770L Lecture 12

- PS 4 - 3 returned - average 52/60 - he will review problems.
- Quick comments on integration by parts and undetermined coefficients.
- Gibb's overshoot via Mathematica (Risch algorithm).
- Convergence & Fourier series.
- Parseval theorem - generalized.
- Gamma function.
- Comments on midterm.

For PS 4, we need integrals like \( \int_a^b x \sin x \, dx \).

- OK to use Mathematica but must know how to do items "by hand."
- Usual approach: integration by parts (see XKCD cartoon).

\[
\int u \, dv = uv - \int v \, du = u x, x, x \quad dv = (\frac{dx}{dx}) \, dx
\]

\[
eq \int_a^b x \sin x \, dx = x(-\cos x) \bigg|_a^b - \int_a^b -\cos x \, dx = [-x \cos x + \sin x]_a^b
\]

\[
\Rightarrow du = dx, \quad v = -\cos x
\]

Integration by parts is usually taught in beginning calculus as one of the "tricks" like trigonometric substitutions.

- But much more important: How to think about it?

\[
\int_a^b (uv) \, dx = uv \bigg|_a^b - \int_a^b \frac{du}{dx} \cdot v \, dx + \int_a^b u \cdot \frac{dv}{dx} \, dx
\]

generalize in vector calculus: eg., \( \int_V \nabla \cdot A \, dx = \int_{\partial V} A \cdot n \, ds \)

- If surface term vanishes (eg, from boundary conditions), then

\[
\int_V \frac{d}{dx} (uv) \, dx = -\int_{\partial V} uv \cdot n \, ds \Rightarrow "\text{more derivative of cost } \frac{d^2}{dx^2}"
\]

In 3D, \( \frac{d^2}{dx^2} \Rightarrow \text{no change} \Rightarrow \text{Hermitean} \)
9/18/13

...later: "self-adjoint" differential operators.

- Alternative to integration by parts ⇒ method of undetermined coefficients (Risch algorithm used by Mathematica).
- Key observations: derivatives are really easy to program symbolically and pattern matching is also easy.

- Use Mathematica to calculate a few (indefinite) integrals:

\[
\int x \sin x \, dx = -x \cos x + \sin x \\
\int x^2 \sin x \, dx = (-x^2 + 3) \cos x + 2x \sin x \\
\int x^2 \cos x \, dx = 2x \cos x + (x^2 - 3) \sin x \\
\text{and so on.}
\]

- Basic idea: The general form of the right side is well defined.
- For integrals of rational functions of polynomials and trig functions, assign coefficients to general answer, differentiate, match coefficients.

\[
eq \int x^2 \sin x \, dx = (a_2 x^2 + a_1 x + a_0) \cos x + (b_2 x + b_0) \sin x
\]

\[
a \frac{d}{dx} \text{ of right side} = (a_2 x + a_1) \cos x - (a_2 x + a_0 + a_1) \sin x + b_2 \sin x + (b_2 x + b_0) \cos x
\]

\[
\text{Matching:}
\begin{align*}
\text{cos x:} & \quad 9a_0 + b_2 = 0 \\
\text{cos x:} & \quad a_1 + b_0 = 0 \\
x^2 \sin x: & \quad -a_2 = 0 \quad \Rightarrow \quad a_2 = -1 \quad \Rightarrow \quad b_4 = 2 \quad \Rightarrow \quad a_0 = 2 \\
x \sin x: & \quad a_2 = 0 \\
\sin x: & \quad -a_0 + 2b_2 = 0
\end{align*}
\]

\[
\int x^2 \sin x \, dx = (-x^2 + 3) \cos x + 2x \sin x + \sqrt{a_0^2 + b_2^2}
\]

- Many other analogous applications, e.g., differential equations.
Gibbs' overshoot

See Appendix 19.3 for detailed analysis.
Here: physics-style analysis using Mathematica.

