In the discussion of Gibb's overshoot, it was claimed that

\[
\lim_{N \to \infty} \frac{1}{N} \left[ \sum_{n=-N}^{N} \frac{\sin \left( \frac{2\pi n}{N} \right)}{n} \right] = \frac{1}{2} + 0.608949
\]

\[
= \int_{0}^{2\pi} \frac{\sin t}{t} \, dt
\]

How do we understand this?

Pause to rederive as a review of Fourier series:

- Looking for series from \(-\pi\) to \(\pi\) \(\Rightarrow\) period \(= 2\pi\)
- Odd about origin \(\Rightarrow\) sines only survive

\[
a_n = \frac{2}{\pi} \int_{0}^{\pi} f(x) \sin nx \, dx
\]

\[
a_0 = \frac{2}{\pi} \int_{0}^{\pi} f(x) \, dx = \int_{0}^{\pi} \sin nx \, dx + \int_{0}^{\pi} (\sin x) \, dx
\]

\[
\Rightarrow \quad a_0 = \begin{cases} \frac{2}{\pi} & \text{n odd} \\ 0 & \text{n even} \end{cases}
\]

Observations on series:
- Arguments of sines are equally spaced \(0, \frac{\pi}{N}, \frac{2\pi}{N}, \ldots\)
- If we let \(t = \frac{\pi}{N} \text{ i.e. } \sin \left( \frac{\pi t}{N} \right) \frac{\pi}{N} \text{ is a general term}
- \(\Delta t = \frac{\pi}{N} \Rightarrow \frac{\pi}{N} \sin \left( \frac{\pi t}{N} \right) \Delta t \text{ is summed } \Rightarrow \text{ integral}\) as \(\Delta t \to 0 \text{ when } N \to \infty

\[
\Rightarrow \quad \sum_{n=-\infty}^{\infty} \sin \left( \frac{\pi n t}{N} \right) \left( \frac{\pi}{N} \right) \int_{0}^{\infty} \sin t \, dt = \int_{0}^{\infty} \sin t \, dt \left[ \text{multiply both sides by } \frac{N}{\pi} \text{ to get our result} \right]
\]

\[
\Rightarrow \quad \text{we are using the "midpoint rule" to approximate the integral.}
\]
9/20/13  
Follow-up to post 4

Fourier series problem (from midterm 2 last year)  
Given Fourier sine/cosine series for 0 ≤ x ≤ L:

\[ f(x) = \sum_{n=1}^{\infty} a_n \sin \frac{n\pi x}{L} \quad \text{or} \quad f(x) = b_0 + \sum_{n=1}^{\infty} b_n \cos \frac{n\pi x}{L} \]

where \( a_0 = \frac{1}{L} \int_{0}^{L} f(x) \sin \left( \frac{n\pi x}{L} \right) \, dx \quad b_0 = \frac{1}{2L} \int_{0}^{L} f(x) \, dx \quad b_n = \frac{1}{L} \int_{0}^{L} f(x) \cos \left( \frac{n\pi x}{L} \right) \, dx \)

\[ \int_{0}^{L} \sin(nx) \, dx = \frac{1}{n} \left[ (-1)^{n+1} \right] \quad \int_{0}^{L} \sin(mx) \, dx = \frac{(1 + (-1)^{m})}{nm} \]

\[ \int_{0}^{L} \cos(mx) \, dx = 0 \quad \int_{0}^{L} \cos(mx) \, dx = \frac{(1 + (-1)^{m})}{nm} \]

Question: Consider the function \( f(x) = 1-x \) on 0 ≤ x ≤ 1. Do you expect the Fourier sine or cosine series to converge more rapidly with the number of terms included?

Sine series: \( a_0 = \frac{1}{L} \int_{0}^{L} (1-x) \sin \left( \frac{n\pi x}{L} \right) \, dx \approx \frac{1}{n} \)

Cosine series: \( b_0 = \frac{1}{L} \int_{0}^{L} (1-x) \cos \left( \frac{n\pi x}{L} \right) \, dx \approx 0 - \frac{1}{n^2} \)

\[ \Rightarrow \text{Cosine coefficients decrease faster (1/n^2 vs 1/n)} \quad \Rightarrow \text{Cosine series converges more rapidly} \]

Sketch the periodic extensions from -2 < x < 2

-2 -1 0 1 2  
Sine series (odd, periodic in [-1, 1])

-2 -1 0 1 2  
Cosine series (even, periodic in [-1, 1])

Compare to sine + cosine series on 0 to L (or -L to L)
Lead in to generalized functions...

What is the Fourier series for a delta function at the origin in -L to L?

\[ f(x) = \sum_{n=-\infty}^{\infty} c_n e^{inx/L} \quad \Rightarrow \quad c_n = \frac{1}{2L} \int_{-L}^{L} f(x) e^{-inx/L} \, dx \]

[How to check? Verify known results: Does \( f(x) = 0 \) \( \Rightarrow c_n = 0 \)? Yes \( \Rightarrow c_n = \delta_{n0} \) ? Yes.]

So let \( f(x) = \delta(x) \) \( \Rightarrow c_n = \frac{1}{2L} \int_{-L}^{L} \delta(x) e^{-inx/L} \, dx = \frac{1}{2L} \]

\( \Rightarrow \) constant coefficients for all \( n \)!!

\[ \delta(x) = \frac{1}{2L} \sum_{n=\infty}^{\infty} e^{inx/L} \]

Looks very dubious for convergence.

\( \Rightarrow \) what we expect for \( \delta \)-function?

How to define more rigorously?

Two choices: i) functional

ii) limit of delta sequence

\( \Rightarrow \) generalized function