e^{-y}}{2} \Rightarrow$



[How do I know these plots? Mathematica or consider $|y|$ small and $|y|$ large, then join. Must do this!]

$\Rightarrow$ Either $\sinh y = 0$ or $\cos x = 0$. If $\sinh y = 0$, then $y = 0$, $\cosh y = 1$ and $\sin x = 6 \Rightarrow$ none!

So $\cos x = 0$, therefore $x = \pm\pi/2, \pm 3\pi/2, \ldots$ But $\sin x$ is then ± 1, and only $+1$ has a solution because $\cosh y > 0$ $\Rightarrow$ $x = (\frac{\pi}{2} + 2\pi n)$, $n = 0, \pm 1, \pm 2, \ldots$ and $y = \cosh^{-1} 6 = \pm 2.478\ldots$ (two y solutions from plot). [Try ArcCosh[6] in Mathematica. Need 6. or ArcCosh[6]//N to get the decimal result.]

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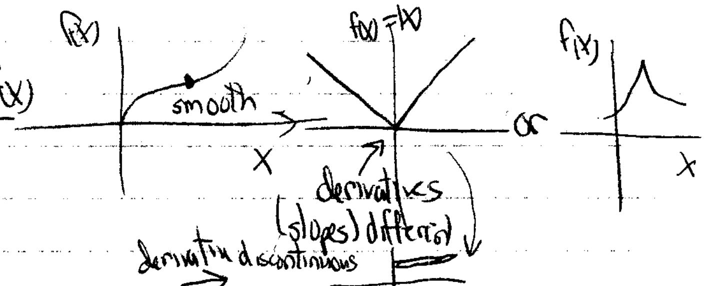
Let's put aside the tricky functions and think of <u>nice</u> ones : continuous and smooth

<u>Smooth</u> for ordinary functions means derivatives exist from both directions :

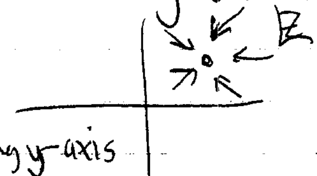
$$\left.\frac{df}{dx}\right|_{\text{at } x} = \lim_{dx\to 0} \frac{f(x+dx)-f(x)}{dx}$$



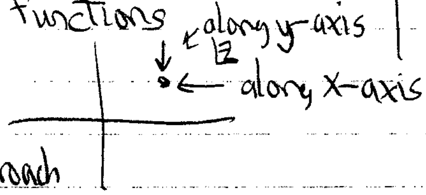
For complex functions, derivative defined similarly

$$\frac{df}{dz} = \lim_{dz\to 0} \frac{f(z+dz)-f(z)}{dz}$$

but $z+dz$ can approach z from many directions



- requiring the same answer is a powerful constraint $\Rightarrow$ analytic functions



- Sufficient to consider just y-axis and x-axis approach

Do it:

$$\frac{df}{dz} \underset{dz=(dx,0)}{\overset{x\text{-axis}}{=}} \lim_{dx\to 0} \frac{u(x+dx,y)+iv(x+dx,y)-[u(x,y)+iv(x,y)]}{dx} = \frac{\partial u}{\partial x} + \frac{i\partial v}{\partial x}$$

$$\underset{dz=(0,dy)}{\overset{y\text{-axis}}{=}} \lim_{dy\to 0} \frac{u(x,y+dy)+iv(x,y+dy)-[u(x,y)+iv(x,y)]}{i\,dy} = \frac{1}{i}\left(\frac{\partial u}{\partial y} + i\frac{\partial v}{\partial y}\right)$$

equate real : $\boxed{\dfrac{\partial u}{\partial x} = \dfrac{\partial v}{\partial y}}$ equate imag : $\boxed{\dfrac{\partial v}{\partial x} = -\dfrac{\partial u}{\partial y}}$ note!

Cauchy-Riemann equations $\Rightarrow$ powerful constraint!

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If differentiable at z_0 and nearby, then function is analytic at $z = z_0$.

