7701 Lecture 10

- Recap and extend:
  - Numerical solution of 2^{nd} order differential equation from page 63. $y'' + p(x)\,y' + q(x) = 0$
  - underlying concepts:
    - rewrite equation (or equations) as a set of coupled first-order differential equation
    - assume smooth functions $\Rightarrow$ if you know a function in one place, you can find it nearby by knowing derivatives and the differential equation tells you derivatives!

$$f(x_{n+1}) = f(x_n + h) = f(x_n) + h\left.\frac{df}{dx}\right|_{x_n} + O(h^2)$$

get from equation!

- Solve for discrete versions of functions: $f(x_0), f(x_1 = x_0 + h), f(x_2), \ldots f(x_N)$ $\equiv f_0, f_1, f_2, \ldots$
- Define $v(x) = y' \Rightarrow v' = y'' = -py' - q$

$$\Rightarrow \quad y_{n+1} = y_n + \left.\frac{dy}{dx}\right|_{x_n} \cdot h = y_n + v_n \cdot h + O(h^2)$$

$$v_{n+1} = v_n + \left.\frac{dv}{dx}\right|_{x_n} \cdot h = v_n + [-p(x_n)v_n - q(x_n)]h + O(h^2)$$

- How do we do better? (eliminate h^2 error, make h adaptive)

- Go over 68 – 71 on finding coefficients of Fourier expansions
  - larger context of orthogonal expansions (monomials are not orthogonal):

$$f(x) = \sum_n c_n \phi_n(x) \quad \text{for a complete orthogonal basis } \{\phi_n\}$$

- General features for sin, cos series:
  - coefficient is (normalization) $\times \int_{x_1}^{x_2} f(x) \begin{Bmatrix} \sin k_n x \\ \cos k_n x \end{Bmatrix} dx$ period $= x_2 - x_1$ (length of interval)

normalization $= \frac{2}{\text{period}}$

(wave number k_n) $\times$ (period) $= n \cdot 2\pi \Rightarrow k_n = \frac{2\pi}{\text{period}} \cdot n$

- Start differential equations with Fourier series

9/13/13

Using Fourier Series to Solve Diff. Eqs.

- We'll consider two representative examples.

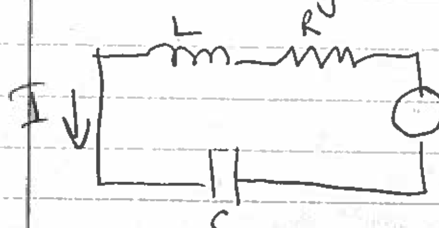
A. An inhomogeneous linear equation with a periodic driving term (periodic but not simply sinusoidal at one frequency)

B. Solution to a wave equation with fixed ends at $x=0, L$ and given an initial condition at $t=0$.

- Examples of A. are a damped, driven, harmonic oscillator:

$$\frac{d^2x}{dt^2} + \alpha\frac{dx}{dt} + k^2 x = f(t)$$

or the same system in electric circuit form: the RLC circuit



$\mathcal{E}(t) \Rightarrow L\frac{dI}{dt} + RI + \frac{Q}{C} = \mathcal{E}(t)$ by voltage drops

($Q(t)$ is the charge, $I(t)$ is the current)

- With $I = \frac{dQ}{dt} \Rightarrow \frac{d^2Q}{dt^2} + \left(\frac{R}{L}\right)\frac{dQ}{dt} + \frac{Q(t)}{LC} = \frac{\mathcal{E}(t)}{L} \equiv \ddot{Q} + 2\alpha\dot{Q} + \bar{\omega}^2 Q$

natural frequency

- First consider if $\mathcal{E}(t)$ has a single frequency. We could take $\mathcal{E}(t) = \mathcal{E}_0\sin(\omega t)$, but this is awkward with both first and second derivatives $\Rightarrow$ use exponentials $\Rightarrow \mathcal{E}(t) = \mathcal{E}_0 e^{i\omega t}$.

