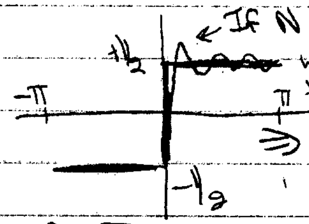
special assignment: watch "Bohemian Gravity" on YouTube

9/20/13 $\boxed{\text{7701 Lecture 13}}$

In the discussion of Gibb's overshoot, it was claimed that

$$\lim_{N\to\infty} \frac{2}{\pi}\left[\frac{\sin\left(\frac{1\cdot\pi}{N}\right)}{1} + \frac{\sin\frac{3\pi}{N}}{3} + \frac{\sin\frac{5\pi}{N}}{5} + \dots \frac{\sin\frac{(N-1)\pi}{N}}{N-1}\right] \doteq \frac{1}{2} + 0.08949$$

$$= \frac{1}{\pi}\int_0^{\pi} \frac{\sin t}{t}\,dt$$



← If N terms, peak at $\frac{\pi}{N}$ with height $\frac{1}{2}+0.08949$ ⇒ ≈9% of discontinuity

How do we understand this?

[Pause to rederive as a review of Fourier series:

- Looking for series from $-\pi$ to π ⇒ period $= 2\pi$
- odd about origin ⇒ sines only survive

$$\Rightarrow f(x) = \sum_{n=1}^{\infty} a_n \sin\left(\frac{2\pi n x}{\text{period}}\right) = \sum_{n=1}^{\infty} a_n \sin(nx) = G(x) - \frac{1}{2}$$

$$a_n = \frac{2}{\text{period}}\int_{\text{interval}} f(x)\sin\left(\frac{2\pi n x}{\text{period}}\right)dx = \frac{2}{2\pi}\left[\int_{-\pi}^{0}\left(-\frac{1}{2}\right)(\sin nx)\,dx + \int_0^{\pi}\left(+\frac{1}{2}\right)(\sin nx)\,dx\right]$$

$$= \frac{1}{\pi}\int_0^{\pi}\sin nx\,dx = \frac{1}{\pi}\,\frac{-1}{n}\cos nx\Big|_0^{\pi} = \frac{1}{n\pi}(1-\cos n\pi)$$

$$\Rightarrow a_n = \begin{cases}\frac{2}{\pi}\frac{1}{n} & n \text{ odd}\\ 0 & n \text{ even}\end{cases}$$
]

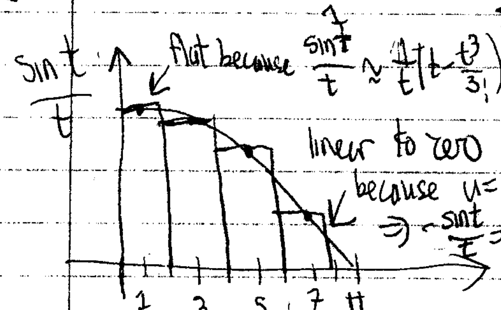
Observations on series:

- arguments of sines are equally spaced $0\ \frac{\pi}{N}\ \frac{3\pi}{N}\ \frac{5\pi}{N} \dots \frac{(N-1)\pi}{N}\ \pi \to t$
- If we let $t = \frac{i\pi}{N}$, i odd, then $(\sin t/t)\cdot\frac{\pi}{N}$ is general term.
- $\Delta t = \frac{2\pi}{N} \Rightarrow \frac{1}{2}(\sin t/t)\Delta t$ is summed ⇒ integral as $\Delta t \to 0$ when $N\to\infty$

You should develop these rough plotting skills

$$\Rightarrow \frac{\sin\left(\frac{1\pi}{N}\right)}{1} + \frac{\sin\frac{3\pi}{N}}{3} + \dots \xrightarrow{N\to\infty} \frac{1}{2}\int_0^{\pi}\frac{\sin t}{t}\,dt$$

[multiply both sides by $\frac{2}{\pi}$ to get our result]



flat because $\frac{\sin t}{t} \approx \frac{1}{t}\left(t - \frac{t^3}{3!}\right) \sim 1 - \frac{t^2}{6}$

linear to zero because $u = t - \pi$

$$\Rightarrow \frac{\sin t}{t} = \frac{\sin(u+\pi)}{\pi+u} = \frac{\sin u}{\pi\left(1+\frac{u}{\pi}\right)} \to u + O(u^2) = (t-\pi) + O(u^2)$$

⇒ we are using the "midpoint rule" to approximate the integral.

