8/28/13

7701 Lecture 4

- Problem set #1 solutions are posted
- PS#2 is posted ⇒ variety of residue, contour integral problems
  - try the first two after today, if possible

- Lecture plan: go back to 21 – 28 or use 21, 22, 30 – 32 + 28
  - probably that's it for Wednesday

- As time permits Wednesday or on Friday, continue with any needed recap and then the catalog of contour integrals.

- Note: I've gotten some great follow-up questions after class or by email. Keep them coming!
  ⇒ you can always email questions and I'll try to respond quickly.

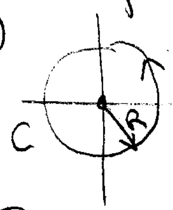
8/28/13

Review as needed 30 – 32 and then proceed.

- Recap from Wednesday: $\int_C f(z)\,dz = \int_C [u(x,y)+iv(x,y)][dx+i\,dy] = \int_C (u\,dx - v\,dy) + i\int_C (v\,dx + u\,dy)$ ← real functions

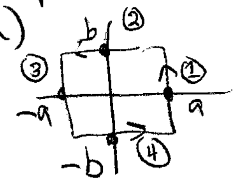
Integral in z plane of $f(z) = z^n$

i)



or

ii)



$$\oint_C z^n\,dz = \begin{cases} 0 & n \neq -1 \\ 2\pi i & n = -1 \end{cases} \quad (1/z \text{ is special!})$$

These cases are easy to do explicitly

i) $$\oint_C z^n\,dz \overset{z=Re^{i\theta}}{=} \int_0^{2\pi} R^n e^{in\theta}(iRe^{i\theta}d\theta) = \begin{cases} \frac{iR^{n+1}}{i(n+1)} e^{i(n+1)\theta}\Big|_0^{2\pi} \propto \left(e^{i(n+1)2\pi} - e^0\right) = 0 & n\neq -1 \\ i\int_0^{2\pi} d\theta = 2\pi i & n = -1 \end{cases}$$

easier →

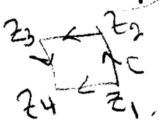
ii) $$\oint_C z^n\,dz \overset{z=x+iy}{=} \overset{①}{\int_{-b}^{b}} (a+iy)^n\,dy + \overset{②}{\int_a^{-a}} (x+ib)^n\,dx + \overset{③}{\int_b^{-b}} (-a+iy)^n\,dy + \overset{④}{\int_{-a}^{a}} (x-ib)^n\,dx$$

$$n\neq 1 \quad = \frac{(a+iy)^{n+1}}{n+1}\Big|_{-b}^{b} + \frac{(x+ib)^{n+1}}{n+1}\Big|_a^{-a} + \frac{(-a+iy)^{n+1}}{n+1}\Big|_b^{-b} + \frac{(x-ib)^{n+1}}{n+1}\Big|_{-a}^{a} = 0$$

cancel in pairs

$$n=-1 \quad \oint_C \frac{1}{z}\,dz \overset{?}{=} \ln(a+iy)\Big|_{-b}^{b} + \ln(x+ib)\Big|_a^{-a} + \ln(-a+iy)\Big|_b^{-b} + \ln(x-ib)\Big|_{-a}^{a}$$

Looks like it cancels in pairs but $\ln z = \ln r + i\theta \Rightarrow \ln r$'s cancel but θ at beginning and end differ by $2\pi \Rightarrow 2\pi i$

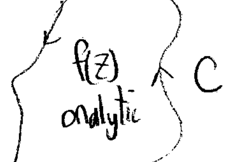


$$\oint_C z^n dz = \frac{z^{n+1}}{n+1}\Big|_{z_1}^{z_2} + \frac{z^{n+1}}{n+1}\Big|_{z_2}^{z_3} + \frac{z^{n+1}}{n+1}\Big|_{z_3}^{z_4} + \frac{z^{n+1}}{n+1}\Big|_{z_4}^{z_1}$$

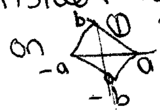
Extrapolation to general C: $$\oint_C z^n\,dz = \begin{cases} 0 & \text{if } n\neq 1 \text{ \underline{independent} of "path" } C \\ 2\pi i & \text{if } n=-1 \text{ and } C \text{ encloses origin (counterclockwise)} \end{cases}$$

- z^n is analytic (Taylor expansion) for $n\neq -1$ while $\frac{1}{z}$ is non-analytic at $z=0$ (a pole). Note that $\oint_C \frac{1}{z^2}dz = 0$!

