9/11/13

2701 Lecture 9

- Loose ends on our first pass at differential equations

(a) On problem 4: $(1+x)y'' + (3+2x)y' + (2+x)y = 0$

$\xrightarrow{\text{asymptotic}}$ $y_\infty'' + 2y_\infty' + y_\infty = 0$. Use $e^{sx} \rightarrow s^2+2s+1=0$ or $s=-1$.

So $y_\infty^{(1)}(x) = e^{-x}$. Do we need an independent solution for this problem?

- No, because we write $y(x) = e^{-x} v(x)$ and $v(x)$ will have two independent solutions and that is all we need.
- What is another y_∞ solution?

  $y_\infty^{(2)}(x) = xe^{-x}$

How do we know? Here we can write $y_\infty^{(2)}(x) = e^{-x} u(x)$ and substitute into the y_∞ equation:

$$e^{-x}u'' - 2e^{-x}u' + e^{-x}u + 2e^{-x}u' - 2e^{-x}u + e^{-x}u = 0$$

Cancel e^{-x} and others: $u'' = 0 \Rightarrow u = ax+b \Rightarrow e^{-x}$ and xe^{-x}.

- How can we tell that these are linearly independent solutions? Calculate the Wronskian: $W(y_1, y_2) \equiv y_1 y_2' - y_2 y_1'$
  - If W is nonzero, then independent.

Check $W(y_\infty^{(1)}, y_\infty^{(2)}) = e^{-x}(e^{-x} - xe^{-x}) + xe^{-x}e^{-x} = e^{-2x} \neq 0$ ✓

- From W a 2nd solution can be found from $y_2 = y_1 \int \frac{W(x)}{y_1^2} dx$ where for

$$y'' + f(x)y' + g(x)y = 0 \Rightarrow W(x) = W(x_0)\, e^{-\int_{x_0}^{x} f(\xi)d\xi}$$

- What if we looked for $y(x) = xe^{-x}v(x)$? Messier, but we would find the same two solutions and with $y(x) = e^{-x}v(x)$.
  - e.g., the equation for v: $x(1+x)v'' + (3x+2)v' + v = 0$ has for $v(x) = x^\alpha$ a solution with $\alpha = -1$.

9/11/13

(b) Convergence of series and termination as polynomials
· Use Legendre polynomial example on (54)–(55) as test case.

Ratio test: $\sum_{n=0}^{\infty} c_n \Rightarrow \lim_{n\to\infty}\left|\frac{c_{n+1}}{c_n}\right| = \begin{cases} <1 & \text{convergent} \\ >1 & \text{divergent} \\ =1 & \text{can't tell} \end{cases}$

For power series, apply at given x:

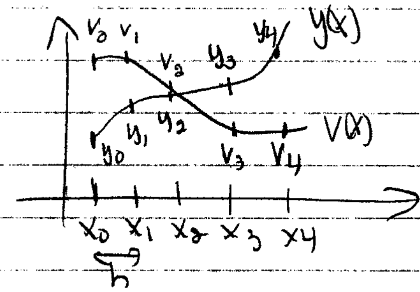
$$\lim_{n\to\infty}\frac{|a_{n+1}(x-x_0)^{n+1}|}{|a_n(x-x_0)^n|} \quad \text{or} \quad |x-x_0| < \lim_{n\to\infty}\left|\frac{a_n}{a_{n+1}}\right| \text{ is radius of convergence}$$

· apply to Legendre.

(c) Numerical solution: pass 1

Solve $\frac{d^2y}{dx^2} + p(x)\frac{dy}{dx} + q(x) = 0$

on a discrete grid (here, equally spaced by $h \Rightarrow$ could be variable).



· basic idea $\Rightarrow$ a derivative tells you how to take a step:

$$\frac{dy}{dx} = f(x) \Rightarrow \Delta y \doteq f(x)\Delta x \Rightarrow y_{n+1} \doteq y_n + \Delta y \doteq y_n + f(x_n)\cdot h$$

· Write 2nd order equation as two first order, introducing $v(x)$:

$$\frac{dy}{dx} = v(x) \Rightarrow y_1 = y_0 + v_0\Delta x = y_0 + v_0 h$$

$$\frac{dv}{dx} = \frac{d^2y}{dx^2} = -p(x)v(x) - q(x) \Rightarrow v_1 = v_0 + (-p(x_0)v_0 - q(x_0))\cdot h$$

} first step requires two values: y_0 and $v_0 = \left.\frac{dy}{dx}\right|_{x_0}$

· General step:

$$\boxed{\begin{aligned} y_{n+1} &= y_n + v_n h \\ v_{n+1} &= v_n + [-p(x_n)v_n - q(x_n)]\,h \end{aligned}}$$

$y_0 \to y_1 \to y_2 \to \ldots$
$v_0 \to v_1 \to v_2 \to \ldots$

· Error is $O(h^2) \Rightarrow$ we can do much better (eg. 4th order Runge-Kutta $O(h^4)$)
  · only uses derivative at left end.
  · can make h adaptive $\to$ change according to right side of diff eq.