9/20/13 · Followup to PS#4

· Fourier series problem (from Midterm 2 last year)

Given Fourier sine/cosine series for $0 \le x \le L$:

$$f(x) = \sum_{n=1}^{\infty} a_n \sin\frac{n\pi x}{L} \quad \text{or} \quad f(x) = b_0 + \sum_{n=1}^{\infty} b_n \cos\frac{n\pi x}{L}$$

where $a_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)dx \quad b_0 = \frac{1}{L}\int_0^L f(x)\,dx,\ b_n = \frac{2}{L}\int_0^L f(x)\cos\left(\frac{n\pi x}{L}\right)dx$

$$\int_0^1 \sin(n\pi x)\,dx = \frac{1+(-1)^{n+1}}{n\pi} \qquad \int_0^1 x\sin(n\pi x)\,dx = \frac{(-1)^{n+1}}{n\pi}$$

$$\int_0^1 \cos(n\pi x)\,dx = 0 \qquad \int_0^1 x\cos(n\pi x)\,dx = -\frac{(1+(-1)^{n+1})}{n^2\pi^2}$$

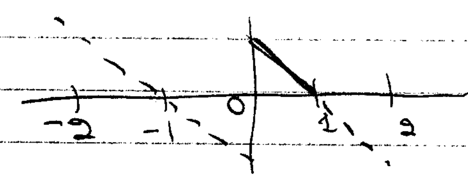
Question: Consider the function $f(x) = 1-x$ on $0 \le x \le 1$. Do you expect the Fourier sine or cosine series to converge more rapidly with the number of terms included?

sine series $\quad a_n = \frac{2}{L}\int_0^L (1-x)\sin\left(\frac{n\pi x}{L}\right)dx \propto \frac{1}{n}$

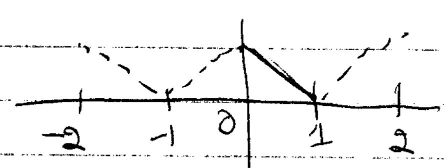
cosine series $\quad b_n = \frac{2}{L}\int_0^L (1-x)\cos\left(\frac{n\pi x}{L}\right)dx \propto 0 - \frac{1}{n^2}$

$\Rightarrow$ cosine coefficients decrease faster $\left(\frac{1}{n^2} \text{ vs. } \frac{1}{n}\right) \Rightarrow$ cosine series converges more rapidly

Sketch the periodic extensions from $-2 < x < 2$



sine series
(odd, periodic in [-1, 1])



cosine series
(even, periodic in [-1, 1])

· compare to sine + cosine series on 0 to L (or -L to L)

9/20/13

Lead in to generalized functions . . .

What is the Fourier series for a delta function at the origin in $-L$ to L?

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{\frac{in\pi x}{L}} \Rightarrow c_n = \frac{1}{2L}\int_{-L}^{L} f(x)\, e^{-in\pi x/L}\, dx$$

correct? correct?

[How to check? Verify known results: Does $f(x)=1 \Rightarrow c_n = \delta_{n0}$? yes
Does $f(x) = e^{\frac{i\pi x}{L}} \Rightarrow c_n = \delta_{n1}$? yes]

So let $f(x) = \delta(x) \Rightarrow c_n = \frac{1}{2L}\int_{-L}^{L} \delta(x)\, e^{-in\pi x/L}\, dx = \frac{1}{2L}$

$\Rightarrow$ constant coefficients for all n!!

$$\delta(x) = \frac{1}{2L}\sum_{n=-\infty}^{\infty} e^{in\pi x/L}$$

looks very dubious for convergence!

$\Rightarrow$ what we expect for δ-function,
How to define more rigorously?
Two choices: i) functional
ii) limit of delta sequence
$\rightarrow$ generalized function