initially: constant coefficient equation solved by e^{st} with $\mathrm{Re}(s)<0 \Rightarrow$ damped and $\Rightarrow$ transients,

  - The equation is linear, so we expect the response to be at the same frequency in steady state $\therefore Q(t) \propto e^{i\omega t}$ will be a solution to the inhomogeneous equation. (Here: let transients die out; find steady-state solution.)
  - General solution includes many frequencies.
  - Linear is critical: if Q^2 appears, then not all terms have the same exponential $\Rightarrow$ can't factor $e^{i\omega t}$ out.

9/13/13

- But if it is linear, then we can solve for a periodic but otherwise general $\mathcal{E}(t)$ by Fourier decomposing $\mathcal{E}(t)$ into frequency components:

$$\mathcal{E}(t) = \sum_{n=-\infty}^{\infty} \mathcal{E}_n e^{i\omega_n t} \quad \text{where } \omega_n \equiv \frac{2\pi n}{\tau}, \ \tau = \text{the period}$$

So the initial problem is to find the $\mathcal{E}_n$ given the form of $\mathcal{E}(t)$.

E.g., a square wave, as in Lea's example:

$$\mathcal{E}(t) = \mathcal{E}_0\left(\frac{1}{2} + \frac{1}{\pi}\sum_{\substack{n=-\infty \\ n\ \text{odd}}}^{\infty} \frac{i}{n} e^{i\omega_n t}\right)$$

Then we take $Q(t) = \sum_{n=-\infty}^{\infty} q_n e^{i\omega_n t}$

substitute into the diff. eq. and equate terms with the same ω_m.

- In detail, we project the m^{th} term by multiplying the full equation by $e^{i\omega_m t}$ and integrating over t for 0 to τ
  $\Rightarrow$ only $n=m$ survives.

$$\Rightarrow \sum_{n=-\infty}^{\infty} -\omega_n^2 q_n e^{i\omega_n t} + 2\alpha \sum_{n=-\infty}^{\infty} i\omega_n q_n e^{i\omega_n t} + \bar{\omega}^2 \sum_{n=-\infty}^{\infty} q_n e^{i\omega_n t} = \frac{1}{L}\sum_{n=-\infty}^{\infty} \mathcal{E}_n e^{i\omega_n t}$$

$$\Rightarrow \left(-\omega_m^2 + i2\alpha\omega_m + \bar{\omega}^2\right) q_m = \mathcal{E}_m/L \Rightarrow \text{solve for } q_m$$

$$q_m = \frac{\mathcal{E}_m/L}{\bar{\omega}^2 - \omega_m^2 + i2\alpha\omega_m} = \frac{\frac{\mathcal{E}_m}{L}(\bar{\omega}^2 - \omega_m^2 - i2\alpha\omega_m)}{(\bar{\omega}^2-\omega_m^2)^2 + 4\alpha^2\omega_m^2}$$

$\leftarrow$ resonance when $\omega_m \approx \bar{\omega}$ natural frequency

- See Lea for the square wave solution. When the $\mathcal{E}_m$ are inserted in the q_m equation, we get real quantities.
  - $\omega_{-m} = -\omega_m$ and $(\omega_m)^2 = \omega_m^2$ so we separate off
    $(\bar{\omega}^2 - \omega_m^2)(e^{i\omega_m t} + e^{-i\omega_m t}) = 2(\bar{\omega}^2 - \omega_m^2)\cos\omega_m t$ and $-i2\alpha\omega_m(e^{i\omega_m t} - e^{-i\omega_m t}) = 4\alpha\omega_m \sin\omega_m t$

9/13/13

For problems of type B, we consider the wave equation for a string attached (fixed ends) at $x=0$ and $x=L$. At $t=0$ we pluck it in the middle and let it go. This establishes the initial conditions $y(x,t=0)$ and $\dot{y}(x,0)=0$.

