· office hours later Monday afternoon

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7701 Lecture 3

· Reminder: PS#1 due in Furnstahl mailbox in main office before the colloquium on Tuesday, $\Rightarrow$ please attend colloquium when possible!

· Questions on $\delta_{ij} - \varepsilon_{ijk}$ problems?

· Warm-ups:

(a) "Spot the error!" $(\sin z)^* = \dfrac{e^{-iz} - e^{iz}}{-2i} = \dfrac{e^{iz} - e^{-iz}}{2i} = \sin z$?!

· must be wrong because it implies $\sin z$ is real!
· try small z to check: $(z)^* \stackrel{?}{=} z \Rightarrow$ only if z is real
· problem: $z \to z^*$ in $(\sin z)^* \Rightarrow$ answer is $\sin z^*$

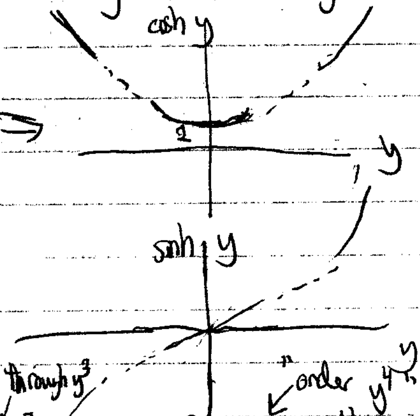
(b) What is $|\sin z|^2$? (ans: $\sin^2 x + \sinh^2 y$)

core competency: (c) What are the zeros of $\sin z$? (ans. $x = 0, \pm n\pi$ and $y = 0$)

(d) Without notes, sketch $\cosh y$ and $\sinh y$ based on $y \to 0$ and $y \to \infty$ limits [also Mathematica]

ans:
$$\cosh y = \frac{e^y + e^{-y}}{2} \xrightarrow[y\to 0]{} \tfrac{1}{2}\left(1 + y + \tfrac{y^2}{2} + 1 - y + \tfrac{y^2}{2}\right) = 1 + \frac{y^2}{2}$$
$$\cosh y \xrightarrow[y \to \pm\infty]{} \tfrac{1}{2} e^{y} \ (+\infty) \text{ or } \tfrac{1}{2} e^{-y} \ (-\infty)$$

$$\sinh y = \frac{e^y - e^{-y}}{2} \xrightarrow[y\to 0]{} \frac{(1+y) - (1-y)}{2} = y$$
$$\sinh y \xrightarrow[y \to \pm\infty]{} \frac{e^y}{2} \ (+\infty) \text{ or } -\tfrac{1}{2} e^{-y} \ (-\infty)$$

$\Rightarrow$ 

Mathematica: small y Series[Cosh[y], {y, 0, 3}] $\to 1 + \frac{y^2}{2} + O[y]^4$ (about 0; through y^3)
large y Series[Cosh[y], {y, Infinity, 2}] $\to$ Cosh[y] doesn't work!

Lecture plan: · Go back to 13, then do 15–17
· Continue with 19+ as time permits

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Analyticity and Taylor expansions ...

- If a function is differentiable (← all orders of derivatives!) at $z = z_0$, then its Taylor expansion exists:

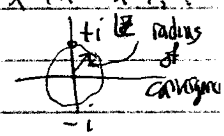
$$f(z) = \sum_{n=0}^{\infty} (z - z_0)^n \frac{f^{(n)}(z_0)}{n!}$$

← note ($n=0$)

← n^{th} derivative with z, evaluated at $z = z_0$

** Existence of Taylor series at z_0 $\Leftrightarrow$ analytic at z_0 (alternative definition)

- Series "works" (that is, it converges) in circular region about z_0 up to first singularity (pole, branch point, essential singularity)
  - $\Rightarrow$ determines "radius of convergence"
    - This is ∞ for e^z, $\sin z$, $\sinh z$, etc.
    - For $\frac{1}{z^2+1}$ about $z=0$, never vanishes for x real: $1 - x^2 + x^4 - x^6 + \ldots$ but only converges for $|z| < 1$ because $\frac{1}{z^2+1} = \frac{1}{z+i}\frac{1}{z-i}$



  - Mathematica notebook complex_series.nb (later)

- Generalization: Laurent expansion or series: $f(z) = \sum_{n=-\infty}^{\infty} a_n (z - z_0)^n$ ← note!!!
  - for example, $f(z) = \frac{1}{e^z - 1} \xrightarrow[z \to 0]{\text{about}} \frac{1}{z} - \frac{1}{2} + \frac{z}{12} + O(z^3)$

$\Rightarrow a_n = 0$ for $n < 0$ except $a_{-1} = 1$, $n = -1$ coefficient is the "residue" (much more to come!)