Cauchy's Theorem: $\oint_C f(z)\,dz = 0$ if analytic inside



Recall that analyticity is a powerful constraint! (C-R equations)

[Aside: How would you explicitly evaluate $\oint z^n dz$ on  ? eg. on ①, use $y = -\frac{b}{a}x + b \Rightarrow dz = dx + i\,dy$
$\Rightarrow dy = -\frac{b}{a}dx \qquad = (1 - i\frac{b}{a})\,dx$

$$\Rightarrow \int_{①} z^n dz = \int_a^0 \left(x + i\left(-\frac{b}{a}x + b\right)\right)^n \left(1 - i\frac{b}{a}\right) dx \text{ or just } \left.\frac{z^{n+1}}{n+1}\right|_a^{bi} !$$ (31)

8/26/13

Recall: there is a theorem that relates surface integrals to line integrals over the region they border: Stoke's theorem. Apply in x-y plane.

In x-y plane $\oint_C \vec{A}\cdot d\vec{r} = \int_S (\vec{\nabla}\times\vec{A})\cdot \underset{\hat{z}}{\hat{n}}\, dS = \iint_S \left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right) dx\,dy$

But $d\vec{r} = \hat{x}\,dx + \hat{y}\,dy$ so $\vec{A}\cdot d\vec{r} = A_x dx + A_y dy$

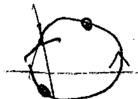
Choose to match $\oint f(z)\,dz = \oint_C (u\,dx - v\,dy) + i\oint_C (v\,dx + u\,dy)$

$A_x = u \quad A_y = -v \qquad\qquad A_x = v \quad A_y = u$

analyticity! $\Rightarrow \iint \left(-\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) dx\,dy = 0!, \qquad \iint \left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right) dx\,dy = 0!$

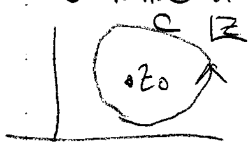
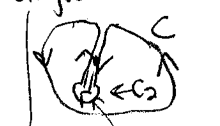
0 by Cauchy-Riemann $\qquad$ 0 by Cauchy Riemann

(See Arfken for more complete proof) [reverse direction ⇒ minus sign]

Note that  $\oint f(z)dz = 0$ means  ← Same result for any different paths..

What if $f(z) = \frac{g(z)}{z - z_0}$ with $g(z)$ analytic?. Eg. $f(z) = \frac{\sin z}{z+1}$ $(z_0 = -1)$

We make a clever adjustment of the contour: add extra pieces!

 $\Rightarrow$  $\Rightarrow \oint_C f(z)dz = 0$ applies to new contour because only non-analytic point, z_0, is now outside!

clockwise

$$\Rightarrow \oint_C \frac{g(z)}{z-z_0}dz + \int_0^{z_0} \frac{g(z)}{z-z_0}dz - \oint_{C_2} \frac{g(z)}{z-z_0}dz = 0$$

desired integral (gap goes away) $\qquad$ in and out straight lines have opposite signs cancel! $\qquad$ C_2 ← small circle (now counterclockwise since we took the −1 out)

• On C_2; Taylor expand $g(z) = g(z_0) + (z-z_0)g'(z_0) + \frac{1}{2}(z-z_0)^2 g^{(2)}(z_0) + \ldots$

change $z' = z - z_0$

$$\oint_{C_2} \frac{g(z)}{z-z_0}dz = g(z_0)\oint_{C_2}\frac{1}{z-z_0}dz = g(z_0)\oint_{C_2}\frac{1}{z'}dz' = g(z_0)\,2\pi i$$

with all other $\oint (z-z_0)^n dz = 0$ from our explicit calculation!

$$\Rightarrow \boxed{\oint_C \frac{g(z)}{z-z_0}dz = 2\pi i\, g(z_0)}$$

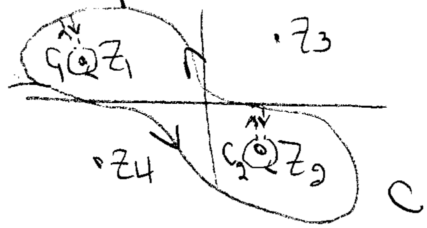
Challenge: What is $\oint_C \frac{g(z)}{(z-z_0)^2}dz \overset{?}{=} g'(z_0)\cdot 2\pi i$

what is $\oint_C \frac{g(z)}{(z-z_0)^n}dz \overset{?}{=} \frac{1}{(n-1)!}g^{(n-1)}(z_0)\cdot 2\pi i$

8/28/13 Recap ← can skip

- What if $f(z)$ has more than one point where it looks like $\frac{g(z)}{z-z_i}$? ← (analytic) (That is, has this form near z_i.)

