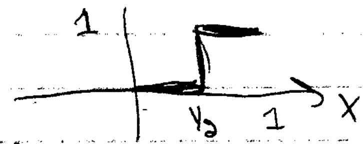
Try out new Fourier series demonstrator

9/16/13

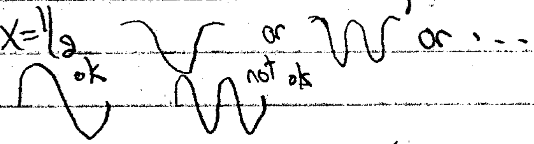
7701 Lecture 11



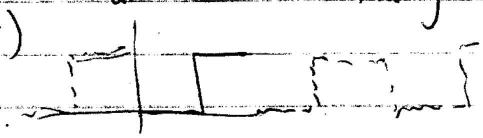
- Recap of step function expansion

$$\theta(x-\tfrac{1}{2}) = \frac{1}{2} + \sum_{\substack{n=0 \\ \text{odd}}}^{\infty} \left(-\frac{2}{\pi}\right)\left(\frac{1}{n}\right) \sin(2\pi n x) = f(x)$$

- only "cos" term is $b_0 = \frac{1}{1} \cdot \int_{1/2}^{1} 1\,dx = \frac{1}{2} \Rightarrow$ constant off set (average)
  (all other terms integrate to zero)
- Other "cos" terms symmetric about $x = 1/2$ … or … or …
- Only every other sine term … ok … not ok



- at $x = \frac{1}{2}, 0, 1$, $\sin(2\pi n \frac{1}{2}) = 0$ for $\underline{n \text{ odd}}$ $\Rightarrow f(\frac{1}{2}) = \frac{1}{2}$ at discontinuity
  $\Rightarrow$ half way between $f(\frac{1}{2}^-)$ and $f(\frac{1}{2}^+)$
- What is the periodic extension?



- Look at other examples in Mathematica fourier_series.nb notebook
  $\Rightarrow$ detailed on 72
  - predict which coefficients will be zero.

- Step through inhomogeneous linear equation (page 80)

- Continue with guitar string example 76 – 77 plus Mathematica notebook fourier_series3.nb

- Continue with extras as time permits

square wave example: $\mathcal{E}(t) = A_0\left(\frac{1}{2} + \frac{1}{\pi}\sum_{n=-\infty}^{\infty} \frac{i}{n} e^{i\omega_n t}\right)$, $\omega_n = \frac{2\pi n}{\tau}$ ← period (80)

$\Rightarrow \mathcal{E}_0 = \frac{A_0}{2}$; $\mathcal{E}_n = iA_0 \frac{1}{\pi n}$

9/16/13

Follow-up on solving inhomogeneous linear equation

$$\ddot{x} + \alpha\dot{x} + k^2 x = f(t) \quad \text{for } x(t)$$

or

$\boxed{\mathcal{E}(t) \text{ is periodic with period } \tau}$

charge in RLC circuit $\ddot{Q} + 2\alpha\dot{Q} + \bar{\omega}^2 Q = \frac{\mathcal{E}(t)}{L}$ for $Q(t)$ $\left(\frac{R}{L} = 2\alpha,\ \frac{1}{LC} = \bar{\omega}^2\right)$

but doesn't mean just one frequency

- If no driving term, then constant-coefficient equation solved by e^{st} and we would find $\mathrm{Re}(s) < 0 \Rightarrow$ damped and therefore transient.