Follow along gibbs_overshoot.nb notebook to explore.

Historical note:
The story is that Albert Michelson, famous for measuring
the speed of light and the Michelson-Morley experiment,
built a machine in 1898 to calculate Fourier coefficients,
when series resumed, worked well except for square wave, which
had persistent ripples near discontinuities which wouldn't go away.
He suspected machine error, but J. Willard Gibbs explained
they were real. The Fourier series overshoots by about 99\% near
the discontinuity, becoming narrower with more terms, but always 99\%.

We do the sum in Mathematica and find the claimed result:
\[
\lim_{N \to \infty} \frac{2}{\pi} \left( \sum_{n=1}^{N} \left( \frac{\sin\left(\frac{2\pi n}{N}\right)}{n} \right) \right) = \frac{1}{6} + 0.08949
\]

Aiken finds this number from:
\[
\int_{0}^{\pi} \frac{\sin(t)}{t} \, dt = \frac{1}{3} + 0.08949
\]

Can you see how the sum is an approximation to the integral
that becomes exact as \(N \to \infty\)?
(answer next time!)
Comments on the upcoming midterm

Tuesday evening, 9/24. Design is for a two-hour exam, but you will not be timed.

Two types: basic problems, synthetic problems
- basic: testing for core competencies
  - what everyone needs to know to pass
  - e.g., S1 and S2, manipulations for pages of
  - vector identities, knowing different types of singularities;
  - complex manipulations (like taking \( n \)th roots), applying Frobenius

Aside: What functions have branch points? \( f(z) = (z-z_0)^{1/2} \), \( \ln(z-z_0) \)
- Non-analytic at \( z_0 \Rightarrow \) Taylor series doesn’t exist. Seen here by
  derivatives (eventually) blowing up at \( z_0 \).

\( f(z) = (z-z_0)^{1/2} \), \( f(z) = 3^{1/2}(z-z_0)^{1/2} \), \( f(z) = 5^{1/2}(z-z_0)^{1/2} \Rightarrow \text{F}^1(z_0) = 0 \)

Synthetic: putting ideas/techniques together; applying to new situations

Based on topics seen in homework problems.

- None, setting up problems, applying tools like \( \delta \), \( \epsilon \), definitions and notation.

You will be given Jackson covers; any needed integrals like \( \int e^{zt} \).
You will not be given (must know these!)
\[ e^z = \cos z + \sin z = 1 + z + \frac{z^2}{2!} + \ldots \quad (1 + z)^k = 1 + k z + \ldots \quad |z| < 1 \]

\[ \text{Ea}: \text{Each} = \sum_{k=0}^{\infty} \frac{z^k}{k!} \quad \text{Ea}: \text{Each} \]
- All numbers in each must be different. If first\( \text{Ea}, \text{Ea}, \text{Ea}, \text{Ea} \)
  - indices are the same, the entries in the first two need to be the same as the second two. In this case, as the 2nd and 3rd of each are the same.

QR, relations, should it have to be code, with a minus sign.

How to prepare: memorize.

- Go over \# 1-4 and make sure you can set up every problem without looking at solutions.
- Check solutions and lecture notes for emphasis points. Remember the picky points taken off on homework and emphasized in class.
Midterm Basic Problems: Examples

What is $\varepsilon_{abc} \varepsilon_{def} = \varepsilon_{abc} \varepsilon_{def}$? (no repeated indices)

$\varepsilon_{abc} \varepsilon_{def} = \varepsilon_{def} \varepsilon_{abc}$ (standard identity)

$\varepsilon_{abc} \varepsilon_{cab} = \varepsilon_{cda} = 0$ because $\varepsilon_{cda} = 0$ for repeated indices.

