· put hypergeometric on board: $x^2y'' - xy'' + 2xy' - \frac{1}{2}y' + \frac{1}{4}y = 0$

· take $y = x^p \sum_{n=0}^{\infty} a_n x^n \Rightarrow$ lowest power of x: $p(p-\frac{1}{2})a_0 = 0$ indicial equation

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$p=0$: $y_1 = a_0 \sum_{n=0}^{\infty} \frac{(2n-1)!!}{2^n n!} x^n$ $\quad$ $p=\frac{1}{2}$ $y_2 = \sqrt{x}\, a_0 \sum_{n=0}^{\infty} \frac{2^n n!}{(2n+1)!!} x^n$

$0 \le |x| < 1$ converges $\quad$ $0 < |x| < 1$ converges

How do we know convergence? Ratio test!

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$\boxed{\text{7701 Lecture 8}}$

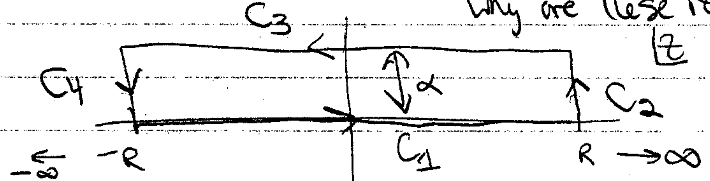
· PS#3 due Tuesday 4pm in Russell Colburn's mailbox

· Follow-ups on contour integration (relevant for PS#3)

1 a) $I = \int_{-\infty}^{\infty} \frac{e^{ax}}{1+e^{bx}}\,dx$ with b real and $b > \mathrm{Re}(a) > 0$

Why are these relevant?

$f(z) = \frac{e^{az}}{1+e^{bz}}$



What is z on each of the four contours?

(class do these)

C_1: $z = x \quad -R \le x \le R$ (with $R \to \infty$)

C_2: $z = R + iy \quad 0 \le y \le \alpha$

C_3: $z = x + i\alpha \quad R \le x \le -R$

C_4: $z = -R + iy \quad \alpha \le y \le 0$

Questions to answer:

① How to choose α? Want $\int_{C_3} \propto \int_{C_1} \Rightarrow \frac{e^{a(x+i\alpha)}}{1+e^{b(x+i\alpha)}} = \frac{e^{ax}e^{ia\alpha}}{1+e^{bx}e^{ib\alpha}}$

What is simplest choice for b?

② Do the integrals on C_2 and C_4 vanish? Why? C_2: $\frac{e^{a(R+iy)}}{1+e^{b(R+iy)}}$

As $R \to \infty$, what wins? (and why do the conditions matter?!)

③ Where are the poles? Does e^{az} have singularities?

Where does $1+e^{bz}$ have zeros for real b?

④ How do you tell Mathematica that $b > \mathrm{Re}(a) > 0$?

Ans: Assumptions -> {b > Re[a] > 0}

Use this in Integrate function.

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Flash back to 43 on dispersion relations (Cahill 5.19)

Use of $\frac{1}{x-x_0-i\varepsilon} = \frac{P}{x-x_0} + i\pi\,\delta(x-x_0)$ ← principal value

pole in upper half plane

note + when $(x-x_0)-i\varepsilon$ has $-i\varepsilon$

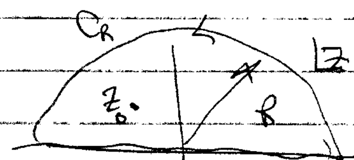
real functions

- Apply to a complex function $f(x) = u(x) + i v(x) = \mathrm{Re}[f(x)] + i\,\mathrm{Im}[f(x)]$ for which $f(z)$ is analytic in the upper half z plane.
  - key example: dielectric constant of a material

  and $f(z) \underset{|z|\to\infty}{\to} 0$ in the upper half plane.

real and imaginary parts describe different physics

- Then for any z_0 in the upper plane,



the integral $\frac{1}{2\pi i}\oint_C \frac{f(z)}{z-z_0}\,dz = f(z_0)$ on this contour as $R\to\infty$, [C_R part vanishes)

