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770 Lecture 14

- Midterm I in Sm1138 (regular room) from 6pm–10pm (or a bit earlier)
- Warm-ups on board: example problems on 88 and 89
- Do gamma function on 60–61 (and 94) — complex integration. Also 45 on analytic continuation and radius of convergence.
- Do generalized Parseval theorem on 82
- Start generalized functions on 95+ if time permits.

Summary of things to know for the midterm:

PS#1:
- δ_{ij}, ε_{ijk} proofs and "spot the error" (vector ≠ component, etc.)
- solutions to complex equations (using $z = re^{i\theta}$ or $z = x + iy$)
- applying Cauchy-Riemann relations
- identifying Laurent expansion and region of convergence
  - know expansions for e^z, $\sin z$, $\cos z$, $(1+z)^\alpha$

PS#2:
- calculating residues: simple poles, a_{-1} term, $g(z)/h(z)$ rule
- Cauchy $\oint (z-z_0)^n dz$, residue theorem, which half-plane to close in, Jordan's lemma or other arguments for ghost contours vanishing. [no tricky contours]
- types of singularities

PS#3:
- $\frac{1}{x-(x_0 \pm i\varepsilon)} = \frac{P}{x-x_0} \mp i\pi\delta(x-x_0)$ and what it means.
- Frobenius method for 2nd order linear equations. Indicial equation, what to do with the a_1 equation.
- finding solution at large x and power series correction (e.g. $y(x) = Cx^\alpha e^{-\beta x}$)

PS#4:
- Applying formulas for Fourier coefficients, implications of coefficients.
- what expansion to use for differential equations (and why)
- solving basis differential equations

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Gamma Function Recap and Follow-Ups

Definitions (see Arfken chap. 8)

$$\Gamma(z) \equiv \int_0^\infty e^{-t} t^{z-1}\, dt \quad \operatorname{Re} z > 0 \;\leftarrow \text{[Why? Integral diverges otherwise]}$$

$= (z-1)!$

Other forms

$$\Gamma(z) = 2\int_0^\infty e^{-t^2} t^{2z-1}\, dt \quad \operatorname{Re} z > 0 \quad \left(u = t^2,\ du = 2t\,dt \quad \int_0^\infty e^{-u} t^{2z-2}\, du = \int_0^\infty e^{-u^2} u^{z-1}\, du\right)$$

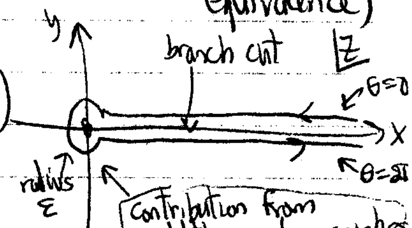
$$\Gamma(z) = \int_0^1 \left[\ln\left(\tfrac{1}{t}\right)\right]^{z-1} dt \quad \operatorname{Re} z > 0$$

or $$\Gamma(z) \equiv \lim_{n\to\infty} \frac{1\cdot 2\cdot 3 \cdots n}{z(z+1)(z+2)\cdots(z+n)} \quad z \neq 0, -1, -2, \ldots$$ (Arfken proves equivalence)

only simple poles for $\Gamma[z]$

Contour representation: $\int_C e^{-z} z^\nu\, dz = \left(e^{2\pi i \nu} - 1\right)\Gamma(\nu+1)$

(not closed) $\rightarrow$ C — bottom of cut — top of cut



Use $\Gamma(z+1) = z\Gamma(z)$ to continue the integral forms to all z except simple poles at $z = 0, -1, -2, \ldots$

Contribution from little circle vanishes as $\varepsilon \to 0$

Keep in mind the form of $\Gamma(z)$ will arise from simple transformations of integrals of interest.

