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7701 Lecture 5

Warm-up: How can you intuitively understand: $|z_1+z_2+\dots+z_n| \le |z_1|+\dots+|z_n|$
(Hint: think about z_i's as vectors in complex z plane)

Recap Theorems:

- If $f(z), g(z)$ analytic inside C, then

$$\oint_C f(z)\,dz = 0, \quad \oint_C \frac{g(z)}{z-z_0}\,dz = g(z_0)\cdot 2\pi i, \quad \oint \frac{g(z)}{(z-z_0)^2}\,dz \stackrel{?}{=} g'(z_0)\cdot 2\pi i$$

- If $f(z) = \sum_{n=-\infty}^{\infty} a_{n,z_i}(z-z_i)^n$ near $z=z_i$, then $\oint_{\substack{\text{circle} \to C \\ \text{around } z_i}} f(z)\,dz = \begin{cases} 0 & \text{if } n\neq -1 \\ a_{-1,z_i} & \text{if } n=-1 \end{cases}$
  - also $a_{n,z_i} = \frac{1}{2\pi i}\oint_{\substack{\text{contour} \\ \text{about } z_i}} \frac{f(z)}{(z-z_i)^{n+1}}\,dz$

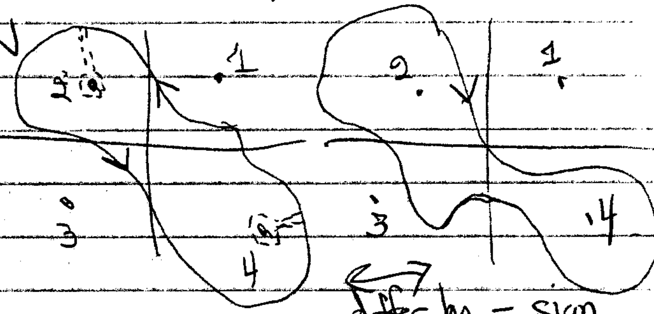
  residue of $f(z)$ at z_i

- $\boxed{\text{Residue Theorem}}$

$$\oint_C f(z)\,dz = (+/-)2\pi i\,(\text{sum of residues of enclosed poles})$$

enclosed CCW/CW

$$= +2\pi i\,(a_{-1,z_2} + a_{-1,z_4})_C$$

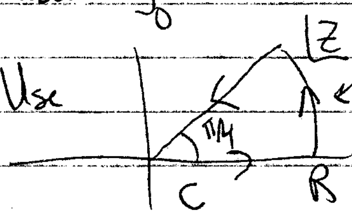


- 6 ways to find a residue → use easiest or more than one to check
  - see (28)
- Applying the residue theorem: (32)–(36) do (2), then (3), (1), (5), (4)

  save

- What about $\int_0^\infty \cos x^2\,dx$?

Plan: Use



and take $R\to\infty$.
What does this help? Why does $C_R \xrightarrow[R\to\infty]{} 0$?

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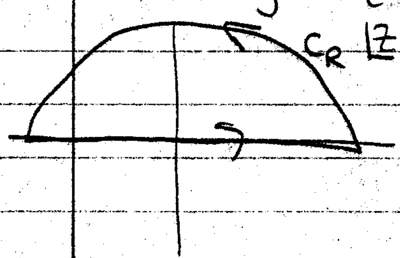
"Spot the Error!" — Jordan's lemma edition

Consider $\oint dz\, f(z)\, e^{ikz}$, with $k>0$, $\lim_{|z|\to\infty} f(z)=0$ on the contour



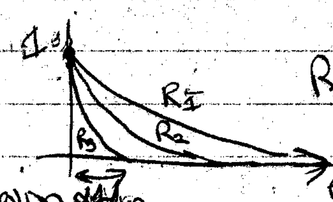
On C_R, $z=Re^{i\theta}$, $dz = iRe^{i\theta}d\theta$

$$\Rightarrow \int_{C_R} dz\, f(z)\, e^{ikz} = \int_0^{\pi} f(Re^{i\theta})\, e^{ikR\cos\theta}\, e^{-kR\sin\theta}\, Rie^{i\theta}\, d\theta$$

Possible Claim: As $R\to\infty$, $e^{-kR\sin\theta}$ dominates R and everything else so the integral $\to 0$.