- But now let $z_0 = x_0 + i\varepsilon$ with x_0 real (certainly this is in upper half plane)

$$\frac{1}{2\pi i}\int_{-\infty}^{\infty} \frac{f(x)}{x-x_0-i\varepsilon}\,dx = f(x_0)$$ (because $f(z)$ analytic so $f(x_0+i\varepsilon) = f(x_0)$ as $\varepsilon\to 0$

$$\Rightarrow \frac{1}{2\pi i} P\int_{-\infty}^{\infty} \frac{f(x)}{x-x_0}\,dx + \frac{1}{2\pi i}\, i\pi \underbrace{\int_{-\infty}^{\infty} f(x)\,\delta(x-x_0)\,dx}_{\frac{1}{2}f(x_0)} = f(x_0)$$

$$\Rightarrow f(x_0) = \frac{1}{\pi i} P\int_{-\infty}^{\infty} \frac{f(x)}{x-x_0}\,dx$$

$$\Rightarrow u(x_0) + i v(x_0) = \frac{1}{\pi i} P\int_{-\infty}^{\infty} \frac{u(x)}{x-x_0}\,dx + \frac{1}{\pi i} P\int_{-\infty}^{\infty} \frac{i v(x)}{x-x_0}\,dx$$

equate real and imaginary: $u(x_0) = \frac{1}{\pi} P\int_{-\infty}^{\infty} \frac{v(x)}{x-x_0}\,dx$ real part is integral over imaginary part

$v(x_0) = -\frac{1}{\pi} P\int_{-\infty}^{\infty} \frac{u(x)}{x-x_0}\,dx$ imaginary part is integral over real part

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Comments on differential equation problems in PS#3:

another example on 54, 55

Problems 2 and 3 are basic applications of the Frobenius method; we'll recap, but remember the indicial equation is the starting point
- one is a power series (you are only to find the regular solution) and the other has power series times non-integer powers of x

problem 3 ⇒
- you are expected to recognize common series, "sum the series" for functions like e^x, $\sin x$, $\cos x$, etc.
- For Mathematica, see the "Solving some PS#3 differential equations" on the example notebook page

Problem 4: $(1+x)y'' + (3+2x)y' + (2+x)y = 0$
- look for $x>0$ solution.

⇒
- do 53 for asymptotic methods
- What does "direct integration" mean?

e.g. $v'' = v'$, let $w = v'$ $\Rightarrow \frac{dw}{dx} = w \Rightarrow \int_{w(0)}^{w(x)} \frac{dw}{w} = \int_0^x dx' \Rightarrow \ln \frac{w(x)}{w(0)} = x$

$\Rightarrow w(x) = w(0)e^x$

$\Rightarrow v'(x) = v'(0)e^x \Rightarrow$ integrate again $v(x) - v(0) = v'(0)e^x$

Problem 5: $y'' + (l + \frac{1}{2} - \frac{x^2}{4})y = 0$ Take $y(x) = e^{-ax^2}v(x)$ and find a choice of "a" that gives a simpler equation for v (i.e., no x^2's)

In general: extract the dominant part of the solution and then solve the rest approximately (or numerically).

Convergence of power series solutions to differential equations
⇒ out to singular points. Check Legendre $y'' - \frac{2x}{1-x^2}y' + \frac{\alpha(\alpha+1)}{1-x^2} = 0$

Ratio test for power series at given x: $\lim_{n\to\infty} \frac{|a_{n+1}(x-x_0)^{n+1}|}{|a_n(x-x_0)^n|} < 1$ converges: $|x-x_0| < \lim_{n\to\infty}\left|\frac{a_n}{a_{n+1}}\right|$

Legendre about $x_0=0$: $\frac{a_{j+2}}{a_j} = \frac{j(j+1)-\alpha(\alpha+1)}{(j+2)(j+1)}$ a_j $\Rightarrow \left|\frac{(1)(1+1/j) - \alpha(\alpha+1)/j^2}{(1+2/j)(1+1/j)}\right| \xrightarrow[j\to\infty]{} 1 \Rightarrow |x| < 1$ ✓

Comment: Many of the most important differential equations in physics are only solved in terms of infinite series, but even so, this is sufficient to derive many of their properties and relations between them.

