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7701 Lecture 7: Frobenius Method

* How do we do series solutions when we have singular points?

If isolated, use Laurent series: $\sum_{n=-m}^{\infty} a_n (x-x_0)^n$

valid for $0 < |x-x_0| < \rho$ ← radius of convergence

If not isolated (e.g. branch point), then allow for non-integer values [could be any number (not just an integer)]

$$y(x) = (x-x_0)^p \sum_{n=0}^{\infty} a_n (x-x_0)^n \leftarrow \text{(these are integers!)}$$

⇒ Frobenius method

* • power p of first non-vanishing term is a parameter to be determined.

Follow examples in Arfken 9.5 (6th) / 7.5 (7th) and Lea 3.3 [practice in PS#3]
- As before, take derivatives (in special region) and substitute into the differential equation
- p determined by finding coefficient of lowest power and setting it to zero. ⇒ indicial equation

Hypergeometric:
$$(x^2-x)\frac{d^2y}{dx^2} + \left(2x-\frac{1}{2}\right)\frac{dy}{dx} + \frac{1}{4}y = 0$$

Suppose we want a power series about $x=0$.
- Write as $y'' + P(x)y' + Q(x)y = 0$

$$\Rightarrow P(x) = \frac{2(x-\frac{1}{4})}{x(x-1)}, \quad Q(x) = \frac{1}{4x(x-1)}$$

⇒ singular at $x=0$, $x=1$ ("regular" singularities) ⇒ choose series as above with p.

definitions: $n! \equiv n(n-1)(n-2)\cdots 1$

$(2n+1)!! = (2n+1)(2n-1)(2n-3)\cdots 1$

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Assume: $y = x^p \sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} a_n x^{n+p}$ with $a_0 \neq 0$

differentiate term-by-term $\Rightarrow$ $y' = \sum_{n=0}^{\infty} (n+p) a_n x^{n+p-1}$ and $y'' = \sum_{n=0}^{\infty} (n+p)(n+p-1) a_n x^{n+p-2}$

Equation is $x^2 y'' - x y'' + 2x y' - \frac{1}{2} y' + \frac{1}{4} y = 0$

$$\Rightarrow \sum_{n=0}^{\infty}(n+p)(n+p-1)a_n x^{n+p} - \sum_{n=0}^{\infty}(n+p)(n+p-1)a_n x^{n+p-1} + 2\sum_{n=0}^{\infty}(n+p)a_n x^{n+p} - \frac{1}{2}\sum_{n=0}^{\infty}(n+p)a_n x^{n+p-1} + \frac{1}{4}\sum_{n=0}^{\infty} a_n x^{n+p} = 0$$

- First find the lowest power of x that appears.
  - In first, third, and fifth, x^p and higher.
  - But 3rd and fourth have x^{p-1} when $n=0$.
  - $\Rightarrow$ This coefficient vanishes: $-p(p-1)a_0 - \frac{1}{2} p a_0 = 0 \Rightarrow \boxed{p(p-\frac{1}{2}) = 0}$ "indicial equation"
    because we already took $a_0 \neq 0$ (otherwise $y \equiv 0$).
- So we have to consider $p=0$ and $p=1/2 \Rightarrow$
- Find coefficient of x^{m+p} and set to zero.
  - In x^{n+p-1} series, let $m = n-1 \Rightarrow n = m+1$. In others, $n=m$!

$$(m+p)(m+p-1)a_m - (m+p+1)(m+p)a_{m+1} + 2(m+p)a_m - \frac{1}{2}(m+p+1)a_{m+1} + \frac{1}{4}a_m = 0.$$

Solve for a_{m+1}: $a_{m+1} = \dfrac{(m+p)(m+p+1) + 1/4}{(m+p+1)(m+p+1/2)} a_m$ so $a_0 \to a_1 \to a_2 \to \ldots$

$\Rightarrow$ given a_0, the entire series is determined, so two independent solutions from $p=0$ and $p=1/2$ here:

$p=0$: $a_{m+1} = \dfrac{m^2+m+1/4}{(m+1)(m+1/2)} a_m = \dfrac{m+1/2}{m+1} a_m = \dfrac{2m+1}{2(m+1)} a_m$

$a_m = \dfrac{2(m-1)+1}{2[(m-1)+1]} a_{m-1} = \dfrac{2m-1}{2m} a_{m-1}$

keep going and combine $\Rightarrow a_{m+1} = \dfrac{1}{2^{m+1}} \dfrac{(2m+1)!!}{(m+1)!} a_0$

solution $\Rightarrow y_1 = a_0 \sum_{n=0}^{\infty} \dfrac{(2n-1)!!}{2^n n!} x^n$ (took $n = m+1$)

- regular at origin and converges for $|x|<1$ (can you show?)

