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7701 Lecture 12

- PS#3 returned — average 52/60 — we will review problems
- Quick comments on integration by parts and undetermined coefficients
- Gibb's overshoot via Mathematica (86) (Risch algorithm)
- Convergence of Fourier series (81)
- Parseval theorem — generalized (82)
- Gamma function (60)
- Comments on midterm (87)

- For PS#4 we needed integrals like $\int_a^b x^n \sin x\, dx$
  - OK to use Mathematica but must know how to do them "by hand."
- Usual approach: integration by parts (see XKCD cartoon).

$$\int u\, dv = uv - \int v\, du \Rightarrow u(x), v(x) \quad dv = \left(\frac{dv}{dx}\right) dx$$

$$\text{eg. } \int_a^b \underbrace{x}_{u} \underbrace{\sin x\, dx}_{dv} = x(-\cos x)\Big|_a^b - \int_a^b (-\cos x)\, dx = \left[-x\cos x + \sin x\right]\Big|_a^b$$

$\Rightarrow du = dx,\ v = -\cos x$

- Integration by parts is usually taught in beginning calculus as one of the "tricks" like trigonometric substitutions.
  - But much more important. How to think about it?

$$\underbrace{\int_a^b \frac{d}{dx}(uv)\, dx}_{\text{volume or interior}} = \underbrace{uv\Big|_a^b}_{\text{surface or interior}} = \int_a^b \frac{du}{dx}\cdot v\, dx + \int_a^b u\cdot\frac{dv}{dx}\, dx$$

generalize in vector calculus: eg., $\underbrace{\int_V \vec{\nabla}\cdot\vec{A}\, d^3x}_{\text{interior of volume}} = \underbrace{\int_S \vec{A}\cdot\hat{n}\, dS}_{\text{surface of volume}}$

- If surface term vanishes (eg. from boundary conditions), then

$$\int u(x)\frac{d}{dx}v(x)\, dx = -\int \frac{du}{dx} v(x)\, dx \Rightarrow \text{"move derivative at cost of minus sign"}$$

- In QM, $-\frac{\hbar^2}{2m}\frac{d^2}{dx^2} \Rightarrow$ no change $\Rightarrow$ "Hermitian"

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- later: "self-adjoint" differential operators.

- Alternative to integration by parts ⇒ method of undetermined coefficients (Risch algorithm used by Mathematica)
  - key observations: derivatives are really easy to program symbolically and pattern matching is also easy.

- Use Mathematica to calculate a few (indefinite) integrals:

$$\int x \sin x \, dx = -x \cos x + \sin x$$
$$\int x^2 \sin x \, dx = (-x^2+2)\cos x + 2x \sin x$$
$$\int x^2 \cos x \, dx = 2x \cos x + (x^2-2)\sin x$$

and so on.

- Basic idea: the general form of the right side is well defined for integrals of rational functions of polynomials and trig functions.
  - assign coefficients to general answer, differentiate, match coefficients

e.g.,
$$\int x^2 \sin x \, dx = (\underbrace{a_2 x^2}_{\text{max } x^2} + a_1 x + a_0)\cos x + (\underbrace{b_1 x}_{\text{max } x} + b_0)\sin x$$

$\frac{d}{dx}$ of right side: $(2a_2 x + a_1)\cos x - (a_2 x^2 + a_1 x + a_0)\sin x + b_1 \sin x + (b_1 x + b_0)\cos x$
$= x^2 \sin x$

Matching:

$x \cos x$: $2a_2 + b_1 = 0$
$\cos x$: $a_1 + b_0 = 0$
$x^2 \sin x$: $-a_2 = 1$
$x \sin x$: $-a_1 = 0$
$\sin x$: $-a_0 + b_1 = 0$

$\Rightarrow$ $a_1 = 0 \rightarrow b_0 = 0$
$\rightarrow a_2 = -1 \rightarrow b_1 = 2 \Rightarrow a_0 = 2$

$\Rightarrow \int x^2 \sin x \, dx = (-x^2+2)\cos x + 2x \sin x$ ✓

- Many other analogous applications!
  - e.g. differential equations.