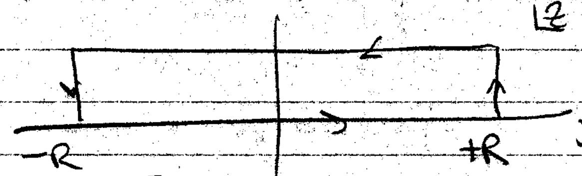
** Why is this claim incorrect?

If we plot $e^{-kR\sin\theta}$ for different R



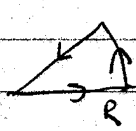
So $e^{-kR\sin\theta} \to 0$ except in a region ~~d~~ $1/kR$ of θ of width $1/R$ $\Rightarrow$ the contribution is $O(\frac{1}{R})$ that only just cancels the R from the measure. $\Rightarrow$ we need $f(Re^{i\theta}) \xrightarrow[R\to\infty]{} 0$ to justify Jordan's lemma. Note: Jordan's lemma with $k<0$ and closing in the lower half plane works just as well $\to$ symmetry.

- Do we have this issue with rectangular contours? No.



No, because $z = R + iy$ and $z = -R + iy$

So $\frac{e^{az}}{1+e^{bz}}$ on the right is determined by $\frac{e^{aR}}{e^{bR}}$ and on the left by e^{-aR}. ($\mathrm{Re}[a]$ is all that matters)

- What about $\oint e^{iz^2} dz$ on  the curved part? There's no $f(z)$ to save us!

Now $z = Re^{i\theta} \to \int_0^{\pi/4} e^{-R^2\sin 2\theta}\, e^{iR^2\cos 2\theta}\, Rie^{i\theta}\, d\theta$

and the $e^{-R^2\sin 2\theta}$ term is now $O(\frac{1}{R^2})$ in the integral so it still wins over the R.

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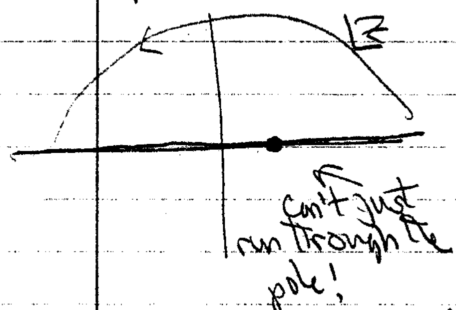
- Why is $\left|\int f(z)\,dz\right| \le \int |f(z)|\,|dz|$?

One last class of integrals to consider:

⑥ Integrals with poles on the real axis

In our examples so far along the real axis, the poles have always been somewhere in the complex plane. What if they are on the axis?