- Mathematica gives you Laurent expansion using Series[ ]
- We can find the Laurent expansion in simple cases, e.g. analytic functions times explicit poles.

Plan: expand functions in Taylor series (about specified point!) and then combine the two parts term by term.

- In problem set: $\frac{\cos z}{z-1}$ about $z=1$ ← not $\frac{1 - \frac{z^2}{2} + \frac{z^4}{4!} + \ldots}{z-1}$!!

$\frac{\ln z}{z-1}$ about $z=1$ "Spot the Error!"

- Where are the singularities for each? (Class question)

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Example: expand $f(z) = \frac{e^z}{(z-2)^3}$ about $z=2$

Spot the Error! $\frac{1+z+\frac{z^2}{2!}+\dots}{(z-2)^3} = \frac{1}{(z-2)^3} + \frac{z}{(z-2)^3} + \frac{z^2/2!}{(z-2)^3} + \dots$ e^z not expanded about 2!

Plan: write $w = z-2$ and expand about $w=0$ (which we can often just write down)

$$e^z = e^{w+2} = e^2 e^w = e^2\left(1 + w + \frac{w^2}{2!} + \dots\right) \underset{\text{for } w}{\overset{\text{substitute}}{=}} e^2\left(1 + (z-2) + (z-2)^2/2! + \dots\right)$$

$\boxed{m^{th} \text{ coefficient is } \frac{1}{m!}}$

$n=-1$ term

$$\Rightarrow f(z) = e^2\left(\frac{1}{(z-2)^3} + \frac{1}{(z-2)^2} + \frac{1}{2(z-2)} + \frac{1}{6} + \dots\right)$$

$$= \sum_{n=-\infty}^{\infty} a_n (z-2)^n \quad \text{where} \quad a_n = \begin{cases} e^2 \frac{1}{(n+3)!} & n \geq -3 \\ 0 & n < -3 \end{cases}$$

general term

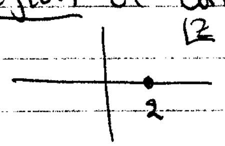
What is the residue? Coefficient of $\frac{1}{z-2} \Rightarrow e^2/2$

- Why is the $n=-1$ term special? (We'll see!)

- Two other ways to find the residue
  - Mathematica Residue[Exp[z]/(z-2)^3), {z, 2}]
  - m^{th} order pole ($m=3$ here)

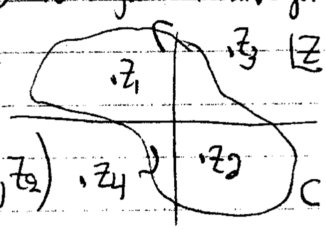
$$\text{Res}(a) = \lim_{z \to a} \frac{1}{(m-1)!} \frac{d^{m-1}}{dz^{m-1}} \left[(z-a)^m f(z)\right] \Rightarrow \frac{1}{2!} \frac{d^2}{dz^2} e^z \Big|_{z=2} = \frac{e^2}{2} \checkmark$$

- Region of convergence?



e^z converges everywhere but we have to exclude the point $z=2$ $\Rightarrow$ $|z-2| > 0$ is region of convergence

- Preview: Residue Theorem



contour in |z plane

$$\oint_C f(z)\,dz = 2\pi i \left(\begin{array}{c}\text{sum of residues at} \\ \text{enclosed poles}\end{array}\right) = 2\pi i \left(a_{-1,z_1} + a_{-1,z_2}\right)$$