- Here: assume it has died out $\Rightarrow$ overall frequency at driving frequency of $f(t)$ or $\mathcal{E}(t)$.
  - If one frequency $\mathcal{E}(t) = \mathcal{E}_0 e^{i\omega t}$ (can always recover $\sin(\omega t)$ by Im part)
  - Linear so response at same ω $\Rightarrow Q(t) \propto e^{i\omega t}$

Note: $\frac{1}{\tau}\int_0^\tau e^{i(\omega_n - \omega_m)\tau} d\tau = \delta_{nm}$

Plan: $\mathcal{E}(t)$ is given, so resolve into Fourier series: $\mathcal{E}(t) = \sum_{n=-\infty}^{\infty} \mathcal{E}_n e^{i\omega_n t}$, $\omega_n = \frac{2\pi n}{\tau}$ ← period

- Substitute and project each mode $Q(t) = \sum_{n=-\infty}^{\infty} q_n e^{i\omega_n t}$

project by multiplying by $e^{-i\omega_m t}$ and integrating over t from 0 to τ $\Rightarrow$ only $n=m$ survives

$\Rightarrow (-\omega_m^2 + i2\alpha\omega_m + \bar{\omega}^2)\, q_m = \mathcal{E}_m / L \Rightarrow$ solve for q_m $\Rightarrow$ done!

Uses $\int_0^\tau e^{-i\omega_m t} e^{i\omega_n t} = \tau\delta_{nm}$. If $n=m$, $\int_0^\tau 1\, dt = \tau$; check $n \neq m$ vanishes.

$$\boxed{q_m \overset{m\neq 0}{=} \frac{\mathcal{E}_m / L}{\bar{\omega}^2 - \omega_m^2 + i2\alpha\omega_m} = \frac{\mathcal{E}_m/L\,(\bar{\omega}^2 - \omega_m^2 - i2\alpha\omega_m)}{(\bar{\omega}^2 - \omega_m^2)^2 + 4\alpha^2\omega_m^2}}$$ and $\boxed{q_0 = \frac{\mathcal{E}_0}{L\bar{\omega}^2} = \frac{A_0 C}{2}}$

$\omega_{-m} = -\omega_m$ and $(\omega_{-m})^2 = \omega_m^2 \Rightarrow$ combine $+m$ and $-m$ for $m > 0$

resonance when $\omega_m \simeq \bar{\omega}$ ← natural frequency

$\Rightarrow (\bar{\omega}^2 - \omega_m^2)(e^{i\omega_m t} + e^{-i\omega_m t}) = 2(\bar{\omega}^2 - \omega_m^2)\cos\omega_m t$

$-2i\alpha\omega_m (e^{i\omega_m t} - e^{-i\omega_m t}) = 4\alpha\,\omega_m \sin\omega_m t$

$\Rightarrow$ Full ~~F~~ series of sin's and cos's $\Rightarrow$ real

9/16/13

Convergence of Fourier Series

When does the Fourier series for a function $f(x)$ converge?
In general terms, if it is sufficiently smooth.

- If $f(x)$ is continuous, it will be uniformly convergent
- But if on $-L<x<L$ it is piecewise "very smooth" then it will still converge pointwise to $\frac{1}{2}[f(x^+)+f(x^-)]$ at each x (this is the average value of discontinuities that we've already seen)
- Piecewise very smooth means at least the first two derivatives exist and 2^{nd} is continuous everywhere except for a finite number of points.

You might think intuitively that the best representation of a function by sines and cosines would depend on how many basis elements N are used.

- This would implies different sets of coefficients $\{a_n, b_n\}$ depending on N. Equivalently, that $\{c_n\}$ could be different.
- But this contradicts our contruction of the basis, which dictates unique coefficients for a given n, independent of the other n or the total N.
- So in what sense can we say we have a best fit?

- Answer: In a least squares sense. If we consider the deviation

details in Lea text

$$R_N = \int_{-L}^{L} \left| f(x) - \sum_{n=-N}^{N} c_n e^{in\pi x/L} \right|^2 dx \quad \text{(sum of deviations squared)},$$

then at any N, minimizing R_N by $\frac{\partial R_N}{\partial c_k}=0$ for all k leads to $c_k = \frac{1}{2L}\int_{-L}^{L} f(x) e^{-ik\pi x/L} dx$, which is the usual coefficient! Also, $R_N \underset{N\to\infty}{\longrightarrow} 0$, although isolated points can differ.