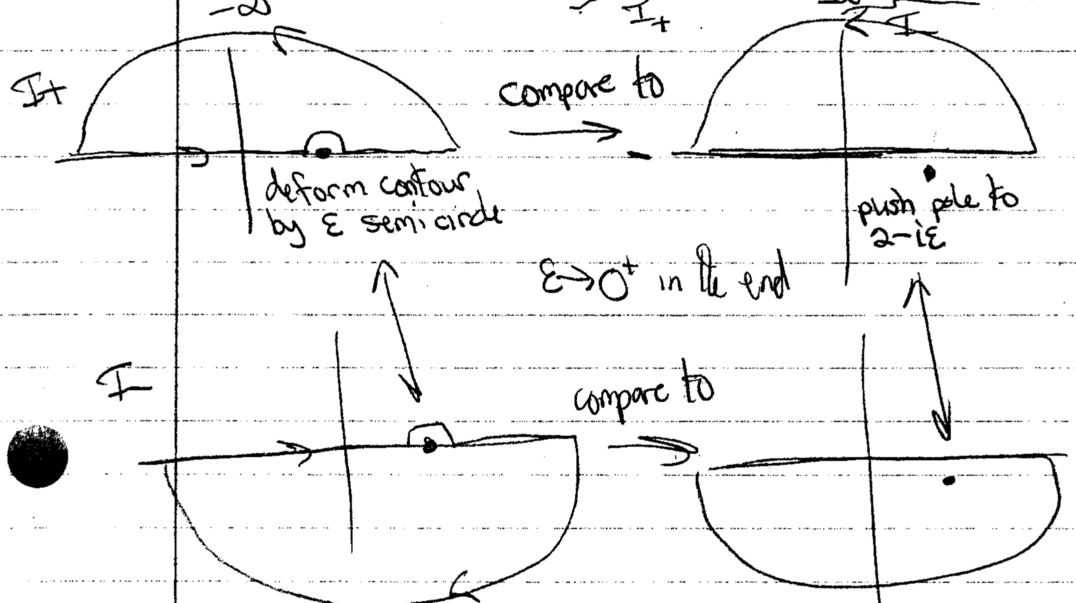


We need to specify, (usually) based on physics, how to go around the pole or how to move the pole.

Let's use the example from Lea, pg 143, $\int_{-\infty}^{\infty} \frac{\sin kx}{x-2}\,dx$ with $k>0$, but we'll move the pole.

Split into e^{ikx} and e^{-ikx} pieces to take advantage of semicircle contributions

$$\int_{-\infty}^{\infty} \frac{\sin kx}{x-2}\,dx = \frac{1}{2i}\left(\underbrace{\int_{-\infty}^{\infty} \frac{e^{ikx}}{x-2}\,dx}_{I_+} - \underbrace{\int_{-\infty}^{\infty} \frac{e^{-ikx}}{x-2}\,dx}_{I_-}\right) \xrightarrow[x\to z]{}$$



I_+: close in upper half plane so Jordan's lemma applies.

I_-: close in lower half plane

implied when using $i\varepsilon$ or $i\eta$ that $\lim_{\substack{\varepsilon\to 0^+ \\ \eta \to 0^+}}$ is taken. Sometimes $\frac{1}{x-2+i0^+}$ used. (40)

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(6) (cont.)

we'll do the pushed pole

$$I_+ = \oint_{C_{R+}} \frac{e^{ikz}}{z-(2-i\varepsilon)}\,dz = 0 \quad \text{(no poles enclosed)}$$

$$= \int_{-\infty}^{\infty} \frac{e^{ikx}}{x-(2-i\varepsilon)}\,dx + \underbrace{\cancel{\oint \frac{e^{ikz}}{z-2}\,dz}}_{\text{semicircle}} \to 0$$

Jordan's lemma since $1/z-2 \xrightarrow[R\to 0]{} 0$ uniformly.

$$I_- = \oint_{C_{R-}} \frac{e^{-ikx}}{z-(2-i\varepsilon)}\,dz \overset{\text{clockwise contour}}{=} -2\pi i\left(e^{-ik\cdot 2}\right) \quad \text{by residue theorem}$$

$$= \int_{-\infty}^{\infty} \frac{e^{-ikx}}{x-(2-i\varepsilon)}\,dx + \cancel{\oint \frac{e^{-ikz}}{z-2}\,dz}^{\,0}$$

$$\Rightarrow \int_{-\infty}^{\infty} \frac{\sin kx}{x-(2-i\varepsilon)}\,dx \underset{\text{variables}}{\overset{\text{change}}{=}} \int_{-\infty+i\varepsilon}^{+\infty+i\varepsilon} \frac{\sin kx}{x-2}\,dx = \frac{1}{2i}\left(0 - \left(-2\pi i\, e^{-i2k}\right)\right)$$

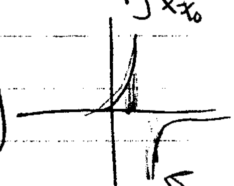
$$\boxed{= \pi e^{-2ki}}$$

Suppose we did $\int_{-\infty}^{\infty} \frac{\sin kx}{x-(2+i\varepsilon)}\,dx = \int_{-\infty-i\varepsilon}^{+\infty-i\varepsilon} \frac{\sin kx}{x-2}\,dx$?