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Recap: Frobenius method and the indicial equation

- General results for series solutions.

① If $x = x_0$ is an ordinary point of
$$y''(x) + P(x)y'(x) + Q(x)y(x) = 0$$
(so $P(x)$ and $Q(x)$ are analytic at x_0), then the general solution is
$$y(x) = c_1 \sum_{n=0} a_n (x-x_0)^n + c_2 \sum_{n=0}^{\infty} b_n (x-x_0)^n$$
linearly independent (what does that mean?)

* The radius of convergence of each series is at least as great as the distance from x_0 to the nearest singular point of the differential equation.

② If we have singular points that are irregular, e.g. $P(x)$ and $Q(x)$ diverge at x_0 worse than $1/x-x_0$ and $1/(x-x_0)^2$, respectively, then life is complicated. E.g. on PS#4 ⇒ essential singularity in solution. Uncommon in equations for physical systems.

③ If $x=0$ is a regular singular point of $y''(x) + p(x)y'(x) + q(x)y(x) = 0$ and ρ_1 and ρ_2 are the roots of the indicial equation with $\rho_1 \geq \rho_2$, then

ⓐ If $\rho_1 - \rho_2 \neq 0$ or an integer then
$$y_1(x) = |x|^{\rho_1} \sum_{n=0}^{\infty} a_n x^n \ (a_0 \neq 0) \quad \text{and} \quad y_2(x) = |x|^{\rho_2} \sum_{n=0}^{\infty} b_n x^n \ (b_0 \neq 0)$$

ⓑ If $\rho_1 = \rho_2$, then
$$y_1(x) = |x|^{\rho_1} \sum_{n=0}^{\infty} a_n x^n \ (a \neq 0) \quad \text{and} \quad y_2(x) = y_1(x) \ln x + |x|^{\rho_1 + 1} \sum_{n=0}^{\infty} b_n x^n$$

ⓒ If $r_1 - r_2 =$ an integer, then
$$y_1(x) = |x|^{\rho_1} \sum_{n=0}^{\infty} a_n x^n \ (a_0 \neq 0) \quad \text{and} \quad y_2(x) = c\, y_1(x) \ln|x| + |x|^{\rho_2} \sum_{n=0}^{\infty} b_n x^n$$
- where c can be zero

- Do not use the 2nd lowest equation in x: $(\ f(\rho)\ ) a_1 = 0$ to find more ρ values. Use it with ρ_1 and ρ_2 to decide if $a_1 = 0$.
Eg. if $f(\rho_1) \neq 0$, then $a_1 = 0$ is required. If $f(\rho_2) = 0$, then a_1 can be anything.

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Gamma Function $\Gamma(z)$ — [see Chap. 8 in Arfken; Sect 2.9 in Lea]

· In the solution to Laguerre's equation in problem 2 of PS#3, you find the series solution:

$$y(x) = a_0 \sum_{n=0}^{\infty} \frac{(-1)^n \alpha(\alpha-1)\cdots(\alpha-(n-1))}{(n!)^2} x^n$$

· How could we write the n^{th} term more generally?

· Consider $\alpha = k$, an integer:

$$\Rightarrow k(k-1)\cdots(k-(n-1)) = \frac{k(k-1)\cdots(k-(n-1))(k-n)(k-(n+1))\cdots 3\cdot 2\cdot 1}{(k-n)(k-(n+1))\cdots 3\cdot 2\cdot 1}$$

$$= \frac{k!}{(k-n)!}$$

Can we generalize? Yes, with $z! = \Gamma(z+1)$, then $\frac{\Gamma(\alpha+1)}{\Gamma(\alpha+1-n)}$ is equal to the $\alpha(\alpha-1)\cdots(\alpha-(n-1))$ part.