Show $\vec{F} \cdot (\vec{B} \times \vec{C}) = C \cdot (\vec{A} \times \vec{B})$ assuming these commute

$= F_0 (\vec{B} \times \vec{C})_0 = \varepsilon_{abc} A_0 B_0 C_0 = \varepsilon_{def} C_0 A_0 B_0$

$= \varepsilon_{def} C_0 A_0 B_0 = C_0 (\vec{A} \times \vec{B})_0$

$= C \cdot (\vec{A} \times \vec{B}) = 0 \, \text{for identical}\,$

$\nabla \times (\nabla \phi) = 0 \Rightarrow (\nabla \times \nabla \phi)_0 = \varepsilon_{abc} A_0 B_0 C_0 = 0$ (or $=-\varepsilon_{abc} A_0 B_0 C_0$)

Fourier series:

- Apply given formulas for coefficients
- Solve differential equations of the two types given

Spot the error:

$\nabla \times (\vec{B} \times \vec{C}) = 0 \, \text{if } \vec{B} \vec{C} \text{ is a scalar, not a vector as needed for cross product}$

$(\vec{A} \times \vec{B})_0 + (\vec{C} \times \vec{D})_0 = \varepsilon_{abc} A_0 B_0 C_0 + \varepsilon_{def} D_0 E_0 F_0$ [The two vectors must have the same index rather than a and d]

What are the singularities of $\sin \frac{z}{z}$ and what type are they:

$z^2 + z + 1 = (z+2)(z+1)$ have branch points at $z = -2$ and $z = -1$

$\sin z$ has zeros on the x-axis for $y = \pi x$, which mean simple poles.

What is radius of convergence about $z = 1$? $z = -1$?
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Would you close in the upper or lower half plane when doing the contour integral for \( \int_{-\infty}^{\infty} \frac{dx}{1+x^2} \)? [Enter: For \( z=Re^{i\theta} \) the integral goes to zero as \( R e^{i\theta}=\frac{\pi}{2} \) for \( R \to \infty \) in either half plane.]

What contour would you choose for \( \int_{-\infty}^{\infty} \frac{\sin x}{x} \, dx \)? (Bonus problem)

[Split into \( \frac{e^{ix}}{x} - \frac{e^{-ix}}{x} \) and define each with a principal value. Jordan's lemma says we can close the first in the upper half plane and the second in the lower half plane and the integral over the large semi-circle vanishes in each case.]

Find the residue of:

a) \( \frac{e^{iTz}}{z(e^{i\alpha} - z)} \) at \( z=0, z=\alpha \):
\[
\begin{align*}
\text{Res}(0) &= \frac{e^{iT\alpha}}{i\alpha}, \\
\text{Res}(\alpha) &= \frac{e^{iT\alpha}}{i\alpha};
\end{align*}
\]

b) \( \frac{ze^{iz}}{z+1} \) at \( z=-1 \):
\[
\text{Res}(-1) = \frac{g(-1)}{h(-1)} = \frac{2i}{4i} = \frac{1}{2};
\]

For differential equations, be prepared to:

a) Apply the Frobenius method to derive recurrence relations (e.g., \( a_{n+2} = \frac{2n}{n+1} a_n \)), and find the first few terms explicitly.

b) Find the asymptotic behavior of \( y(x) \) as \( x \to \infty \), as in class and PSet 4.

Laurent series and radius of convergence (nearest singularity): Laurent expansion of \( f(z) \) \( \frac{z^2}{z^4} \) about \( z=0 \). Residue at \( z=0 \). 

Radius of convergence: About \( z=\alpha \), residue at \( z=\alpha \).

Taylor (power) series:
\[
\begin{align*}
\text{Taylor series: } e^{-3e(z-1)} &= \sum_{n=0}^{\infty} \frac{(-3e)^n}{n!} (z-1)^n \\
&= \sum_{n=0}^{\infty} \frac{(-3e)^n}{n!} (z-1)^n,
\end{align*}
\]

[\( n=1, \ z=1+ \text{Re} e^{i\text{Im}} \), \( f(0) + f(0) + f(0) + f(0) + \ldots \)]