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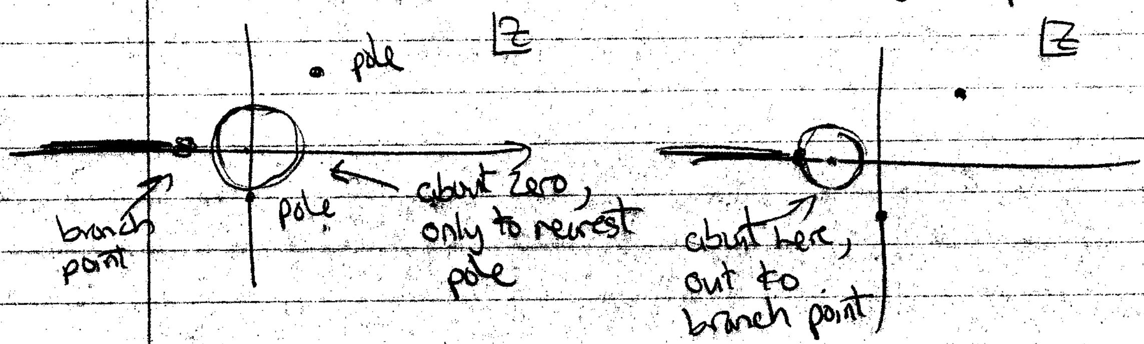
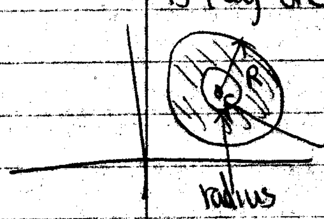
Summary of convergence of Taylor and Laurent series...

* A Taylor series about $z=z_0$ has a convergent expansion in a circle out to the first non-analytic point



- Function is defined by Taylor expansion inside "radius of convergence" (radius of circles in figures)
- converges inside, diverges outside
  - ⇒ Illustrate by Mathematica complex_series.nb notebook for one or two examples. [try just inside and just outside]
- extend definition further by analytic continuation (more in texts and later)

- Try Laurent series in Mathematica as well
- Derivations of Laurent series are in the texts. Key generalization is they are valid in an annular region, rather than filled-in circle.



$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z-z_0)^n \quad \text{unique}, \quad a_n = \frac{1}{2\pi i}\oint_C \frac{f(z)}{(z-a)^{n+1}}\,dz$$

Contour C (in $r<|z-z_0|<R$) ← converge here. Often r is just z_0.

Singularities: Vocabulary

- isolated singular point if not analytic only at $z=z_0$
  - $f(z) = a_{-1}/z-z_0 + a_0 + a_1(z-z_0) + \ldots \Rightarrow$ simple pole with residue a_{-1}
  - if only isolated poles, then "meromorphic"
  - order n pole if $(z-z_0)^n f(z)$ is non singular at z_0 but non-zero.
- essential singularity if all n to $-\infty$ contribute: eg. $e^{1/z} = \sum_{m=-\infty}^{0} z^m \frac{1}{(-m)!}$
- branch point: multivalued $f(z)$ when circling branch point z_0

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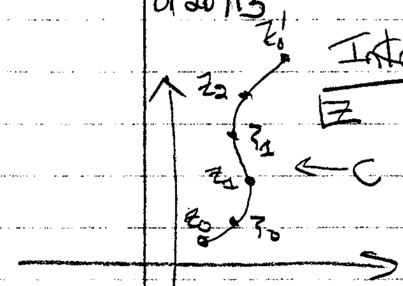
Integrals in the Complex Plane



• In analogy to the "Riemann sum" definition of a integral $\int_a^b f(x)\,dx = \lim_{\Delta x_i \to 0} \sum_i f(x_i)\Delta x_i$, ← could evaluate at any intermediate point.

we define

contour integral of $f(z)$ along C →
$$\int_{z_0}^{z_0'} f(z)\,dz = \lim_{n\to\infty} \sum_{j=1}^{n} f(\xi_j)(z_j - z_{j-1})$$

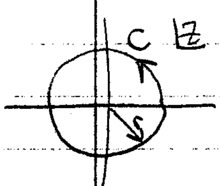
* Where the limit should be independent of how the z_j's are spaced or what intermediate ξ_j's are chosen.