Two examples

$$\int_0^\infty e^{-x^4}\, dx \underset{\substack{x = u^{1/4} \\ dx = 1/4\, u^{-3/4} du}}{\overset{u = x^4}{=}} \int_0^\infty e^{-u} \tfrac{1}{4} u^{\frac{1}{4}-1}\, du = \tfrac{1}{4}\Gamma\left(\tfrac{1}{4}\right) = \Gamma\left(\tfrac{5}{4}\right)$$

$$\int_0^1 x^k \ln x\, dx \underset{dx = -e^{-u} du}{\overset{x = e^{-u}}{=}} \int_\infty^0 (e^{-u})^k (-u)\, e^{-u}\, du = -\int_0^\infty e^{-(k+1)u}\, u\, du \underset{du = dt/(k+1)}{\overset{t = (k+1)u}{=}} \frac{1}{(k+1)^2}\int_0^\infty e^{-t}\, t\, dt$$

$$= \frac{-1}{(k+1)^2}\Gamma(2) = \frac{-1}{(k+1)^2}$$

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Generalized Functions: Core Competencies

① Applying properties of delta/theta functions
② Identification of delta functions in physics context } in appropriate coordinates
③ Fourier representations
④ Differential equations with impulse terms

intuition and applications

- Here focus on applications rather than proofs and mathematical formalism of distributions.

① Applying properties of delta/theta functions (sometimes the 3 is omitted)

- These functions are idealization of physical situations
  - charge density of point charge at $\vec{x}_0$: $\rho(\vec{x}) = q\,\delta^3(\vec{x}-\vec{x}_0)$
    [notation $\delta^3(\ )$ as reminder $\delta(x)\delta(y)\delta(z)$ in Cartesian]
  - impulse driving force at t_0: $F(t) = C\,\delta(t-t_0)$;
    e.g. strike string or rod at $t=t_0$ abruptly
  - Start applying a driving force at $t=t_0$:
    $F(t) = [F_0 \sin \omega t]\,\theta(t-t_0)$

⇒ in the real world, charges are spread out at least a little (or, as for electrons, we can't tell), forces take some time to deliver or start up.

- So it is natural physically and not just mathematically to consider delta sequences. Prove properties for delta sequences!
- In field theory: "regularized" delta functions

$$\lim_{n\to\infty} \phi_n(x) = \delta(x)$$

PS#5 $\quad \phi_n^{(1)}(x) = \dfrac{n}{\pi}\left(\dfrac{1}{1+n^2x^2}\right)$

$$\phi_n^{(2)}(x) = \frac{1-\cos nx}{n\pi x^2}$$

- Look at these with Mathematica: ps5_checks.nb

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Need to demonstrate "sifting property"

check this works for $\{\phi_n\}$ in $n\to\infty$ limit (PS#6: contour integration)

$$\int_{-\infty}^{\infty} f(x)\,\delta(x)\,dx = f(0) \qquad \left[\int_{-\infty}^{\infty} f(x)\,\delta(x-a)\,dx = f(a)\right]$$

from change of variables on delta sequence

[sift out the value at $x=0$ of the test function $f(x)$]

- restrictions on $f(x)$? Depends on ϕ_n sequence
  - we usually require $\int_{-\infty}^{\infty} |f(x)|^2\,dx < \infty \Rightarrow$ "square integrable"

  For physics reasons, this is usually sufficient.

  PS#3: does $f(x)$ have to be analytic for delta sequence?

Another property: $\delta(ax) = \frac{1}{|a|}\delta(x)$

- such an equation is always a shorthand because we'll be applying it in an integral
- Would see to follow from change of variables

$$\int_{-\infty}^{\infty} f(u)\,\delta(au)\,du \underset{dx = a\,du}{\overset{x = au}{=}} \int_{-\infty}^{\infty} f\left(\tfrac{x}{a}\right)\delta(x)\,\tfrac{1}{a}\,dx = \tfrac{1}{a}f(0) \quad ??$$

$$\overset{?}{=} \int_{-\infty}^{\infty} f(x)\,\tfrac{1}{a}\,\delta(x)\,dx$$

But where is $|a|$? We have implicitly assumed $a>0$.
If $a<0$, we also have to flip the limits

$$a<0: \int_{+\infty}^{-\infty} f\left(\tfrac{x}{a}\right)\delta(x)\,\tfrac{1}{a}\,dx = -\tfrac{1}{a}\int_{-\infty}^{\infty} f\left(\tfrac{x}{a}\right)\delta(x) = -\tfrac{1}{a}f(0) \overset{\text{general}}{\Rightarrow} \tfrac{1}{|a|}\delta(x)$$