· For example:

· The wave equation is $v^2 \frac{\partial^2 y}{\partial x^2} = \frac{\partial^2 y}{\partial t^2}$ where the wave velocity v in this case depends on the mass density and the tension.

Goal: find $y(x,t)$ for all t and $0 \leq x < L$.

Plan: Apply separation of variables and then insert a general Fourier series for the time dependence after Fourier analyzing $y(x,0)$

Separation of variables is based on the observation that the ansatz

$$y(x,t) = A(x)\,B(t)$$

results in two separated equations for the x and t dependence.

$$v^2 \frac{\partial^2 y}{\partial x^2} = \frac{\partial^2 y}{\partial t^2} \Rightarrow v^2 A'' B = A B'' \Rightarrow v^2 \underbrace{\frac{A''}{A}}_{\text{only } x \text{ dependence}} = \underbrace{\frac{B''}{B}}_{\text{only } t \text{ dependence}}$$

Key: $\frac{A''}{A}$ only depends on x while $\frac{B''}{B}$ only depends on t, so each separately must be a constant independent of x and t!

$\Rightarrow \frac{A''}{A} = -k^2 \Rightarrow A = \sin k_n x$ with $k = k_n = \frac{n\pi}{L}$ to vanish at $x=0$, $x=L$.

Then $\frac{B''}{B} = -k_n^2 v^2 = -\left(\frac{n\pi}{L}\right)^2 v^2 \equiv -\omega_n^2 \Rightarrow \sin(\omega_n t), \cos(\omega_n t)$ are solutions. $\boxed{\omega_n = \frac{n\pi v}{L}}$

$\Rightarrow$ general solution $\boxed{y(x,t) = \sum_{n=1}^{\infty} \sin\left(\frac{n\pi x}{L}\right)\left[a_n \sin(\omega_n t) + b_n \cos(\omega_n t)\right]}$

boundary conditions fix this

9/13/13

Note that a_n, b_n absorb any overall coefficient that we might have put with $\sin(k_n x)$.

· The coefficients a_n, b_n are determined completely by the initial conditions at $t=0$!

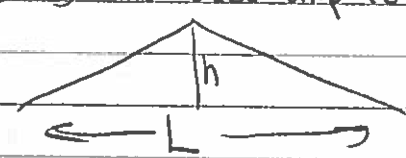
· Pluck from rest $\Rightarrow \left.\frac{dy}{dt}\right|_{t=0} = 0 \Rightarrow a_n = 0$ for all n.

Find b_n from: $y(x,0) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right)$

(same Fourier decomposition problem considered earlier!)

· Suppose we start with a simple half sine wave: $\sin\left(\frac{\pi x}{L}\right)$.
 · We expect $\cos(\omega t)$, with $\frac{\omega}{k} = v$ or $\omega = kv$ ← $k = \frac{\pi}{L}$
 · It stays a sine wave, with $\cos(\omega t)$ modulated amplitude.

· Now example 4.4 in Lea.



at $t=0$, at rest

See the Mathematica notebook Fourier series.nb for the calculation of the coefficients.

Full solution!

$$\boxed{y(x,t) = \frac{8h}{\pi^2} \sum_{n=1}^{\infty} \frac{(-1)^{(n-1)/2}}{n^2} \sin\left(\frac{n\pi x}{L}\right) \cos\left(n\frac{\pi v t}{L}\right) \quad n \text{ odd}}$$

· only harmonics even about the middle of the string
 · Note: we could put the origin in the middle. Might be more convenient, but it is not necessary to solve the problem.