- If true for all z, then function is entire

A basic problem:

Check $f(z) = z^3$ for analyticity. Are C-R equations satisfied?

$$f(z) = z^3 = (x+iy)^3 = x^3 + 3x^2 iy + 3x(iy)^2 + (iy)^3$$

$$= \underbrace{x^3 - 3xy^2}_{u} + i(\underbrace{3x^2y - y^3}_{v})$$

$\boxed{\text{Find } u \text{ and } v \text{ first}}$

$$\frac{\partial u}{\partial x} = 3x^2 - 3y^2 \qquad \frac{\partial v}{\partial y} = 3x^2 - 3y^2 \qquad \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \ \checkmark$$

$$\frac{\partial u}{\partial y} = -6xy \qquad \frac{\partial v}{\partial x} = 6xy \qquad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \ \checkmark$$

$\Rightarrow$ C-R satisfied and partial derivatives continuous everywhere $\Rightarrow$ analytic. All polynomials work.

- consider e^z and related functions. $\Rightarrow$ analytic

What is not analytic?

Another type of problem

Find analytic function $w(z) = u(x,y) + iv(x,y)$ given that $u(x,y) = x^3 - 3xy^2$ [so find $v(x,y)$]

$$\frac{\partial u}{\partial x} = 3x^2 - 3y^2 = \frac{\partial v}{\partial y} \xrightarrow{\text{integrate}} v = 3x^2y - y^3 + f(x)$$

$\leftarrow$ any function of x is integration constant (check $\frac{\partial v}{\partial y}$ explicitly)

$$-\frac{\partial u}{\partial y} = 6xy \text{ and } \frac{\partial v}{\partial x} = 6xy + \frac{\partial f}{\partial x} \Rightarrow \text{equal if } \frac{\partial f}{\partial x} = 0 \text{ or } f(x) = \text{const.}$$

$$\Rightarrow w(z) = x^3 - 3xy^2 + i(3x^2y - y^3 + \text{const.}) \text{ is analytic!}$$

Very particular — can't just choose u and v!

Is $f(z) = \text{Re } z = x$ analytic?

$u = x$, $v = 0$

$\Rightarrow \frac{\partial u}{\partial x} = 1$, $\frac{\partial v}{\partial y} = 0$ not equal! So no, not analytic.

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Spot the error! Find u and v; NOT $u \neq \frac{1}{x}$, $v \neq \frac{1}{y}$

Try $f(z) = \frac{1}{z}$ for satisfying C-R equations

$$\frac{1}{z} = \frac{1}{x+iy} = \frac{1}{x+iy}\,\frac{x-iy}{x-iy} = \frac{x-iy}{x^2+y^2} \Rightarrow u = \frac{x}{x^2+y^2},\ v = \frac{-y}{x^2+y^2}$$

$$\frac{\partial u}{\partial x} = \frac{1}{x^2+y^2} + x\frac{-1}{(x^2+y^2)^2}2x = \frac{1}{(x^2+y^2)^2}\left(x^2+y^2-2x^2\right) = \frac{y^2-x^2}{(x^2+y^2)^2}$$

$$\frac{\partial v}{\partial y} = -\frac{1}{x^2+y^2} + (-y)\frac{-1}{(x^2+y^2)^2}2y = \frac{1}{(x^2+y^2)^2}\left(2y^2-(x^2+y^2)\right) = \frac{y^2-x^2}{(x^2+y^2)^2}$$

$$\Rightarrow \boxed{\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}}\ \checkmark$$

easier with Mathematica!

Similarly, $\frac{\partial u}{\partial y} = \frac{-x}{(x^2+y^2)^2}2y$ $\quad \frac{\partial v}{\partial x} = \frac{+y}{(x^2+y^2)^2}2x \Rightarrow \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$ ✓

$\Rightarrow$ analytic <u>except</u> when $x=y=0 \Rightarrow$ at $z=0$

$f(z=0) = \infty$, called a "simple pole" (more later)

• If $\frac{1}{z-z_0}$, then pole at z_0.

But now try $f(z) = \frac{1}{z^*} = \frac{1}{x-iy} = \frac{x+iy}{x^2+y^2}$ $\quad u = \frac{x}{x^2+y^2},\ v = \frac{y}{x^2+y^2}$

$$\frac{\partial u}{\partial x} = \frac{y^2-x^2}{(x^2+y^2)^2}, \quad \frac{\partial v}{\partial y} = \frac{x^2-y^2}{(x^2+y^2)^2} \neq \frac{\partial u}{\partial x}$$

so <u>not</u> analytic. Close doesn't count!

minus sign - difference

Can't just assign u and v functions and get analytic w!

Next: Integrals of $f(z)$!