An alternative definition as a line integral uses $f = u + iv$:

$$\int_{z_0}^{z_0'} f(z)\,dz = \int_{(x_0,y_0)}^{(x_0',y_0')} [u(x,y) + iv(x,y)][dx + i\,dy] \qquad dz = dx + i\,dy$$

$$= \int_{(x_0,y_0)}^{(x_0',y_0')} [u(x,y)\,dx - v(x,y)\,dy] + i\int_{(x_0,y_0)}^{(x_0',y_0')} [v(x,y)\,dx + u(x,y)\,dy]$$

Consider $f(z) = z^n$, n integer and contour C a circle of radius r, traversed counterclockwise (increasing θ)



First $n \neq -1$: $\oint_C z^n\,dz \overset{z=re^{i\theta}}{\underset{dz = ire^{i\theta}d\theta}{=}} \int_0^{2\pi} r^n e^{in\theta}(ire^{i\theta}\,d\theta) = ir^{n+1}\int_0^{2\pi} e^{i(n+1)\theta}\,d\theta$

$\oint$ means a closed path

$$= \frac{r^{n+1}}{(n+1)} e^{i(n+1)\theta}\Big|_0^{2\pi} = 1 - 1 = \boxed{0}$$

for $n = -1$: $\oint_C \frac{1}{z}\,dz = \int_0^{2\pi} \left(\frac{1}{r}e^{-i\theta}\right)(ire^{i\theta})\,d\theta = i\int_0^{2\pi} d\theta = \boxed{2\pi i}$

• Note that $n = -2$, $n = -3$, etc. are zero; just $n = -1$ is different. *

• also, these answers hold for any r.

• If $f(z) = (z - z_0)^n$ and radius r about z_0, then write $z = z_0 + re^{i\theta}$ ⇒ same result
← constant

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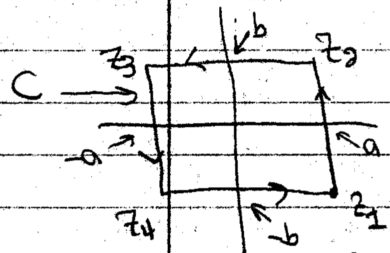
Now consider a different path: around a rectangle.



$$\oint_C z^n\,dz = \int_{-b}^{b} (a+iy)^n\,dy + \int_{0}^{-a} (x+ib)^n\,dx + \int_{b}^{-b} (-a+iy)^n\,dy + \int_{-a}^{a} (x-ib)^n\,dx$$

If $n \neq 1$: $\oint_C z^n\,dz = \left.\frac{(a+iy)^{n+1}}{n+1}\right|_{-b}^{b} + \left.\frac{(x+ib)^{n+1}}{n+1}\right|_{0}^{-a} + \left.\frac{(-a+iy)^{n+1}}{n+1}\right|_{b}^{-b} + \left.\frac{(x-ib)^{n+1}}{n+1}\right|_{-a}^{a}$

$$= \left[\overset{1}{\cancel{\frac{(a+ib)^{n+1}}{n+1}}} - \overset{4}{\cancel{\frac{(a-ib)^{n+1}}{n+1}}}\right] + \left[\overset{2}{\cancel{\frac{(-a+ib)^{n+1}}{n+1}}} - \overset{1}{\cancel{\frac{(a+ib)^{n+1}}{n+1}}}\right] + \left[\overset{3}{\cancel{\frac{(-a-ib)^{n+1}}{n+1}}} - \overset{2}{\cancel{\frac{(-a+ib)^{n+1}}{n+1}}}\right]$$

$$+ \left[\overset{4}{\cancel{\frac{(a-ib)^{n+1}}{n+1}}} - \overset{3}{\cancel{\frac{(-a-ib)^{n+1}}{n+1}}}\right]_{-a}^{a} = \boxed{0} \text{ again!}$$

Quicker: $\oint z^n\,dz = \left.\frac{z^{n+1}}{n+1}\right|_{z_1}^{z_2} + \left.\frac{z^{n+1}}{n+1}\right|_{z_2}^{z_3} + \left.\frac{z^{n+1}}{n+1}\right|_{z_3}^{z_4} + \left.\frac{z^{n+1}}{n+1}\right|_{z_4}^{z_1} = 0$ (cancel in pairs again)

What if $n = -1$?

$$\oint \frac{1}{z}\,dz \overset{?}{=} \ln(a+iy)\Big|_{-b}^{b} + \ln(x+ib)\Big|_{a}^{-a} + \ln(-a+iy)\Big|_{b}^{-b} + \ln(x-ib)\Big|_{-a}^{a} \overset{?}{=}$$

Looks like canceling in pairs to get zero again, but we need to be more careful!