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Gibb's Overshoot

- See Arfken 19.3 for detailed analysis.
- Here: physics-style analysis using Mathematica.

- Follow along gibbs_overshoot.nb notebook to explore.

- Historical note:
  - The story is that Albert Michelson, famous for measuring the speed of light and the Michelson-Morley interferometer experiment, built a machine in 1898 to calculate Fourier coefficients. When series resummed, worked well except for square wave, which had persistent wiggles near discontinuities, which wouldn't go away. He suspected machine error, but J. Willard Gibbs explained they were real: The Fourier series overshoots by about 9% near the discontinuity, becoming narrow with more terms, but always there!

- We do the sum in Mathematica and find the claimed result:

$$\lim_{N\to\infty} \frac{2}{\pi}\left[\frac{\sin\left(\frac{1\cdot\pi}{N}\right)}{1} + \frac{\sin\left(\frac{3\cdot\pi}{N}\right)}{3} + \frac{\sin\left(\frac{5\pi}{N}\right)}{5} + \dots \frac{\sin\frac{(N-1)\pi}{N}}{N-1}\right] \doteq \frac{1}{2} + 0.08949$$

9% of 1 overshoot

- Arfken finds this number from

$$\frac{1}{\pi}\int_0^{\pi} \frac{\sin(t)}{t}\,dt \doteq \frac{1}{2} + 0.08949$$

- Can you see how the sum is an approximation to the integral that becomes exact as $N\to\infty$?
  (answer next time!)

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Comments on the upcoming midterm

- Tuesday evening, 9/24. Design is for two-hour exam, but you will not be timed.

- Two types: basic problems, synthetic problems
  - basics: testing for core competencies
    - what everyone needs to know to pass
    - e.g., δ_{ij} and ε_{ijk} manipulations for proofs of vector identities; knowing different types of singularities; complex manipulations (like taking n^{th} roots); applying Frobenius

      what does "analytic" entail?
    - "Spot the error!"

  Aside: What functions have branch points? $f(z) = (z-z_0)^\alpha$ ($\alpha \neq$ integer), $\ln(z-z_0)$
  - Non-analytic at $z_0 \Rightarrow$ Taylor series doesn't exist. Seen here by derivatives (eventually) blowing up at z_0 eg.

    $f(z) = (z-z_0)^{3/2}$, $f'(z) = 3/2(z-z_0)^{1/2}$ $f''(z) = \frac{3}{2}\cdot\frac{1}{2}(z-z_0)^{-1/2} \Rightarrow f''(z_0) = \infty$

  - synthetic: putting ideas/techniques together; applying to new situations

- Based on topics seen in homework problems
  - themes, setting up problems, applying tools like δ_{ij}, ε_{ijk}, definitions and notation

- You will be given Jackson covers; any needed integrals like $\int_0^\infty e^{-x^2}dx$
- You will not be given (must know these!)

  $e^{iz} = \cos z + i\sin z = 1 + z + \frac{z^2}{2!} + \ldots$ $\qquad (1+z)^\alpha = 1 + \alpha z + \ldots \quad |z| < 1$

  $\varepsilon_{abc}\varepsilon_{ade} = \delta_{bd}\delta_{ce} - \delta_{be}\delta_{cd}$ "All numbers in ε_{abc} must be different. If first indices are the same, then either the next two and the last two are the same; or the 2^{nd} and 3^{rd} of each are the same, with a minus sign."

  - CR relations, shouldn't have to memorize.

- How to prepare:
  - Go over PS#1-4 and make sure you can set up every problem without looking at solutions
  - Check solutions and lecture notes for emphasis points. Remember the picky points taken off on homework and emphasized in class

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Midterm Basic Problems: Examples

What is?

$\varepsilon_{abc}\varepsilon_{def}$ = [$\varepsilon_{abc}\varepsilon_{def}$ no repeated indices!]