Parseval's Theorem: $\frac{1}{2L}\int_{-L}^{L} |f(x)|^2 dx = \sum_{n=-\infty}^{\infty} |c_n|^2 \underset{\text{RLC circuit}}{\overset{\text{physics}}{\Longrightarrow}} U = \frac{1}{\tau}\int_{0}^{\tau} \frac{Q(t)^2}{2C} dt = \frac{1}{2C}\sum_n |Q_n|^2$ ← add up energy in each mode to get total slightly altered

9/16/13

Lea problem 22: Generalized Parseval Theorem:

If $f(x)$ is represented by the series $\sum_n f_n e^{inx}$ over the interval $0<x<2\pi$ and $g(x)=\sum_n g_n e^{inx}$ over the same range, prove that (for f and g real)

$$\frac{1}{2\pi}\int_0^{2\pi} f^*(x)\, g(x)\, dx = \sum_{n=-\infty}^{\infty} f_n^* g_n$$

Proof:

Substitute expansions in to the integral

$$\frac{1}{2\pi}\int_0^{2\pi}\left(\sum_n f_n^* e^{-inx}\right)\left(\sum_m g_m e^{imx}\right) dx$$

assume we can interchange sums and integrals

$$= \sum_n\sum_m f_n^* g_m \frac{1}{2\pi}\int_0^{2\pi} e^{-inx} e^{imx}\, dx$$

$$\frac{1}{i(m-n)} e^{i(m-n)x}\Big|_0^{2\pi} \quad \text{if } m\neq n$$

$$x\Big|_0^{2\pi} \quad \text{if } m=n$$

$$\frac{1}{2\pi}\frac{1}{i(m-n)}\left(e^{i(m-n)2\pi}-e^0\right) \quad \text{if } m\neq n$$

0

integer

$$2\pi \quad \text{if } m=n$$

$$= \sum_n\sum_m f_n^* g_m \delta_{mn}$$

$$= \sum_n f_n^* g_n \qquad \text{QED}$$

Suppose $f(x)=g(x) \Rightarrow \frac{1}{2\pi}\int_0^{2\pi} |f(x)|^2 dx = \sum_{n=-\infty}^{\infty} |f_n|^2$

cf. normalization and coefficients of basis in quantum mechanics

More abstract $\langle f|g\rangle = \frac{1}{2\pi}\int_0^{2\pi}\langle f|x\rangle\langle x|g\rangle dx = \sum_n \langle f|\phi_n\rangle\langle\phi_n|g\rangle$

$\frac{1}{2\pi}\int_0^{2\pi}|x\rangle\langle x|\, dx = 1$ $\quad 1=\sum_n|\phi_n\rangle\langle\phi_n|$ $\quad f_n^*$ $\quad g_n$

9/16/13

Fourier Series as Basis Expansion (Recap)

Recall the generic procedure we're thinking about:
If we have a complete basis $\{\phi_n(x)\}$ then we can expand

$$f(x) = \sum_n c_n \phi_n(x)$$

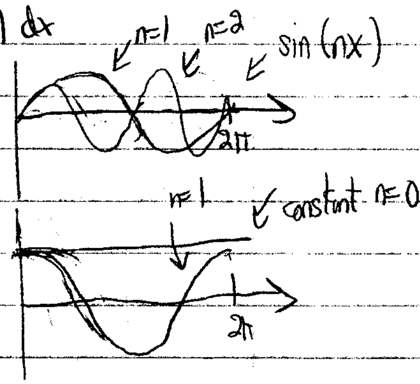
with unique constant coefficients c_n.