- What else? Derivatives! (prove with delta sequences. Here: mnemonic)

$$\int_{-\infty}^{\infty} \delta'(x)\,f(x)\,dx = \int_{-\infty}^{\infty}\left[\tfrac{d}{dx}\delta(x)\right]f(x)\,dx = -\int_{-\infty}^{\infty}\delta(x)\,\tfrac{d}{dx}f(x)\,dx + \cancel{\text{surface}} \text{ since } \delta(\pm\infty)=0$$

switching derivative

$$\Rightarrow \int_{-\infty}^{\infty}\delta'(x)\,f(x)\,dx = -f'(0) \underset{\text{keep doing it}}{\Rightarrow} \int_{-\infty}^{\infty}\delta^{(n)}(x)\,f(x)\,dx = (-1)^n f^{(n)}(0)$$

switch sign every time

looks like Taylor series coefficient!

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- delta function of a function. Prove with delta sequences. Result

$$\int_{-\infty}^{\infty} \delta(g(x))\, f(x)\, dx = \sum_{i=1}^{N} \frac{f(x_{0_i})}{|g'(x_{0_i})|}$$

x_{0_i} are N zeros of $g(x)$: $g(x_{0_i}) = 0,\ i=1,\dots,N$

e.g. PS#5 $\displaystyle\int_{-\infty}^{\infty} e^{-x^2}\delta(x^2+x-6)\, dx$

What is the idea?

Suppose $g(x) = x^2 - (a+b)x + ab = (x-a)(x-b)$

Near $x=a$, variation of $x-b$ part is slow so replace by $a-b$

$$\Rightarrow \delta((x-a)(x-b)) \xrightarrow[x=a]{\text{near}} \delta((a-b)(x-a)) = \frac{1}{|a-b|}\delta(x-a) \qquad a = x_{0_1}$$

$\leftarrow$ from $\delta(cx) = \frac{1}{|c|}\delta(x)$

$$\xrightarrow[x=b]{\text{near}} \delta((b-a)(x-b)) = \frac{1}{|b-a|}\delta(x-b) \qquad b = x_{0_2}$$

$$\Rightarrow \int_{-\infty}^{\infty} \delta((x-a)(x-b))\, f(x)\, dx = \frac{1}{|a-b|}f(a) + \frac{1}{|b-a|}f(b)$$

Note that $|g'(a)| = |2x-(a+b)x|\Big|_{x=a} = |2a-(a+b)| = |a-b|$

and $|g'(b)| = (b-a) = |a-b|$.

- More generally, near $x = x_{0_i}$, $g(x) = \underbrace{\cancel{g(x_{0_i})}}_{0 \text{ by construction}} + (x-x_{0_i})g'(x_{0_i})$ $\leftarrow$ determines behavior

$+ \frac{1}{2}(x-x_{0_i})^2 g''(x_{0_i}) + \dots$

$\nwarrow$ why can we neglect? (look at $\{\phi_n\}$'s)

What if there is a double (or higher) root?

Eg. $(x-a)^2$?

Then the formula does not apply!

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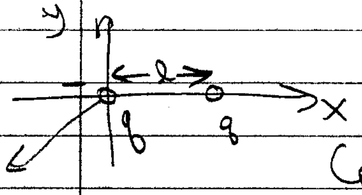
② Identification of delta functions in physics contexts

potential from charge density $\Phi(\vec{x}) = \frac{1}{4\pi\varepsilon_0}\int \frac{\rho(\vec{x}')}{|\vec{x}-\vec{x}'|}\, d^3x$

Suppose dipole: point charges q at separation ℓ



$$\rho(\vec{x}) = -q\,\delta(x)\,\delta(y)\,\delta(z) + q\,\delta(x-\ell)\,\delta(y)\,\delta(z)$$

Coefficient correct? Check total charge from + charge

$$q \stackrel{?}{=} \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy \int_{-\infty}^{\infty} dz\; q\,\delta(x-\ell)\,\delta(y)\,\delta(z) \cancel{dx\,dy\,dz} = q \;\checkmark$$

What about ideal dipole? $\ell \to 0$, $\vec{p} = p\hat{x}$, $p = q\ell$ finite as $\ell \to 0$