E.g. $\frac{1}{z^2+1} = \frac{1}{(z+i)(z-i)}$



Add canceling straight lines and little circles $z_1, z_2, \ldots$ to all enclosed poles.

Expand $f(z)$ at each z_i in a Laurent expansion:

$$f(z) = \sum_{n=-\infty}^{\infty} a_{n,z_i}(z-z_i)^n$$

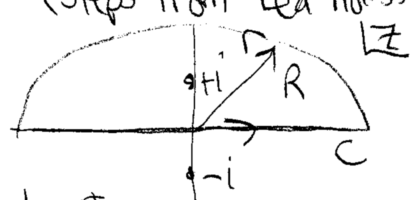
note! → $n=-\infty$

$n=-1$, a_{-1,z_i} is $\frac{1}{z-z_i}$ coefficient ⇒ "residue"

$$\Rightarrow \oint_C f(z)\,dz = 2\pi i\left(a_{-1,z_1} + a_{-1,z_2}\right)$$ ← only n^{-1} parts survive c_1, c_2 integrals

general ⇒ $= 2\pi i$ (sum of "residues" of enclosed poles)

Example application to $\int_{-\infty}^{\infty} \frac{1}{x^2+1}\,dx$ (steps from Lea notes)



$\oint_C \frac{1}{z^2+1}\,dz$

1. Draw complex z plane with contour C chosen to include integral of interest. Mark poles or other singularities, including branch points.
2. If there is a branch cut, "deform" the contour so it doesn't hit the cut
3. Note poles inside C (to left) ⇒ here $z_0 = +i$ only
4. Evaluate the residue of $f(z)$ at each enclosed pole $\left[\text{Res } f(+i) = \frac{1}{2i}\right]$
5. Apply the residue theorem: $\oint f(z)\,dz = 2\pi i \frac{1}{2i} = \pi$
6. Evaluate other non-vanishing parts of integral (here, need to show semi circle part vanishes as $R \to \infty$)

- So to proceed we need to know how to evaluate residues, examples of how to choose contours, and how to show if pieces vanish.

8/26/13

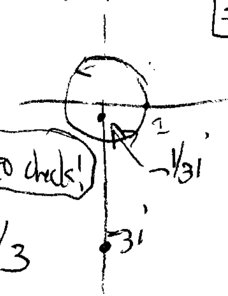
Catalog of contour integrals

① $\int_0^{2\pi} f(\sin\theta, \cos\theta)\, d\theta \Rightarrow$ strategy: take C to be unit circle in z-plane

$$z = re^{i\theta} \overset{\text{on } C}{=} e^{i\theta} \Rightarrow dz = ie^{i\theta}d\theta = iz\,d\theta \Rightarrow d\theta = \frac{dz}{iz}$$

$$\sin\theta = \frac{e^{i\theta}-e^{-i\theta}}{2i} = \frac{z-1/z}{2i}, \quad \cos\theta = \frac{e^{i\theta}+e^{-i\theta}}{2} = (z+\tfrac{1}{z})/2.$$

e.g. $$\int_0^{2\pi} \frac{d\theta}{5+3\sin\theta} = \oint_{\substack{\text{unit circle}\\ C}} \frac{dz/iz}{5+3\left(\frac{z-z^{-1}}{2i}\right)} \overset{*\frac{2iz}{2iz}}{=} \oint \frac{2\,dz}{3z^2+10iz-3}$$



(plug in to check!)

quadratic formula yields simple poles $z = \frac{-10i \pm\sqrt{-100+36}}{6} = \frac{-10i \pm 8i}{6} = -3i, -i/3$