Cf. spatial vectors $\vec{V} = \hat{x} V_1 + \hat{y} V_2 + \hat{z} V_3 \rightarrow (V_1, V_2, V_3)$
$\Rightarrow$ c_n are like the coordinates V_i
· the ϕ_n are like orthogonal vectors $\hat{x}, \hat{y}, \hat{z}$

$$\hat{x}\cdot\vec{V} = \cancel{\hat{x}\cdot\hat{x}}^{\,1} V_1 + \hat{x}\cdot\hat{y} V_2 + \hat{x}\cdot\hat{z} V_3 = V_1$$

As in QM: $|f\rangle = \underbrace{\sum_n |\phi_n\rangle\langle\phi_n|}_{\text{completeness} \Rightarrow \text{identity}} f\rangle \Rightarrow \langle\phi_n|f\rangle = \int \phi_n^*(x) f(x)\, dx$

$\int |x\rangle\langle x|\, dx$



· for fourier series on $x \in [0, 2\pi]$

$$f(x) = \sum_{n=0}^{\infty} a_n \sin nx + b_n \cos nx$$

$$\sin nx = \frac{e^{inx} - e^{-inx}}{2i} \qquad \cos nx = \frac{e^{inx} + e^{-inx}}{2}$$

$$= \sum_{n=0}^{\infty} \underbrace{\left(\frac{a_n}{2i} + \frac{b_n}{2}\right)}_{c_n} e^{inx} + \underbrace{\left(\frac{-a_n}{2i} + \frac{b_n}{2}\right)}_{c_{-n}} e^{-inx} = \sum_{n=-\infty}^{\infty} c_n e^{inx}$$

(note $c_{-n} = c_n^*$ if a_n, b_n are real)

and $a_m = \frac{1}{\pi}\int_0^{2\pi} f(x) \sin mx\, dx \quad m \geq 1$

$b_m = \frac{1}{\pi}\int_0^{2\pi} f(x) \cos mx\, dx \quad m \geq 1, \quad b_0 = \frac{1}{2\pi}\int_0^{2\pi} f(x)\, dx$

$\Bigg\}\ c_n = \frac{1}{2\pi}\int_0^{2\pi} f(x) e^{-inx}\, dx$

9/16/13

Consider Lea problem 4.4:

An odd function $f(x)$ on the range $(-L, L)$ has the additional property that $f(x+L) = -f(x)$

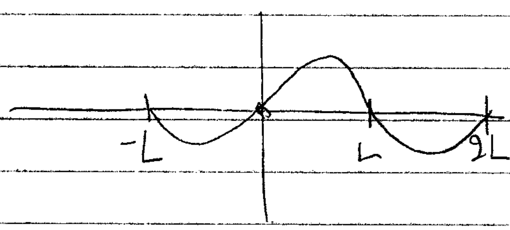
a) make a sketch showing important features of the function

b) Which kind on Fourier series (sine, cosine, or full) represents this function on the range $-L \le x < L$?

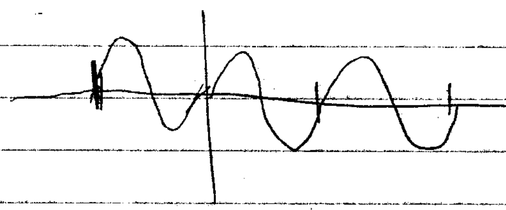
c) Show that the series has only terms of odd order $(n=2m+1)$ and find a formula for the coefficients as an integral over $0 \le x \le L/2$.

d) Repeat with $f(x+L) = +f(x)$

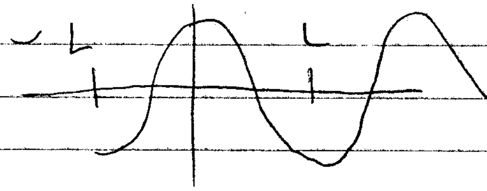
e) Repeat with even functions and $f(x+L) = -f(x)$.

a) 

b),c) sine series with only odd terms since even about $x = \frac{L}{2}$
$\Rightarrow$ determined by $[0, \frac{L}{2}]$ integral

d) 

$f(x+L) = +f(x)$
$\Rightarrow$ sine series with only <u>even</u> terms

e) 

$f(x+L) = -f(x)$ but even
$\Rightarrow$ cosine series with odd terms
(~~to~~ about $f(x+L) = f(x)$)