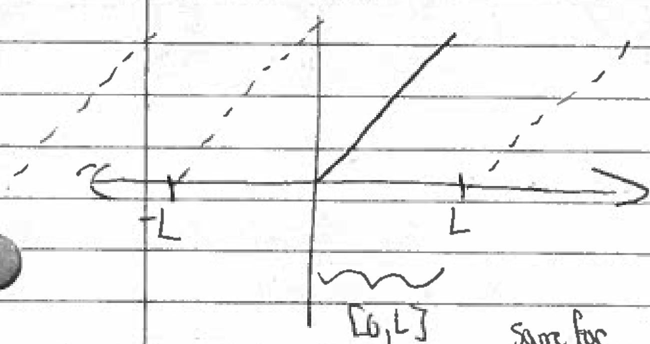
9/13/13 Periodic Extensions

For other intervals, just change variables to get $\sin k_n x$, $\cos k_n x$, $e^{\pm i k_n x}$

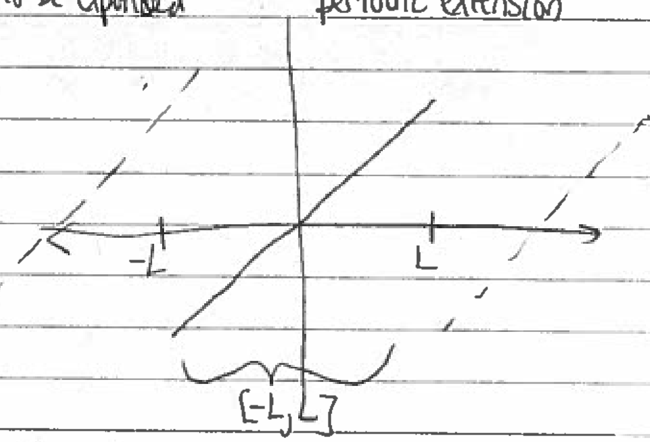
If $0 < x < L$ $\quad a_m = \frac{2}{L}\int_0^L f(x)\sin\left(n\frac{2\pi x}{L}\right)dx$ $\quad$ period L

If $-L < x < L$ $\quad a_m = \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right)dx$ $\quad$ period $2L$

Consider $f(x) = x$ — function to be expanded - - - periodic extension



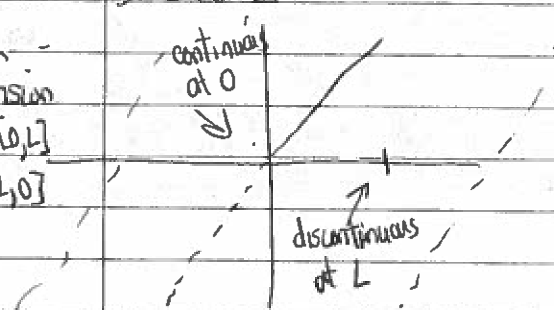
Same for odd function



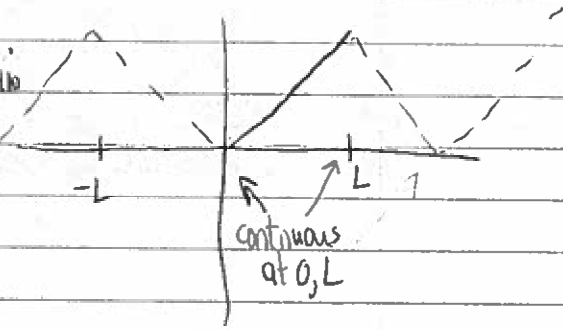
Sine series

Cosine series

odd extension from [0,L] to [-L,0]



even extension from [0,L] to [-L,0]



$f(x) = \sum_{n=1}^{\infty} a_n \sin k_n x \quad k_n = \frac{\pi n}{L}$

$a_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}\,dx$

$b_n = 0$

$f(x) = b_0 + \sum_{n=1}^{\infty} b_n \cos\frac{n\pi x}{L}$

$b_n = \frac{2}{L}\int_0^L f(x)\cos\frac{n\pi x}{L}\,dx$

$b_0 = \frac{1}{L}\int_0^L f(x)\,dx$

Look at fourier_series2.nb examples.