$p=\frac{1}{2}$: $a_{m+1} = \dfrac{m^2+2m+1}{(m+3/2)(m+1)} a_m = \dfrac{m+1}{m+3/2} a_m$

$= \dfrac{2(m+1)}{2m+3} a_m = \dfrac{2^2(m+1)m}{(2m+3)(2m+1)} a_{m-1}$

keep going $= \dfrac{2^{m+1}(m+1)!}{(2m+3)!!} a_0$ [used $a_m = \dfrac{2(m) a_{m-1}}{2(m-1)+3}$]

$\Rightarrow y_2 = \sqrt{x}\, a_0 \sum_{n=0}^{\infty} \dfrac{2^n n!}{(2n+1)!!} x^n \Leftarrow$ any a_0 here

$\Rightarrow$ converges for $|x|<1$ (can you show?) but branch point at origin.

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Further comments on differential equations:

- Lea lists some methods of solution
  1. Guess the form (eg. undetermined coefficients)
  2. Power-series type solution (Frobenius) — this can be useful for numerical calculations as well
  3. Asymptotic solution
  4. Relate (possibly with change of variable) to known form (eg. hypergeometric equation)
  5. Integrate numerically

- Much of the rest of the course will touch on different features of differential equations!

- Sometimes Frobenius doesn't give us a 2nd solution (indicial equation has repeated root or roots differ by an integer).
  - Problem is that there is a logarithm, and Frobenius doesn't allow for it.
  - So look for

$$y_2 = \underbrace{y_1}_{\text{first solution } y_1(x)} \ln x + \sum_{n=0}^{\infty} \overbrace{b_n}^{\text{to be determined}} x^{n+p}$$

- If we want to expand around a non-zero singular point (eg. $x=1$ for the Legendre equation), simply change to $w=x-1$ so that the singularity is at $w=0$, and apply Frobenius to get $y(w)$. See Lea Example 3.7.

- The indicial equation may have complex roots. Just do it!

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Asymptotic methods $\Rightarrow$ look at large values of x

- This is often helping in checking or starting numerical solutions.

Consider the example in Lea chapter 3, (example 3.9)

Modified Bessel equation is:

$$y'' + \frac{1}{x}y' - \left(1 + \frac{m^2}{x^2}\right)y = 0.$$

Find $y(x)$ as $x \to \infty$.

- For large x, the $\frac{1}{x}y'$ and $-\frac{m^2}{x^2}y$ term get small, so y_∞ satisfies

$$y''_\infty - y_\infty = 0$$

- We know the solutions: $y_\infty = (\text{const})e^{\pm x}$

- Now we can extract this dominant large x behavior by writing

$$y(x) = V(x)\,y_\infty(x) = V(x)\,e^{\pm x}$$

$$y' = V'e^{\pm x} \pm Ve^{\pm x}, \quad y'' = V''e^{\pm x} \pm 2V'e^{\pm x} + Ve^{\pm x}$$

Substituting in the equation

$$\Rightarrow V''e^{\pm x} \pm 2V'e^{\pm x} + Ve^{\pm x} + \frac{1}{x}(V'e^{\pm x} \pm Ve^{\pm x}) - \left(1 + \frac{m^2}{x^2}\right)Ve^{\pm x} = 0$$

cancel $e^{\pm x}$ $\Rightarrow$ $V'' + \left(\frac{1}{x} \pm 2\right)V' + \frac{1}{x}\left(\pm 1 - \frac{m^2}{x^2}\right)V = 0$

Again, look at x large, assuming $V = x^\alpha(1 + \text{corrections})$.

$$\Rightarrow \alpha(\alpha-1)x^{\alpha-2} + \alpha x^{\alpha-2} \pm 2\alpha x^{\alpha-1} \pm x^{\alpha-1} - m^2 x^{\alpha-3} = 0$$

Keep leading terms: $x^{\alpha-1} \Rightarrow \pm 2\alpha \pm 1 = 0 \Rightarrow \alpha = -1/2$

$\Rightarrow \boxed{y = \frac{e^{-x}}{\sqrt{x}}}$ or $\boxed{y = \frac{e^{x}}{\sqrt{x}}}$ are the asymptotic forms.

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Another Frobenius Method Example

$$y(x) = x^k \sum_{n=0}^{\infty} a_n x^n$$

· origin of special polynomials

(A) Legendre equation: $(1-x^2)y'' - 2xy' + \alpha(\alpha+1)y = 0$

Check singularities $\Rightarrow y'' - \frac{2x}{1-x^2}y' + \frac{\alpha(\alpha+1)}{1-x^2}y = 0 \Rightarrow$ isolated singular points $x = \pm 1$

· Expand about $x=0$ (see Lea Example 3.7 for an expansion about 1)

$$y = \sum_{n=0}^{\infty} a_n x^{n+k} \qquad y' = \sum_{n=0}^{\infty}(n+k)a_n x^{n+k-1} \qquad y'' = \sum_{n=0}^{\infty}(n+k)(n+k-1)a_n x^{n+k-2}$$

Separate terms with different powers of x:

$$\sum_{n=0}^{\infty}(n+k)(n+k-1)a_n x^{n+k-2} - \sum_{n=0}^{\infty}(n+k)(n+k-1)a_n x^{n+k} - 2\sum_{n=0}^{\infty}(n+k)a_n x^{n+k} + \sum_{n=0}^{\infty}\alpha(\alpha+1)a_n x^{n+k} = 0$$

lowest powers of x: x^{k-2}, x^{k-1}, only from the first sum.