9/11/13

Differential Equations in *Mathematica*: PS#3 examples

PS#3 Problem 2

Try to find a general solution to PS #3, Problem 2 (note that we need y''[x], not just y''):

```
DSolve[x y''[x] + (1 - x) y'[x] + α y[x] == 0, y, x]
```

```
{{y → Function[{x}, C[1] HypergeometricU[-α, 1, x] + C[2] LaguerreL[α, x]]}}
```

Perhaps not so helpful because these are general functions (look them up in the Documentation Center). We can evaluate LaguerreL for integer α:

```
LaguerreL[2, x]
```

$$\frac{1}{2}\left(2 - 4x + x^2\right)$$

We can also do a series expansion of LaguerreL to check our answer. (Hint: you will need to use FullSimplify to get it in a useful form.)

To see the polynomial solutions from the diff. eq. we can solve with a specific integer value for α, such as α=2:

```
DSolve[x y''[x] + (1 - x) y'[x] + 2 y[x] == 0, y, x]
```

$$\left\{\left\{y \to \text{Function}\left[\{x\},\ (2 + (-4 + x)\, x)\, C[1] + \frac{1}{4} C[2] \left(3\, e^x - e^x x + 2\, \text{ExpIntegralEi}[x] - 4\, x\, \text{ExpIntegralEi}[x] + x^2\, \text{ExpIntegralEi}[x]\right)\right]\right\}\right\}$$

Ok, that gives our polynomial and another cryptic function: ExpIntegralEI (look it up!).
Define a function that lets us easily change the integer and test it out:

```
problem2[n_] := DSolve[x y''[x] + (1 - x) y'[x] + n y[x] == 0, y, x]
```

```
problem2[2]
```

$$\left\{\left\{y \to \text{Function}\left[\{x\},\ (2 + (-4 + x)\, x)\, C[1] + \frac{1}{4} C[2] \left(3\, e^x - e^x x + 2\, \text{ExpIntegralEi}[x] - 4\, x\, \text{ExpIntegralEi}[x] + x^2\, \text{ExpIntegralEi}[x]\right)\right]\right\}\right\}$$

Good, but awkward. Let's strip out the excess baggage:

```
problem2full[n_] := (y[x] /. DSolve[x y''[x] + (1 - x) y'[x] + n y[x] == 0, y, x])[[1]]
```

The [[1]] gets rid of the final {}'s. It refers to a part of the equation.
Look up Part in the help for examples of this very useful capability.

```
problem2full[2]
```

$$(2 + (-4 + x)\, x)\, C[1] + \frac{1}{4} C[2] \left(3\, e^x - e^x x + 2\, \text{ExpIntegralEi}[x] - 4\, x\, \text{ExpIntegralEi}[x] + x^2\, \text{ExpIntegralEi}[x]\right)$$

9/11/13

(6B)

```
? ExpIntegralEi
```

ExpIntegralEi[z] gives the exponential integral function Ei(z). ≫

Expand to see the logarithm (the solution with this log is discussed in the Arfken and Lea texts):

```
Simplify[Series[ExpIntegralEi[x], {x, 0, 4}], Assumptions → {x > 0}]
```

$$(\text{EulerGamma} + \text{Log}[x]) + x + \frac{x^2}{4} + \frac{x^3}{18} + \frac{x^4}{96} + O[x]^5$$

```
EulerGamma // N
```