. only $-\frac{i}{3}$ inside. $\mathrm{Res}(-\frac{i}{3}) = \lim_{z\to -i/3} (z+\frac{i}{3}) \frac{2/3}{(z+i/3)(z+3i)} = \frac{2/3}{+i(\frac{1}{3}+3)} = \frac{1}{4i}$

$$\Rightarrow \oint_C \frac{2\,dz}{3z^2+10iz-3} = 2\pi i \left(\frac{1}{4i}\right) = \frac{\pi}{2}. \quad \text{Mathematica? } \checkmark$$

Integrate[1/(5+3Sin[t]), {t,0,2Pi}]

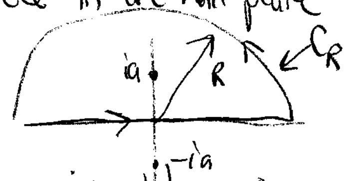
② $\int_{-\infty}^{\infty} f(x)\,dx$ where $f(z) \to 0$ like $\frac{1}{z^2}$ or faster

What if $\int_0^{\infty} \frac{1}{(x^2+a^2)^2}dx$? $\left(= \frac{1}{2}\int_{-\infty}^{\infty} \frac{1}{(x^2+a^2)^2}dx\right)$

strategy: Take C to be real axis and semi-circle in one half plane

$$I = \int_{-\infty}^{\infty} \frac{1}{(x^2+a^2)^2}dx \Rightarrow \oint_C \frac{1}{(z^2+a^2)^2}dz = I + \lim_{R\to\infty} \int_{C_R} \frac{1}{(z^2+a^2)^2}dz$$

$a > 0$



* always check that this doesn't contribute *

integral on C_R: $z = Re^{i\theta}$, $0 \le \theta < \pi$, as $R\to\infty$, integral $\propto \frac{R}{R^4} \times (\text{bounded integral}) \xrightarrow[R\to\infty]{} 0$

• poles of order 2 at $\pm ia \Rightarrow$ only $+ia$ in contour.

$$\mathrm{Res}\, f(ia) = \lim_{z\to ia} \frac{d}{dz}\left[(z-ia)^2 \frac{1}{(z^2+a^2)^2}\right] = \lim_{z\to ia} \frac{d}{dz}\frac{1}{(z+ia)^2} = \lim_{z\to ia} \frac{-2}{(z+ia)^3} = \frac{-2}{(2ia)^3} = \frac{1}{4ia^3}$$

$\boxed{\text{should be real}}$ $\Rightarrow I = 2\pi i \left(\frac{1}{4ia^3}\right) = \boxed{\frac{\pi}{2a^3}}$ Mathematica? $\checkmark$ (with assumptions) ← show this

• What if closed in lower half plane? Then $\mathrm{Res}\, f(-ia) = \lim_{z\to -ia} \frac{-2}{(z-ia)^3} = \frac{2}{(2ia)^3} = \frac{-1}{4ia^3}$

But $I = -2\pi i \cdot -1/4ia^3 = \pi/2a^3$ same answer.

↖ clockwise contour

8/28/13

③ Fourier transforms: $\int_{-\infty}^{\infty} e^{ikx} f(x)\,dx$ or $\int_{-\infty}^{\infty} \begin{Bmatrix}\cos kx\\ \sin kx\end{Bmatrix} f(x)\,dx$

May need Jordan's Lemma:
If $f(z) \underset{z\to\infty}{\Longrightarrow} 0$ <u>uniformly</u>, then $\underbrace{\lim_{R\to\infty} \int_{C_R}}$ $f(z)e^{ikz}\,dz = 0$ for $k>0$ and C_R upper half of $|z| = R$ circle

$\Rightarrow$ less restrictive than $\frac{1}{z^2}$ behavior.

· for $\cos kx$ and $\sin kx$, split into $e^{\pm ikx}$ and close with semi-circle in appropriate half plane.

· Here an example — we'll see more later

If $k>0$ $\quad I = \int_{-\infty}^{\infty} \frac{e^{ikx}}{x^4+1}\,dx \Rightarrow \oint \frac{e^{ikz}}{z^4+1} = I + \int_{C_R} \frac{e^{ikz}}{z^4+1}$ → 0 because $e^{ikz} = e^{ik(x+iy)} = e^{-ky}e^{ikx}$ (0 as $y\to\infty$, because $y = R\cos\theta$)

Poles at 4th roots of -1. $z^4 = -1 = 1\cdot e^{i\pi} e^{2\pi i n}$
$\Rightarrow e^{i\pi/4}, e^{i3\pi/4}, e^{i5\pi/4}, e^{i7\pi/4}$. First two are in upper half plane.
· details in example 2.20 in Lea
answer $I = \pi \frac{\sqrt{2}}{2} e^{-\frac{\sqrt{2}k}{2}} \left(\cos \frac{k}{\sqrt{2}} + \sin \frac{k}{\sqrt{2}}\right)$

For $\int_{-\infty}^{\infty} \cos kx\, f(x)\,dx = \mathrm{Re} \int_{-\infty}^{\infty} e^{ikx} f(x)\,dx$ <u>usually</u> works.

