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Physics 7701: Lecture 6

Contour integral plan summary:

1. Choose contour C in z-plane to include integral of interest and have do-able remainder. Mark poles, branch points and cuts.
2. Deform contour to avoid any branch cuts.
3. Note poles inside C.
4. Evaluate residues at poles from 3.
5. Apply the residue theorem $\oint_C f(z)\,dz = \pm 2\pi i$ (sum of residues enclosed) {counterclockwise / clockwise}

Our catalog:

1. $\int_0^{2\pi} F(\sin\theta, \cos\theta)\,d\theta$
2. $\int_{-\infty}^{\infty} f(x)\,dx$ when $f(z) \to 0$ fast enough
3. $\int_{-\infty}^{\infty} \{e^{ikx}$ or $\cos kx$ or $\sin kx\}\, f(x)\,dx$
4. → $\int_{-\infty}^{\infty} f(x)\,dx$ with hyperbolic functions (rectangular contour)
5. $\int_0^{\infty} x^\alpha f(x)\,dx$, $\alpha \neq$ integer (but rational)
6. → integrals with poles on the real axis

- Comments on PS#1 (from grader)
  - vectors ≠ components, $\vec{A} = (\vec{B}\times\vec{C})_i$; components matching $A_i = B_i + C_j$
  - $z = re^{i\theta}$ has $r \geq 0$
  - $n(x,t) = n_0 e^{i(kx-\omega t)}$ but n_0 already used
  - integration constants when solving CR equations
  - simplify sums
  - ** Don't include scratch work, staple pages, use one side of paper (one problem per page)
- Follow-up comments
  - (3) From cos/sin ↔ exp or Re/Im; Jordan's lemma and $\int_{C_R} e^{iz^2}\,dz$ ← why upper or lower?
  - (4) go through hyperbolic functions
  - (5) recap compared to homework
  - (6) Integrals with poles on real axis

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Brief mention: Dispersion relations

- You'll see these in E+M as the Kramers-Kronig relations, which relate the real and imaginary parts of the dielectric constant of a material ⇒ dispersive (index of refraction) and absorptive properties are related.
  - They show up in nonrelativistic and relativistic scattering

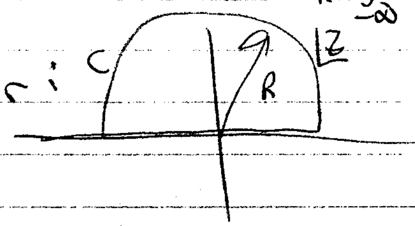
Claim: Given $f(z)$ analytic in the upper half plane with $|f(z)| \xrightarrow[z\to\infty]{} 0$, in the upper half plane (physics tells us!)

$$\mathrm{Re}[f(x_0)] = \frac{1}{\pi} P\int_{-\infty}^{\infty} \frac{\mathrm{Im}[f(x)]}{x-x_0}\,dx \quad \text{or} \quad u(x_0) = \frac{1}{\pi}P\int_{-\infty}^{\infty}\frac{v(x)\,dx}{x-x_0}$$

(principal value)

$$\mathrm{Im}[f(x_0)] = -\frac{1}{\pi} P\int_{-\infty}^{\infty} \frac{\mathrm{Re}[f(x)]}{x-x_0}\,dx \qquad v(x_0) = -\frac{1}{\pi}P\int_{-\infty}^{\infty}\frac{u(x)\,dx}{x-x_0}$$

Consider Cauchy integral formula over:



$$f(z_0) = \frac{1}{2\pi i}\oint \frac{f(z)}{z-z_0}\,dz$$

$$= \frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{f(x)}{x-z_0}\,dx + \text{vanishing semicircle if } z_0 \text{ in upper half plane.}$$

If in lower half plane, the integral is zero.

Now let z_0 approach the axis from above $z_0 = x_0 + i\varepsilon$ and below $z_0 = x_0 - i\varepsilon$ and average:

$$P\int_{-\infty}^{\infty}\frac{f(x)}{x-x_0}dx = \frac{1}{2}\left[\int_{-\infty}^{\infty}\frac{f(x)\,dx}{x-x_0-i\varepsilon} + \int_{-\infty}^{\infty}\frac{f(x)\,dx}{x-x_0+i\varepsilon}\right] = \frac{1}{2}\left[2\pi i\, f(x_0+i\varepsilon) + 0\right] = \pi i\, f(x_0)$$

analytic so $f(x_0+i\varepsilon) = f(x_0)$ as $\varepsilon\to 0$

discontinuity across the axis

$$\Rightarrow \boxed{f(x_0) = \frac{1}{\pi i}P\int_{-\infty}^{\infty}\frac{f(x)}{x-x_0}\,dx}$$

Now let $f(x) = \mathrm{Re}[f(x)] + \mathrm{Im}[f(x)] = u + iv$ and equate real and imaginary parts, and we're done! (The $1/i$ exchanges real ⇔ imag on right side.)