0.577216

```
problem2full[1]
```

$$(-1+x)\,C[1] + C[2]\,(-e^x - \text{ExpIntegralEi}[x] + x\,\text{ExpIntegralEi}[x])$$

```
problem2full[2]
```

$$(2+(-4+x)\,x)\,C[1] + \frac{1}{4}C[2]\left(3\,e^x - e^x x + 2\,\text{ExpIntegralEi}[x] - 4\,x\,\text{ExpIntegralEi}[x] + x^2\,\text{ExpIntegralEi}[x]\right)$$

```
problem2full[3]
```

$$(-6+(-6+x)\,(-3+x)\,x)\,C[1] + \frac{1}{36}C[2]\left(-11\,e^x + 8\,e^x x - e^x x^2 - 6\,\text{ExpIntegralEi}[x] + 18\,x\,\text{ExpIntegralEi}[x] - 9\,x^2\,\text{ExpIntegralEi}[x] + x^3\,\text{ExpIntegralEi}[x]\right)$$

```
problem2full[4]
```

$$(24+(-4+x)\,x\,(24+(-12+x)\,x))\,C[1] + \frac{1}{576}C[2]\left(50\,e^x - 58\,e^x x + 15\,e^x x^2 - e^x x^3 + 24\,\text{ExpIntegralEi}[x] - 96\,x\,\text{ExpIntegralEi}[x] + 72\,x^2\,\text{ExpIntegralEi}[x] - 16\,x^3\,\text{ExpIntegralEi}[x] + x^4\,\text{ExpIntegralEi}[x]\right)$$

What about non-integers?

```
problem2full[1 / 2]
```

$$C[1]\,\text{HypergeometricU}\left[-\frac{1}{2}, 1, x\right] + C[2]\,\text{LaguerreL}\left[\frac{1}{2}, x\right]$$

PS#3 Problem 3

Ok, let's try problem 3

```
ans2 = DSolve[4 x^2 y''[x] + 4 x y'[x] + (4 x^2 - 1) y[x] == 0, y, x]
```

$$\left\{\left\{y \to \text{Function}\left[\{x\}, \frac{e^{-i x}C[1]}{\sqrt{x}} - \frac{i\,e^{i x}C[2]}{2\sqrt{x}}\right]\right\}\right\}$$

Pull out the function and convert to sines and cosines:

66

9/11/13

```
ExpToTrig[y[x] /. ans2 [[1]]] // Simplify
```

$$\frac{(2\,C[1] - i\,C[2])\,Cos[x] + (-2\,i\,C[1] + C[2])\,Sin[x]}{2\sqrt{x}}$$

Fourier Series : Pass 0:
- Use PhET Fourier : Making Waves
  - consider triangle, square wave, saw tooth
  - demonstrate the Wave Game ⇒ unique decomposition
- Demonstrate Mario's Fourier Scribble in Mathematica

9/17/13

Fourier Series : Pass 1

In our brief look at series solutions to differential equations, all of the equations were linear : only single powers of y and its derivatives.
- Jackson, in the introduction to his text on Classical Electrodynamics, argues that linear systems are all he needs to consider.
- More generally we have to consider scales and ensure that $y(x)$ doesn't get "unnaturally" large.

When does it work?

Linearity implies that given two solutions $y_1(x)$ and $y_2(x)$, the linear sum $c_1 y_1 + c_2 y_2$ is also a solution. Eg. for $y'' + \alpha y = 0$
- conversely, if we have something like $y'' + \alpha y^2 = 0$, the linear combination doesn't work because we have cross terms.

You recognize the physics of this property of linearity as the principle of superposition.
- It means we can make complicated solutions by adding together simple ones
- But also that we can decompose complicated solutions (essentially any we would encounter in a physics problem) into simple ones.

One choice of simple functions for $f(x)$ in $0 \le x \le 2\pi$, [also written $x \in [0, 2\pi]$] are sines and cosines, or e^{ixn}'s

$$\ast \quad \Rightarrow f(x) = \sum_{n=0}^{\infty} (a_n \sin nx + b_n \cos nx) = \sum_{n=-\infty}^{\infty} c_n e^{inx}$$

real

complex coefficients
- can also put $\frac{1}{\sqrt{2\pi}}$ explicitly here

Core competencies:
1. finding coefficients $\{a_n, b_n\}$ or $\{c_n\}$
2. basic examples: square wave, saw tooth → behavior of series (Gibb's overshoot)
3. solving certain diff eqs (which ones, how to do it)

Lead in!
We've used a series expansion of monomials (x^n), possibly with an overall non-integral x^α, for solutions to differential equations. The x^n are a basis. What about other choices for the basis?