$\Rightarrow \ln z = \ln r + i\theta \Rightarrow$ the $\ln r$ parts will cancel

but θ increases by 2π in going from z_1 back to z_1

$\Rightarrow$ integral gives $i \cdot 2\pi = \boxed{2\pi i}$ from imaginary parts of $\ln z$

Bold extrapolation:

- We always get $\oint_C z^n\,dz = \begin{cases} 0 & \text{if } n \neq -1 \\ 2\pi i & \text{if } n = -1 \end{cases}$ for any C around origin (if counterclockwise)

- Turns out to be true for $n \neq -1$ generally and for $n = -1$ if the point $z = 0$ is inside C.

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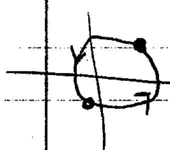
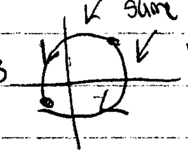
More generally we have Cauchy's Theorem:

$$\boxed{\oint_C f(z)\,dz = 0}$$ if $f(z)$ is analytic in a "simply connected" region R that includes C.

 $= 0$ means  same result for different paths $\Rightarrow$ like conservative force

Analytic in R means the Cauchy-Riemann relations hold inside of C. These are derivatives in the interior of a region so they should tell us about what happens on an integral around the boundary $\Rightarrow$ Stokes's theorem!

- Integral of derivatives in interior is equal to function on boundary (we'll come back to this later in the semester)

Let's see that this is just Stokes's theorem in xy plane

Suppose $\vec{A} = \hat{x}A_x + \hat{y}A_y + [\hat{z}0]$ [choose $\vec{A}$ and S to match!]

counterclockwise in x-y plane

$$\oint_C \vec{A}\cdot d\vec{r} = \int_S (\vec{\nabla}\times\vec{A})\cdot\underbrace{\hat{n}}_{\hat{z}}\,dS = \iint_S \left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right) dx\,dy$$

Recall $$\oint f(z)\,dz = \oint_C (u\,dx - v\,dy) + i\oint_C (v\,dx + u\,dy)$$

Looks like $\vec{A}\cdot d\vec{r} = A_x dx + A_y dy$ if we identify A_x, A_y with u, v

$d\vec{r} = \hat{x}\,dx + \hat{y}\,dy$

$$\oint v\,dx + u\,dy \underset{A_y = u}{\overset{A_x = v}{=}} \iint \cancel{\left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right)}^{0 \text{ by C-R}} dx\,dy = 0\ !$$

$$\int u\,dx - v\,dy \underset{A_y = -v}{\overset{A_x = u}{=}} \iint \cancel{\left(-\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)}^{0 \text{ by C-R}} dx\,dy = 0\ !$$

- More complete proof given in Arfken. See also Cahill.

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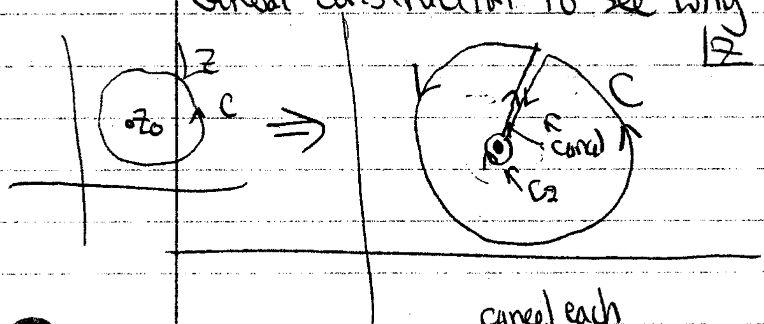
What about have $\frac{1}{z}$ or, more generally, $\frac{1}{z-z_0}$?

Cauchy's Integral Formula:

$$\boxed{\frac{1}{2\pi i}\oint \frac{g(z)}{z-z_0}\,dz = g(z_0)}$$

where z_0 is any point in region bounded by C while $g(z)$ is analytic on and in C.