The Gamma function is defined as

$$\Gamma(x) = \int_0^{\infty} e^{-t} t^{x-1}\, dt \quad \text{for } x>0 \text{ (real)}$$

· $\Gamma(1) = \int_0^{\infty} e^{-t}\, dt = -e^{-t}\Big|_0^{\infty} = 1$

· We can integrate by parts to show that $\Gamma(x+1) = x\Gamma(x) = x(x-1)\Gamma(x-1)$

$$\xrightarrow[x=n \text{ integer}]{} \Gamma(n+1) = n!$$

[eg. $u = e^{-t} \Rightarrow du = -e^{-t}dt$. $dv = t^{x-1}dt \Rightarrow v = \frac{t^x}{x}$

$$\Rightarrow \Gamma(x) = \cancel{e^{-t}\frac{t^x}{x}\Big|_0^{\infty}}^{\,0} + \int \frac{t^x}{x} e^{-t}\, dt = \frac{1}{x}\Gamma[x+1]$$]

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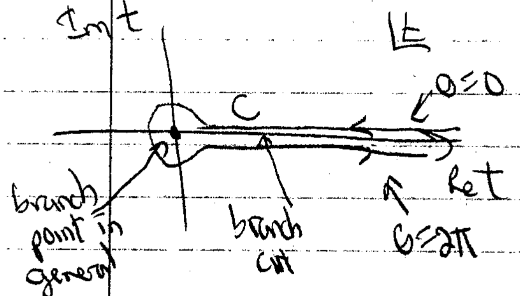
Note that $\int_{-\infty}^{\infty} e^{-u^2}\,du = 2\int_0^{\infty} e^{-u^2}\,du = 2\cdot\frac{1}{2}\int_0^{\infty} e^{-t}\, t^{-1/2}\,dt = \Gamma(1/2)$!

$t = u^2$, $u = t^{1/2}$, $du = \frac{1}{2}\frac{dt}{t^{1/2}}$

· Claim $\int_0^{\infty} e^{-x^4}\,dx = \frac{1}{4}\Gamma(\frac{1}{4}) = \Gamma(\frac{5}{4})$ [switch to $u = x^4$] and $\int_0^1 x^k \ln x\,dx \overset{[x=e^{-y}]}{=} -\frac{1}{(k+1)^2}\Gamma(2) = -\frac{1}{(k+1)^2}$

How do we define the integral for $\mathrm{Re}(z) > 0$? Use a contour integral.



Use this contour so that we include the top of the branch backwards and the bottom from $\mathrm{Re}\,t = 0$ to ∞, with $\theta = 2\pi$.

· The little circle around the branch point vanishes!

$$\int_{C_\varepsilon} t^{z-1} e^{-t}\,dt = \int_0^{2\pi} (\varepsilon e^{i\theta})^{z-1}\, e^{-(\varepsilon e^{i\theta})}\, i\varepsilon e^{i\theta}\,d\theta$$

$$\xrightarrow[\varepsilon \to 0]{} \varepsilon^z \int_0^{2\pi} e^{iz\theta}\, i\,d\theta$$

$$= \frac{\varepsilon^z}{z}\left(e^{2\pi i z} - 1\right) \xrightarrow[\varepsilon\to 0]{} 0 \quad \text{for } \mathrm{Re}(z) > 0$$

· On the bottom of the branch, we get an extra $e^{2\pi i z}$, so

$$\boxed{\Gamma(z) = \frac{1}{e^{2\pi i z} - 1}\int_C t^{z-1} e^{-t}\,dt}$$

bottom of cut · top of cut · $\mathrm{Re}\, z > 0$ · not closed!

When $z = n$ positive integer, define as limit. There are simple poles for $z =$ non-negative integer.

Ok, but what about $\Gamma(-1/2)$? Define by analytic continuation using

$\Gamma(z) = \Gamma(z+1)/z$ sufficiently many times. Eg. $\Gamma(-1/2) = \Gamma(+1/2)/{-1/2}$

(repeat until $\mathrm{Re}(z+1) > 0$)