$n=0$, x^{k-2}: $k(k-1)a_0 = 0 \Rightarrow k=0$ or $k=1 \Rightarrow$ consider each.

$n=1$, x^{k-1}: $(k+1)\cdot k\, a_1 = 0$.

For general term demand that coefficient of x^{k+j}, $j \geq 0$ vanish

Set $n=j+2$ in first term and $n=j$ in others and switch to sum over j:

$$(j+2+k)(j+1+k)a_{j+2} - (j+k)(j+k-1)a_j - 2(j+k)a_j + \alpha(\alpha+1)a_j = 0$$

$\boxed{k=0}$ $(j+2)(j+1)a_{j+2} = [j(j-1)+2j - \alpha(\alpha+1)]a_j$ or $\boxed{a_{j+2} = \frac{j(j+1)-\alpha(\alpha+1)}{(j+2)(j+1)}a_j}$

$$\Rightarrow a_2 = \frac{-\alpha(\alpha+1)}{2\cdot 1}a_0, \quad a_4 = \underbrace{\frac{2\cdot 3 - \alpha(\alpha+1)}{4\cdot 3}}_{\text{"}-(\alpha+3)(\alpha-2)\text{"}}a_2, \quad a_6 = \underbrace{\frac{4\cdot 5 - \alpha(\alpha+1)}{6\cdot 5}}_{= -(\alpha+5)(\alpha-4)}a_4, \ldots$$

$$\Rightarrow \boxed{y(x) = a_0\left[1 - \frac{\alpha(\alpha+1)}{2!}x^2 + \frac{\alpha(\alpha+1)(\alpha+3)(\alpha-2)}{4!}x^4 - \frac{\alpha(\alpha+1)(\alpha+3)(\alpha-2)(\alpha+5)(\alpha-4)}{6!}x^6 + \ldots\right]}$$

even series. The general term is easy to imagine (but maybe not to write!)

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(4) (cont.)

Now we can develop an odd series starting with a_1 either from $k=0$ or $k=1$.

For $k=0$, we have the same relation between a_{j+2} and a_j, but now we start with a_1!

$$a_3 = \frac{\overbrace{1\cdot 2-\alpha(\alpha+1)}^{=-(\alpha+2)(\alpha-1)}}{3\cdot 2}a_1,\quad a_5=\frac{\overbrace{3\cdot 4-\alpha(\alpha+1)}^{=-(\alpha+4)(\alpha-3)}}{5\cdot 4}a_3,\quad a_7=\frac{5\cdot 6-\alpha(\alpha+1)}{7\cdot 6}a_5,\ldots$$

$$\Rightarrow \boxed{y(x) = a_1\left[x - \frac{(\alpha-1)(\alpha+2)}{3!}x^3 + \frac{(\alpha-3)(\alpha-1)(\alpha+2)(\alpha+4)}{5!}x^5 + \ldots\right]}$$

For $k=1$, the recurrence is: $(j+3)(j+2)a_{j+2} = [(j+1)j+2(j+1)-\alpha(\alpha+1)]a_j$

or $a_{j+2} = \dfrac{(j+1)(j+2)-\alpha(\alpha+1)}{(j+3)(j+2)}a_j$. Starting from $j=0$ gives the same series (remembering that $k=1$ means we start from x and have odd powers)

Do these series converge? Not for $x\geq 1$ for general α.
However, if α is an integer l, then the series truncates $\Rightarrow$ polynomial
Check (with $a_0=1$, $a_1=1$)

$l=0$ $\quad a_2=0$ and all higher $\Rightarrow y(x)=1$ $\quad [P_0(x)=1]$

$l=1$ $\quad a_3 \propto (1\cdot 2-1\cdot 2)/3\cdot 2=0 \Rightarrow y(x)=x$ $\quad [P_1(x)=x]$

$l=2$ $\quad a_2=-2(3)/2$, $a_4\propto(2\cdot 3-2\cdot 3)=0 \Rightarrow y(x)=1-3x^2$ $\quad [P_2(x)=\frac{1}{2}(3x^2-1)]$

$l=3$ $\quad a_3=-2\cdot 5/3!$, $a_5\propto(3\cdot 4-3\cdot 4)=0 \Rightarrow y(x)=x-\frac{5}{3}x^3$ $\quad [P_3(x)=\frac{x}{2}(5x^2-3)]$

- The polynomials in [ ]'s are the Legendre polynomials, which we'll see again!
  $\Rightarrow$ They are normalized on $[-1,1]$ by $\int_{-1}^{1} P_l(x)P_{l'}(x)\,dx = \frac{2}{2l+1}\delta_{ll'}$