$\Longrightarrow$ pick up pole in <u>upper</u> half plane $= \frac{1}{2i}\left(+2\pi i\, e^{+i2k} - 0\right)$

$$\boxed{= \pi e^{+2ki}}$$

Define <u>principal</u> <u>value</u> integral!



$$P\int_{-\infty}^{\infty} \frac{\sin kx}{x-2}\,dx \equiv \lim_{\varepsilon\to 0}\left(\int_{-\infty}^{2-\varepsilon} \frac{\sin kx}{x-2}\,dx + \int_{2+\varepsilon}^{+\infty} \frac{\sin kx}{x-2}\,dx\right)$$

contributions cancel near pole

$$\xrightarrow[\substack{+i\varepsilon \text{ and} \\ -i\varepsilon \text{ results!}}]{\text{average of}} \frac{1}{2}\left(\pi e^{-2ki} + \pi e^{2ki}\right) = \boxed{\pi \cos 2k}$$

(like integrating above and below to remove the piece near the pole.)

- Example: on PS#3

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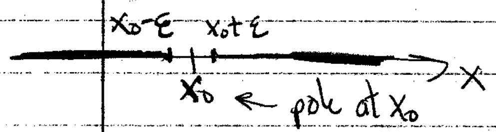
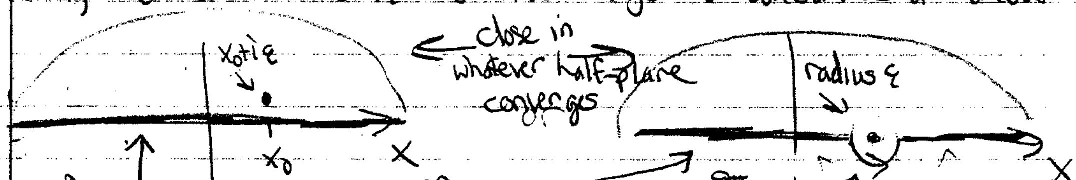
Principal value restated...

If we integrate a function with a pole on the real axis: $g(x) = \frac{f(x)}{x-x_0}$, the principal value integral is defined as $P\int_{-\infty}^{\infty} \frac{f(x)}{x-x_0}\,dx = \lim_{\varepsilon\to 0} \int_{-\infty}^{x_0-\varepsilon} \frac{f(x)}{x-x_0}\,dx + \int_{x_0+\varepsilon}^{\infty} \frac{f(x)}{x-x_0}\,dx$



So we can calculate the principal value directly from this definition.

Or, we can relate it to two ways to calculate a contour



same answer for integral:

$$\int_{-\infty}^{\infty} \frac{f(x)}{x-(x_0+i\varepsilon)}\,dx = P\int_{-\infty}^{\infty} \frac{f(x)}{x-x_0}\,dx + \underbrace{\lim_{\varepsilon\to 0} \int_{\pi}^{2\pi} \frac{f(x_0+\varepsilon e^{i\theta})}{\varepsilon e^{i\theta}}\,\varepsilon i e^{i\theta}\,d\theta}_{+i\pi f(x_0)} \quad (\text{or } -\pi<\theta<0)$$

, the integrals over the large semi-circles are assumed go to zero.

$$+i\pi f(x_0) = i\pi\int_{-\infty}^{\infty} f(x)\,\delta(x-x_0)\,dx$$

Symbolically: $\boxed{\frac{1}{x-x_0-i\varepsilon} = \frac{P}{x-x_0} + \pi i\,\delta(x-x_0)}$

Similarly,

$$\int_{-\infty}^{\infty} \frac{f(x)}{x-(x_0-i\varepsilon)}\,dx = P\int_{-\infty}^{\infty} \frac{f(x)}{x-x_0}\,dx + \underbrace{\lim_{\varepsilon\to 0} \int_{\pi}^{0} \frac{f(x_0+\varepsilon e^{i\theta})}{\varepsilon e^{i\theta}}\,\varepsilon i e^{i\theta}\,d\theta}_{-i\pi f(x_0)}$$

$\Rightarrow \boxed{\frac{1}{x-x_0+i\varepsilon} = \frac{P}{x-x_0} - \pi i\,\delta(x-x_0)}$

Finally, $\frac{P}{x-x_0} = \frac{1}{2}\left(\frac{1}{x-(x_0+i\varepsilon)} + \frac{1}{x-(x_0-i\varepsilon)}\right)$, which is more convenient sometimes.