More generally, use
$$\int_{-\infty}^{\infty} \cos kx\, f(x)\,dx = \frac{1}{2}\int_{-\infty}^{\infty} e^{ikx} f(x)\,dx + \frac{1}{2}\int_{-\infty}^{\infty} e^{-ikx} f(x)\,dx,$$

close in upper half-plane $\qquad$ close in lower half plane

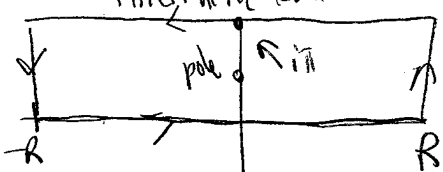
8/26/13

(4) $\int_{-\infty}^{\infty} f(x)\,dx$ with hyperbolic functions (because C_R integrand doesn't vanish),

e.g. $I = \int_0^{\infty} \frac{\cos x}{\cosh x}\,dx \underset{\text{to } -\infty}{\overset{\text{extend}}{=}} \frac{1}{2}\int_{-\infty}^{\infty} \frac{\cos x}{\cosh x}\,dx = \frac{1}{2}\mathrm{Re}\int_{-\infty}^{\infty} \frac{e^{ix}}{\cosh x}\,dx$

Looks like $f(z) = \frac{e^{iz}}{\cosh z}$ should be good. But this doesn't vanish on our usual semicircle because of the denominator.

Alternative contour



Now we can write $\oint_C f(z)\,dz = \sum$ integrals all around. (so 4 integrals to look at)

The key is that the second horizontal contour is closely related to the integral on the real axis:

$\cosh(x+i\pi) = \frac{1}{2}(e^x \underbrace{e^{i\pi}}_{-1} + e^{-x}\underbrace{e^{-i\pi}}_{-1}) = -\frac{1}{2}(e^x + e^{-x}) = -\cosh x$

and

$e^{i(x+i\pi)} = e^{ix}\cdot e^{-\pi}$ so we get $-e^{-\pi}$ times the original integral.

As shown on Lea pg. 142, the side pieces vanish. Let's check the right side:

$z = R+iy, \quad 0 \le y < \pi$

$\Rightarrow \left|\int_{\text{side}} \frac{e^{iz}}{\cosh z}\,dz\right| = \left|\int_0^{\pi} \frac{e^{iR}e^{-y}}{e^R e^{iy} + e^{-R}e^{-iy}}\,i\,dy\right| = \frac{1}{e^R}\left|\int_0^{\pi} \frac{e^{iR}e^{-y}}{e^{iy}+e^{-2R}e^{-iy}}\,dy\right|$

$\le \frac{1}{e^R}\pi \to 0$ as $R\to\infty$

bounded by integral of magnitude of integrand

So we have that (in $R\to\infty$ limit)

$\oint_{\text{rectangle}} \frac{e^{iz}}{\cosh z}\,dz = (1+e^{-\pi})\int_{-\infty}^{\infty} \frac{e^{ix}}{\cosh x}\,dx = 2\pi i(\text{sum of residues enclosed})$

opposite direction

Now where is $\cosh z = 0$? $\cosh(x+iy) = \cosh x\cos y + i\sinh x\sin y \Rightarrow 0$ at $x=0$, $y = i(2n+1)\pi/2$