9/11/13:

Fourier Series: Pass 2

- Consider the expansion of the function $f(x)$ first in the interval $0 \le x \le 2\pi$, later generalized to $0 \le x \le L$ or $-L \le x \le L$ (simple change of variable): [this is Lea's notation]

$$f(x) = \sum_{n=0}^{\infty} (a_n \sin nx + b_n \cos nx) = \sum_{n=-\infty}^{\infty} C_n e^{inx}$$

Unfortunately, Arfken uses a different notation (also Cahill):

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$, switching $a_n \Leftrightarrow b_n$, $n \geq 1$,

splitting out the constant term explicitly, and with a $1/2$ so that the formula below for a_n applies to all.
$\Rightarrow$ be careful when using formulas from the books!

- The function we want to Fourier analyze does not have to be periodic. If it is not, our Fourier series will only be a good representation in $[0, 2\pi]$.
  - The conditions on $f(x)$ sufficient for an expansion to exist is that it is "piecewise regular": only a finite # of finite discontinuities and a finite number of maxima and minima in $[0, 2\pi]$.
  - These hold for most physical problems.
  - The ability of a Fourier series to represent discontinuities is in contrast to Taylor series.
  - * Given a Fourier series, we can integrate term-by-term but be careful differentiating — it may no longer converge.

- Think of a Fourier series expansion in the larger context of orthogonal expansions: $f(x) = \sum_n c_n \phi_n(x)$ for a complete basis $\{\phi_n\}$.
More abstractly:
$|f\rangle = \sum_n |\phi_n\rangle\langle\phi_n|f\rangle$ with the coefficient $\langle\phi_n|f\rangle = \int \phi_n^*(x) f(x)\, dx$
calculated here in coordinate representation.

9/11/13

In the present case, the $\langle \phi_n | \phi_m \rangle$ integrals are:

m, n integers

$$\begin{cases} \int_0^{2\pi} (\sin nx)(\sin mx)\, dx = \pi \delta_{nm} \\ \int_0^{2\pi} (\cos nx)(\cos mx)\, dx = \pi \delta_{nm} \\ \int_0^{2\pi} (\sin nx)(\cos mx)\, dx = 0 \end{cases}$$

0 if $n \neq m$

- See the Mathematica notebook fourier_series.nb to verify these results. Other ways to do these integrals:
  - convert to exponentials [eg. $\sin \alpha = (e^{i\alpha} - e^{-i\alpha})/2i$] and integrate directly
  - use trig. identities to reduce to single sine or cosine, eg. $\sin\alpha \sin\beta = \frac{1}{2}[\cos(\alpha-\beta) - \cos(\alpha+\beta)]$

- Check the same plots (and do others) so the intuition of <u>why</u> they are orthogonal becomes clear.

- Observation: The coefficients of the Fourier series are like components of the vector $|f\rangle$ in a particular basis, so naturally we isolate the coefficient by the dot product $\langle \phi_n | f \rangle$.
  - We will see many more examples of basis functions later.

9/11/13

If we can exchange the order of summation and integration, then

$$a_m = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin mx\,dx \quad m \geq 1$$

$$b_m = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos mx\,dx \quad m \geq 1$$

$m=0 \quad b_0 = \frac{1}{2\pi}\int_0^{2\pi} f(x)\,dx$ [in Arfken this term is $\frac{1}{2}a_0$]

For a complex series,

$$c_n = \frac{1}{2\pi}\int_0^{2\pi} f(x)\,e^{-inx}\,dx$$ [same idea to project]

[If $f(x)$ is real, $c_n^* = c_{-n}$ holds.]

If we switch to $x \in [0, L]$, then $[x \to \frac{2\pi x}{L}]$

$$a_m = \frac{2}{L}\int_0^{L} f(x)\sin\left(n\frac{2\pi x}{L}\right)dx$$

$$b_m = \frac{2}{L}\int_0^{L} f(x)\cos\left(n\frac{2\pi x}{L}\right)dx$$

$b_0 = \frac{1}{L}\int_0^L f(x)\,dx$ average of $f(x)$

- normalization constant for a_m, b_m is 2/(length of interval)
- argument of sines, cosines is $k_n x$ where $k_n \equiv n\cdot\frac{2\pi}{L} = 2\pi/(L/n)$, $n \geq 1$

For a complex series $f(x) = \sum_{n=-\infty}^{\infty} c_n e^{in2\pi x/L}$

$$\Rightarrow c_n = \frac{1}{L}\int_0^L f(x)\,e^{-in2\pi x/L}\,dx$$

Recall: $x \in [0,L]$ $\quad f(x) = b_0 + \sum_{n=1}^{\infty} a_n \sin\left(\underbrace{n\frac{2\pi x}{L}}_{k_n x}\right) + \sum_{n=1}^{\infty} b_n \cos\left(\underbrace{n\frac{2\pi x}{L}}_{k_n x}\right)$

full cycles:

71

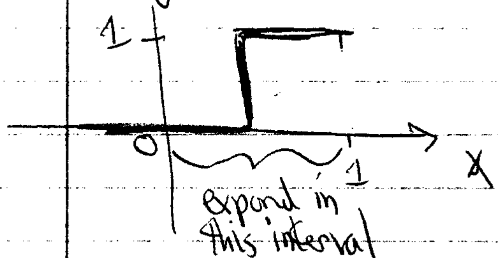
9/11/13

Apply to a step function: $\theta(x-\frac{1}{2})$ in $[0,1]$:

$a_n = \frac{2}{L}\int_0^L f(x)\sin\left(n\frac{2\pi x}{L}\right)dx$

$b_n = \frac{2}{L}\int_0^L f(x)\cos\left(n\frac{2\pi x}{L}\right)dx$

$b_0 = \frac{1}{L}\int_0^L f(x)\,dx$



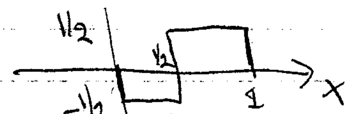
Check the following in the Mathematica fourier_series.nb notebook.

$$a_n = \frac{2}{1}\int_0^1 \theta(x-\tfrac{1}{2})\sin(2\pi n x) = 2\int_{1/2}^{1}\sin(2\pi n x)\,dx \overset{\text{Mathematica}}{=} \frac{(-1)+(-1)^n}{n\pi} = \begin{cases} 0 & \text{if } n \text{ even} \\ -\frac{2}{n\pi} & \text{if } n \text{ odd}\end{cases}$$

$$b_n = 2\int_{1/2}^{1}\cos(2\pi n x)\,dx \overset{\text{Mathematica}}{=} 0$$

[These are all even about $x=\frac{1}{2}$ but $f(x)$ is odd $\Rightarrow$ vanish].

$$b_0 = 1\int_{1/2}^{1} 1\,dx = 1/2$$ $\Rightarrow$ there is a constant offset (remove the average value)

$\Rightarrow$ 

is expanded in sines. Note what happens at $x=0,\frac{1}{2},1 \Rightarrow$ discontinuities

$$\Rightarrow f(x) = \theta(x-\tfrac{1}{2}) = \frac{1}{2} + \sum_{\substack{n=0\\ \text{odd}}}^{\infty}\left(-\frac{2}{\pi}\right)\frac{\sin(2\pi n x)}{n}$$

$$= \frac{1}{2} - \frac{2}{\pi}\sum_{m=0}^{\infty}\frac{\sin(2\pi(2m+1)x)}{2m+1}$$

- Look at Mathematica notebook to see how the function is built up. Try $n_{max} = 1, 3, 5, 10, 50, \ldots$

- Where there is a discontinuity in the function, the Fourier series evaluates to the midpoint of the jump $\Rightarrow$ $1/2$ at $x=0, 1/2, 1$

- Look at Gibbs overshoot $\Rightarrow$ value of peak is 1.179 (see Arfken $= \frac{2}{\pi}\int_0^{\pi}\frac{\sin t}{t}dt$) and doesn't improve. From the discontinuity $\Rightarrow$ be careful if your numerics in such places must be accurate.

9/11/13

We have two other examples in the Mathematica notebook to consider:

i) An inverted parabola $f(x) = 1-x^2$ in $[-1,1]$
ii) a half circle $f(x) = \sqrt{1-x^2}$ in $[-1,1]$

For these examples,

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right)dx$$

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right)dx$$

$$b_0 = \frac{1}{2L}\int_{-L}^{L} f(x)\,dx \quad \text{(average again)}$$

- Notice the patterns of convergence.
  $\Rightarrow$ next time consider the nature of the convergence of the series toward $f(x)$.

- Depending on whether $f(x)$ is even or odd, we can consider separate sine and cosine series.
  - The real issue is the <u>periodic extension</u> by the expansion outside the interval. See the discussion in Lea.

- <u>Parseval's theorem</u> says how the integral of the square of $f(x)$ is related to the sum of squares of coefficients. E.g.,

$$\frac{1}{2L}\int_{-L}^{L}|f(x)|^2 = \sum_{n=-\infty}^{\infty}|c_n|^2$$

cf. $\langle f|f\rangle = \sum_n \langle f|\phi_n\rangle\langle\phi_n|f\rangle \propto \sum_n |c_n|^2$