General construction to see why this works: (shows we can deform contours)



By adding extra pieces to our desired contour C (the straight line in and back to z_0 and integral C_2 in a circle around z_0) we can now apply Cauchy.

cancel each other

clockwise

$$\underbrace{\oint_C \frac{g(z)}{z-z_0}\,dz}_{\text{desired integral}} + \underbrace{\int \frac{g(z)}{z-z_0}}_{\text{in and out straight lines}} - \oint_{C_2} \frac{g(z)}{z-z_0}\,dz = 0$$

← counterclockwise circle

$$\text{But } \oint_{C_2} \frac{g(z)}{z-z_0}\,dz \overset{z=z_0+re^{i\theta}}{=} \int_0^{2\pi} \frac{g(z_0+re^{i\theta})}{\cancel{re^{i\theta}}}\,\cancel{r}i\cancel{e^{i\theta}}\,d\theta$$

by changing variables to $z = z_0 + re^{i\theta}$

$$= i\int_0^{2\pi} g(z_0+re^{i\theta})\,d\theta \xrightarrow[r\to 0]{g\text{ is smooth!}} ig(z_0)\int_0^{2\pi} d\theta = 2\pi i g(z_0)$$

$$\Rightarrow \oint_C \frac{g(z)}{z-z_0}\,dz = 2\pi i\, g(z_0) \qquad \text{QED}$$

- Once you know values on the boundary of an analytic region, you know them in the interior!

Generalize: $f'(z_0) = \frac{1}{2\pi i}\oint \frac{f(z)}{(z-z_0)^2}\,dz$, $f^{(n)}(z_0) = \frac{n!}{2\pi i}\oint \frac{f(z)}{(z-z_0)^{n+1}}\,dz$

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Apply Cauchy's formula to simple case:

$$\oint_C (z-z_0)^n \, dz = \begin{cases} 0 & n \neq -1 \text{ because analytic} \\ 2\pi i & n = -1 \quad \text{take } g(z) = 1 \end{cases}$$

$\Rightarrow$ If we have a function with a Taylor series, then term by term the $(z-z_0)^n$ for $n \geq 0$ terms vanish and $\oint f(z)\,dz = 0$ $\Rightarrow$ Cauchy's Theorem!

existence of $f(z) = \sum_{n=0}^{\infty} (z-z_0)^n \frac{f^{(n)}(z_0)}{n!} \Longleftrightarrow$ analytic function

$\Rightarrow$ works up to first non-analytic point in a region

$\Rightarrow$ determines "radius of convergence"

- This is ∞ for e^z, $\sin z$
- for $\frac{1}{z^2+1}$ about $z=0$, $|z|<1$ because $\frac{1}{z^2+1} = \frac{1}{z+i}\frac{1}{z-i}$

  ← even though never vanishes for $z=x$ real, doesn't converge for $|x|\geq 1$.

$\Rightarrow$ check with Mathematica (see notebook complex_series.nb)

- If we have poles, as before we generalize with a Laurent series or expansion

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z-z_0)^n \quad \text{where } a_n = \frac{1}{2\pi i}\oint \frac{f(z')\,dz'}{(z'-z_0)^{n+1}}$$

↖ note

in annular region

- e.g. $f(z) = (e^z-1)^{-1} = \frac{1}{z} - \frac{1}{2} + \frac{z}{12} + O(z^3)$ about $z=0$ (more later on finding a_n)

So now term-by-term will give zero except the $a_{-1}\oint (z-z_0)^{-1} dz = (a_{-1}) \times 2\pi i$ term

$\star \Rightarrow$ $\boxed{a_{-1} \text{ is the residue.}}$

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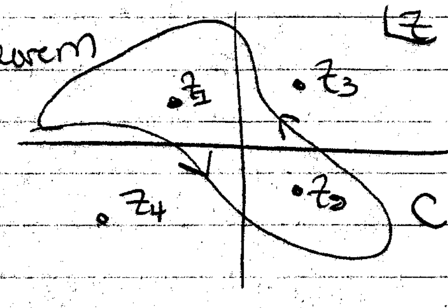
This is all we need for the residue theorem



$$\oint_C f(z)\,dz = 2\pi i\left(a_{-1,z_1} + a_{-1,z_2}\right)$$

$= 2\pi i$ (sum of "residues" of enclosed poles)

Here $f(z) = \sum_{n=-\infty}^{\infty} a_{n,z_i}(z-z_i)^n$ about $z_i \Rightarrow a_{-1,z_i}$ is $\frac{1}{z-z_i}$ coefficient.