Aside: Mathematica Integrate has PrincipalValue -> True option

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Follow-up: If u and v are the real and imaginary parts of an analytic function w(z), then they are harmonic: they satisfy $\nabla^2 u = 0$, $\nabla^2 v = 0$ Laplace's equations (in 2 dimensions)

Proof is simple from C-R equations:

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

differentiate again:
$$\frac{\partial}{\partial x}\frac{\partial u}{\partial x} = \frac{\partial^2 u}{\partial x^2} \overset{CR}{=} \frac{\partial^2 v}{\partial x \partial y} \overset{\text{exchange}}{=} \frac{\partial}{\partial y}\frac{\partial v}{\partial x} \overset{CR}{=} -\frac{\partial}{\partial y}\frac{\partial u}{\partial y} = -\frac{\partial^2 u}{\partial y^2}$$

$$\Rightarrow \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 = \nabla^2 u$$ (and similarly for v).

An application to fluid flow is given in Lea 2.4.1, but also relevant to E+M.

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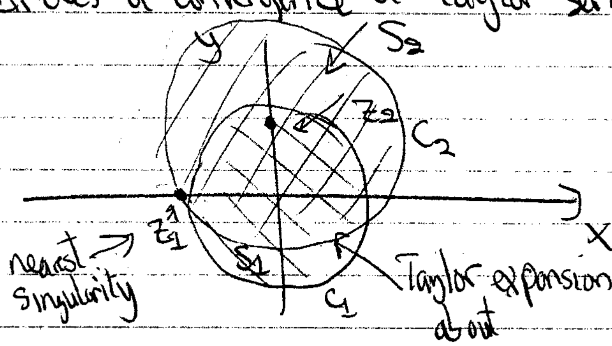
Brief mention: Analytic continuation (more later in course)

Footnote in Arfken, p434: "If two analytic functions coincide (i.e. have the same value) in any region or on any line segment, then they are the same function — that is, in regions where they are both well defined, they'll give the same answer."

- This enables us to extend functions to regions in z beyond where they are originally defined.

One way to do this analytic continuation is by overlapping circles of convergence of Taylor series.



Arfken 6.11

The circle of convergence C_1 is where the expansion of $f(z)$ about zero converges. So f in C_1 is defined by the Taylor series but not outside. However, there is an expansion about z_2 that is good within C_2.

- In the overlap region, f is uniquely defined, so it is the same function within both C_1 and C_2. We have "continued" the series from C_1 to C_2. And repeat.

- Alternative methods for analytic continuation will be considered later. (E.g. gamma function)

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Differential Equations: Pass 1

- In PS#3 you are asked to find solutions to Laguerre's differential equation

$$x y'' + (1-x) y' + \alpha y = 0$$

Special functions!

and the Bessel equation

$$4x^2 y'' + 4xy' + (4x^2 - 1) y = 0 \quad \text{(or divide by 4)}$$

by using series. Notation: $y = y(x)$, $y' \equiv \frac{dy}{dx}$, $y'' \equiv \frac{d^2y}{dx^2}$

- Recall the terminology (knowing these is a core competency)

  order: highest derivative of $y \Rightarrow$ both are 2nd order diff eqs.

  linear or nonlinear: does $y, y', y'', \ldots$ appear as more than first power? Here: no, so linear. (Implications?)

  $\Rightarrow$ Jackson says he will consider linear equations only.

  Nonlinear example $y y' = 2$. Note: functions of x don't matter.

  Ordinary or partial: Does y depend on more than one variable (eg. x and t) with partial derivatives for each?

  $\Rightarrow$ partial diff eq or PDE. Here, $y(x)$ so ordinary.

  - Wave equations are PDE's: $\frac{\partial^2 y(x,t)}{\partial t^2} = v^2 \frac{\partial^2 y}{\partial x^2}$ (separation of variables is one way to attack PDE's)

  Homogeneous or inhomogeneous:

  Does each term depend on y or derivatives $\Rightarrow$ homogeneous

$$\frac{d^2y}{dt^2} + 2\alpha \frac{dy}{dt} + \omega_0^2 y = 0 \quad \text{(damped harmonic oscillator)}$$

  - We'll come back to inhomogeneous ("driving terms" $\Rightarrow$ Green's functions, etc.)