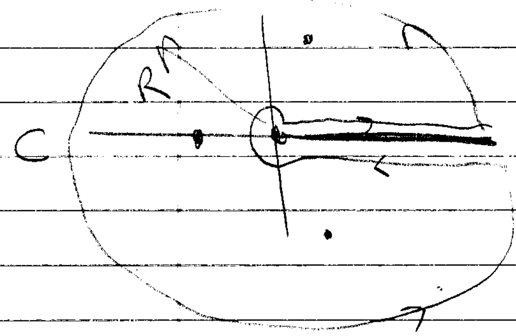
$\Rightarrow$ only $z = i\frac{\pi}{2}$ is inside. Residue is: $\mathrm{Res}\,f(i\tfrac{\pi}{2}) = \lim_{z\to i\frac{\pi}{2}} \frac{e^{iz}}{\sinh z} = \frac{e^{-\frac{\pi}{2}}}{\sinh i\frac{\pi}{2}} = \frac{e^{-\pi/2}}{i\sin\pi/2} = \frac{e^{-\pi/2}}{i}$

$\Rightarrow \int_{-\infty}^{\infty}\frac{e^{ix}}{\cosh x} = \frac{1}{1+e^{-\pi}}\,2\pi i\,\frac{e^{-\pi/2}}{i} = \frac{2\pi}{e^{\pi/2}+e^{-\pi/2}} = \frac{\pi}{\cosh\frac{\pi}{2}}$ and $\int_0^{\infty}\frac{\cos x}{\cosh x}\,dx = \frac{1}{2}\mathrm{Re}\,\frac{\pi}{\cosh\frac{\pi}{2}} = \boxed{\frac{\pi}{2\cosh\frac{\pi}{2}}}$ ✓

8/28/13

(5) $\int_0^\infty x^\alpha f(x)\,dx$ with $\alpha \neq$ integer $\Rightarrow$ branch point at the origin.

Let's choose the branch cut along the real positive axis and the contour shown.



Why does this help? Because now the integral contour is entirely outside of the branch cut.

Let's try $I = \int_0^\infty \frac{\sqrt{x}}{x^3+1}\,dx$ (like Lea 2.23 only x^3+1 instead of x^2+1)

$f(z) = \frac{\sqrt{z}}{z^3+1}$ is analytic inside of C except at the 3 poles (cube roots of -1).

So to carry out this integral, we need to sum $2\pi i \,*$ the residues, check the large and small circles for contributions, and then include both the top and bottom integrals along the x-axis.

- The roots are $z^3 = e^{\pi i}\cdot e^{2\pi i n} \Rightarrow z_1 = e^{\pi/3 i}, e^{\pi i}, e^{\frac{5}{3}\pi i}$ (for $0 \le \theta < 2\pi$)
- The residues are $-\frac{i}{3}, \frac{i}{3}$, and $-\frac{i}{3}$ (how do you find these?)

$$\Rightarrow \oint_C \frac{\sqrt{z}}{z^3+1} = 2\pi i\left(-\frac{i}{3}+\frac{i}{3}-\frac{i}{3}\right) = +\frac{2\pi}{3}$$

Note: Mathematica Residue command gets this wrong because different $\sqrt{x}$ branch!

- On the large circle $\left|\frac{\sqrt{z}}{z^3+1}\right| = \frac{\sqrt{R}}{R^3\left|1+e^{-3i\theta}/R^3\right|} \le \frac{1}{R^{5/2}(1-1/R^3)} \le \frac{8}{7R^{5/2}}$ for $R \ge 2$

so integral is less than $2\pi R \cdot \frac{8}{7R^{5/2}} \xrightarrow[R\to\infty]{} 0$

- On the small circle, set $z = \varepsilon e^{i\theta}$ and take $\varepsilon \to 0$.

$$\oint_{C_\varepsilon} \frac{\sqrt{z}}{z^3+1}dz = i\varepsilon^{3/2}\int_{2\pi}^{0} \frac{e^{i3\theta/2}}{\varepsilon^3 e^{3i\theta}+1}\,d\theta \to 0 \text{ as } \varepsilon \to 0$$

- The upper x-axis integral is what we want.

Below we have $\theta = 2\pi$

$$\int_\infty^0 \frac{\sqrt{r e^{2\pi i}}}{r^3 e^{6\pi i}+1}dr = -\int_0^\infty \frac{\sqrt{r}\,\overset{-1}{e^{i\pi}}}{r^3+1}dr = \int_0^\infty \frac{\sqrt{r}}{r^3+1}dr = I \text{ so we get } 2I \text{ total}$$

not always true! See PS#3

$$\Rightarrow \int_0^\infty \frac{\sqrt{x}}{x^3+1}dx = \frac{1}{2}\,\frac{2\pi}{3} = \frac{\pi}{3}$$