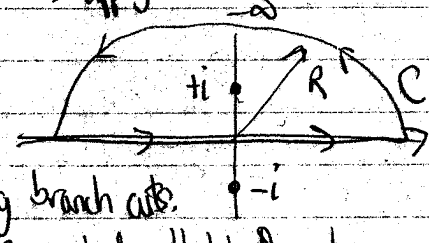
Apply this to calculate many definite integrals that arise in mathematical physics.

Aside: List of methods for definite integrals from Arfken:

1. contour integration
2. convert to gamma or beta function
3. numerical quadrature
4. integral transforms
5. series expansion and term-by-term integration

· Mathematica might use any of these methods (at your request or automagically)

Lea's steps for evaluating an integral → apply to $\int_{-\infty}^{\infty} \frac{1}{x^2+1}dx$ as simple example



$\boxed{\oint_C \frac{1}{z^2+1}dz}$

1. Draw complex z plane with contour C chosen to include integral of interest. Mark poles or other singularities, including branch cuts.
2. If there is a branch cut, "deform" the contour so it doesn't hit the cut.
3. note poles inside C (here $z = +i$)
4. evaluate the residue of f at each enclosed pole: $\frac{1}{z^2+1} = \frac{1}{z+i}\left(\frac{1}{z-i}\right) = \frac{1}{(z-i)+2i}\frac{1}{z-i}$
   $= \frac{1}{2i}\left(\frac{1}{1+\frac{z-i}{2i}}\right)\frac{1}{z-i}$ since $\frac{z-i}{2i}$ sm
   $= \frac{1}{2i}\frac{1}{z-i}\left(1-\frac{z-i}{2i}+\left(\frac{z-i}{2i}\right)^2+\dots\right)$ so res $= \frac{1}{2i}$
5. apply the residue theorem $\oint f\,dz = 2\pi i \frac{1}{2i} = \pi$
6. evaluate other non-vanishing parts (eg, integrals on both side of branch cut)

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Ways to find a residue

0. Use Mathematica Residue[f[z], {z, z_0}] [careful of branches!]
1. Find the Laurent series and pick out a_{-1} coefficient
   - often with simple Taylor expansion of non-pole piece
2. Simple pole at a:

$$\text{Res}\, f(a) = \lim_{z \to a} (z-a) f(z)$$

"cancel the pole then set $z=a$ in the rest"

- most common rule

$$f(z) = \frac{1}{z^2+1}, \quad a = i \Rightarrow \text{Res}\, f(i) = \lim_{z \to i} (z-i) \frac{1}{z^2+1}$$

$$= \lim_{z \to i} \frac{1}{z+i} = \frac{1}{2i}$$

If $f(z) = \frac{\sin z}{z^2+1}$, $\Rightarrow \text{Res}\, f(i) = \frac{\sin i}{2i}$

3. Pole of order m $\quad \text{Res}\,(a) = \lim_{z \to a} \frac{1}{(m-1)!} \frac{d^{m-1}}{dz^{m-1}} \left[(z-a)^m f(z) \right]$

E.g., $f(z) = \frac{e^z}{(z-1)^2} \Rightarrow \text{Res}\,(z=1) = \lim_{z \to 1} \frac{1}{(2-1)!} \frac{d}{dz} \left[\cancel{(z-1)^2} \frac{e^z}{\cancel{(z-1)^2}} \right] = e^z \Big|_{z=1}$

$= e$

cf. $f(z) = \frac{1}{(z-1)^2} \Rightarrow \text{Res}\,(z=1) = 0!$

check Mathematica Residue[Exp[z]/(z-1)^2, {z, 1}] = e ✓

4. If $f(z) = \frac{g(z)}{h(z)}$ and $h(z)$ has simple zero at a, $g(z)$ analytic at a

$$\Rightarrow \text{Res}\, f(a) = \lim_{z \to a} \frac{g(z)}{h'(z)} \qquad f(z) = \tan z = \frac{\sin z}{\cos z}$$

(e.g., if non-factorizable denominator) $\qquad \text{Re}\, f\left(\frac{\pi}{2}\right) = \lim_{z \to \frac{\pi}{2}} \frac{\sin z}{-\sin z} = -1$

5. evaluate $\frac{1}{2\pi i} \oint_c f(z)\, dz$

- More examples from integrals later.