  Are coefficients constant? (yes here)

  How to solve this? (eg. proportional to exponential in x)

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Many methods to solve differential equations, including numerical methods that are very important.

- Here we'll consider power series solutions — expand around a point.

- Distinguish between expanding about an ordinary and singular point of the differential equation.
  - Solve for $y'' = f(x, y, y')$ [nothing multiplying y'']

  - If homogeneous, then

$$y'' + P(x)\, y' + Q(x)\, y = 0$$

Just for confusion, called a regular point in Lea

  - Cases
    i) If $P(x)$, $Q(x)$ finite at $x = x_0$, x_0 is ordinary point
    ii) If either diverge, then singular
      - "regular" if $(x-x_0)P(x)$ and $(x-x_0)^2 Q(x)$ stay finite

Bessel equation

$$x^2 y'' + x y' + (x^2 - n^2) y = 0$$

$$\Rightarrow \quad y'' + \frac{1}{x} y' + \left(1 - \frac{n^2}{x^2}\right) y = 0$$

$\Rightarrow P(x) = \frac{1}{x}$, $Q(x) = 1 - \frac{n^2}{x^2}$ $\Rightarrow$ $x = 0$ is regular singularity and no other singular points for finite x.

- Almost always true in physics equations that we have no worse than a regular singular point
  $\Rightarrow$ can do series expansion (not necessarily a Taylor series!)

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Basic principles of series solutions
- We can expand the desired solution(s) in a series
  - it may be a Laurent series or overall fractional powers
- We can take derivatives of the series term by term (why?)
- We can equate coefficients of equal powers after we plug in the series into our equation. (why?) [uniqueness of power series]

- Solution about an ordinary point.
  - Do an example where we know the answer by inspection:

$$\frac{d^2y}{dx^2} + k^2 y = 0$$ (Helmholtz or simple harmonic oscillator equation)

$\Rightarrow$ know $y = \sin kx$ or $\cos kx$ are solutions.

- No singular points, so $x=0$ is an ordinary point

<u>Assume</u> $$y = \sum_{n=0}^{\infty} a_n x^n$$

$$y' = \sum_{n=0}^{\infty} n a_n x^{n-1}$$ (really starts at $n=1$, first term 0)

$$y'' = \sum_{n=0}^{\infty} n(n-1) a_n x^{n-2}$$ (really starts at $n=2$)

<u>Substitute</u>:

$$\sum_{n=0}^{\infty} n(n-1) a_n x^{n-2} + k^2 \sum_{n=0}^{\infty} a_n x^n = 0$$

Key: only satisfied if the net coefficient of each power of x is <u>separately</u> zero. (why?)

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Consider the first couple of x^n terms:

x^0: $n=2$ in first, $n=0$ in 2nd sum:

$$2\cdot 1\cdot a_2 + k^2 a_0 = 0 \Rightarrow \boxed{a_2 = -k^2\frac{a_0}{2\cdot 1}}$$

x^1: $n=3$ in first, $n=1$ in 2nd sum:

$$3\cdot 2\, a_3 + k^2 a_1 = 0 \Rightarrow \boxed{a_3 = -k^2\frac{a_1}{3\cdot 2}}$$

Generalize (we have sufficient terms to do that in this case)!

x^{m-2}: take $n=m$ in first, $n=m-2$ in 2nd

$$m(m-1)a_m + k^2 a_{m-2} = 0 \Rightarrow a_m = -k^2\frac{a_{m-2}}{m(m-1)}$$

and repeat for a_{m-4} (and so on)

$$a_m = \frac{-k^2}{m(m-1)}\,\frac{-k^2 a_{m-4}}{(m-2)(m-3)} = \frac{(-k^2)^2 a_{m-4}}{m(m-1)(m-2)(m-3)}$$ ← looks like factorial building up!

Continue

$$a_m = \begin{cases} (-1)^{m/2}\dfrac{k^m}{m!}a_0 & m \text{ even} \\ (-1)^{(m-1)/2}\dfrac{k^{m-1}}{m!}a_1 & m \text{ odd} \end{cases}$$

$$\Rightarrow y_1 = a_0\left(1 - \frac{(kx)^2}{2!} + \frac{(kx)^4}{4!} + \dots\right) = a_0 \cos kx \ \checkmark$$

$$y_2 = a_1\left(x - \frac{k^2x^3}{3!} + \frac{k^4x^5}{5!} + \dots\right) = \frac{a_1}{k}\sin kx \ \checkmark$$

useful, but our examples seem to have singular points ⇒ generalize!