$\varepsilon_{abc}\varepsilon_{dec}$ = [$(\delta_{ad}\delta_{be} - \delta_{ae}\delta_{bd})$ standard identity]

$\delta_{ab}\varepsilon_{cab}$ = [$\varepsilon_{caa} = 0$ because $\varepsilon_{ijk} = 0$ for repeated indices]

Show

$\vec{A}\cdot(\vec{B}\times\vec{C}) = \vec{C}\cdot(\vec{A}\times\vec{B})$ assuming these commute

$$= A_a(B\times C)_a = \varepsilon_{abc}A_aB_bC_c = \varepsilon_{abc}C_cA_aB_b$$
$$= \varepsilon_{cab}C_cA_aB_b = C_c(\vec{A}\times\vec{B})_c$$
$$= \vec{C}\cdot(\vec{A}\times\vec{B}) \quad \text{QED}$$

$\vec{\nabla}\times(\vec{\nabla}\phi) = 0 \Rightarrow (\nabla\times\vec{\nabla}\phi)_a = \underbrace{\varepsilon_{abc}}_{\text{antisymmetric}}\underbrace{\partial_b\partial_c}_{\text{symmetric}}\phi = 0$ (or $= \varepsilon_{acb}\partial_c\partial_b\phi = -\varepsilon_{abc}\partial_b\partial_c\phi = 0$)

Fourier series:

- apply (given) formulas for coefficients
- solve differential equations of the two types on PS#4, including carrying out separation of variables.

Spot the Error!

$\nabla\times(\vec{B}\cdot\vec{C}) = 0$ [$\vec{B}\cdot\vec{C}$ is a scalar, not a vector as need for cross product]

$(A\times B)_a + (C\times D)_d = \varepsilon_{abc}A_bB_c + \varepsilon_{def}C_eD_f$ [The two vectors must have the same index rather than a and d]

What are the singularities of $\dfrac{\sqrt{z^2+3z+2}}{\sin z}$ and what type are they?

[$z^2+3z+2 = (z+2)(z+1) \Rightarrow \sqrt{z+2}\sqrt{z+1}$ have branch points at $z=-2$ and $z=-1$

$\sin z$ has zeros on the x axis for $y=0, x=n\pi$, which mean simple poles]

What is radius of convergence about $z=i$? $z=-1+i$?

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Would you close in the upper or lower half plane when doing the contour integral for $\int_{-\infty}^{\infty} \frac{dx}{1+x^2}$?

[Either. For $z = Re^{i\theta}$ the integral over a semicircle goes to zero as $R/R^2 = 1/R$ for $R \to \infty$ in either half plane]

What contour would you choose for $\int_{-\infty}^{\infty} \frac{\sin x}{x}\, dx$? (bonus problem)

[Split into $\left(\frac{e^{ix}}{x} - \frac{e^{-ix}}{x}\right)/2i$ and define each with a principal value. Jordan's lemma says we can close the first in the upper half plane and the second in the lower half plane and the integral over the large semi-circle vanishes in each case.]

Find the residue of a) $\frac{e^{\pi i z}}{z(z-ia)}$ at $z=0$, $z=ia$ $\left[Res(0) = \frac{1}{-ia} = \frac{i}{a};\ Res(ia) = e^{\pi a}/ia \right]$

b) $\frac{2z}{z^4-1}$ at $z=i$ $\left[\text{Use } Res(i) = \frac{g(i)}{h'(i)} = \frac{2i}{4i^3} = -\frac{1}{2}\right]$

- For differential equations, be prepared to
  a) apply the Frobenius method to derive a recurrence relations (e.g. $a_{m+2} = \{\ \} a_m$) and find the first few terms explicitly
  b) find the asymptotic behavior of $y(x)$, as in class and PS#4.

- Laurent series and radius of convergence (nearest singularities)

  Laurent expansion of e^z/z^4 about $z=0$. Residue at $z=0$? $\frac{1}{6}$

  Radius of convergence? About $z=1$? Residue at $z=1$? 0

  Taylor (power) series: $e - 3e(z-1) + \frac{13}{2}e(z-1)^2 + \ldots$

  $\left[w = z-1,\ z = w+1,\ f(w) = e^{w+1}/(w+1)^4,\ f(0) + wf'(0) + \frac{w^2}{2}f''(0) \right.$

  $\left. \frac{e}{1} + (z-1)\left(\frac{1}{1} - 4\frac{1}{1}\right) + \ldots \right]$