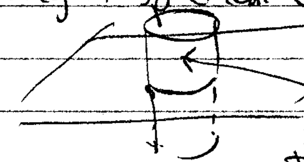
$$\Rightarrow \rho(\vec{x}) \xrightarrow[\ell \to 0]{} -q\,\delta(y)\,\delta(z)\left[\delta(x) - \delta(x-\ell)\right]$$

$$= \lim_{\ell \to 0} -\underbrace{q\ell}_{p}\,\delta(y)\,\delta(z)\left[\frac{\delta(x)-\delta(x-\ell)}{\ell}\right] = -p\,\delta(y)\,\delta(z)\,\delta'(x)$$

derivative of delta function!

Example 6.2 in Lea: (more on PS#5!)

- Sheet of charge in $z=0$ plane with surface charge density σ_0 (charge/area). Find volume charge density in spherical coordinates.



Only at $z=0 \Rightarrow \rho(\vec{x}) = k\,\delta(z)$. What is k?

cylinder of cross sectional area dA

find charge in cylinder $dq = \int_{\text{cylinder}} \rho(\vec{x})\, dV = \left(\int_{-\infty}^{\infty} k\,\delta(z)\, dz\right) dA = k\, dA$
(should be $\sigma_0\, dA$) $= \sigma_0\, dA \Rightarrow \boxed{k = \sigma_0}$

- change z to spherical $\rho(\vec{x}) = \sigma_0\,\delta(r\cos\theta)$

$$= \frac{\sigma_0\,\delta(\cos\theta)}{r} = \frac{\sigma_0\,\delta(\theta - \frac{\pi}{2})}{r\,\left|[-\sin\theta]_{\theta=\frac{\pi}{2}}\right|} = \frac{\sigma_0}{r}\,\delta\!\left(\theta - \frac{\pi}{2}\right)$$

$\delta(cx) = \frac{1}{|c|}\delta(x)$; find zeros in $0 \le \theta \le \pi$; only zero in $0 \le \theta \le \pi$

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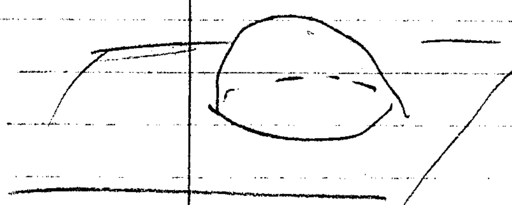
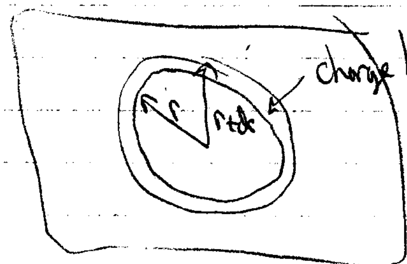
How to check our result? Integrate in spherical coordinates over shell of thickness dr





$$dq = \int_{shell} \rho(\vec{x})\, dV = \int_0^{2\pi} d\phi \int_0^{\pi} d\theta\, \frac{\sigma_0}{r}\delta\left(\theta - \frac{\pi}{2}\right) r^2 \sin\theta\, dr$$

$$= 2\pi \frac{\sigma_0}{r} r^2 \sin\frac{\pi}{2}\, dr = \sigma_0 2\pi r\, dr \checkmark$$

③ Fourier representations

For $-L < x < L$ $\quad \phi_n(x) = \sum_{m=-\infty}^{\infty} c_m^{(n)} e^{im\pi x/L}$

Find c_m for given $\{\phi_n\}$ and then take limit (do this in Mathematica and cps if possible. dirac_delta.nb)

Result $c_m^{(n)} \xrightarrow[n\to\infty]{} \frac{1}{2L} \Rightarrow \delta(x) = \frac{1}{2L}\sum_{m=-\infty}^{\infty} e^{im\pi x/L}$

* All Fourier modes contribute with the same amplitude! every harmonic is equal.

$\xrightarrow[\text{transform}]{\text{Fourier}}$ $\boxed{\delta(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\omega t}\, d\omega}$

cf. $\langle x|x'\rangle = \delta(x-x') = \sum_{m=-\infty}^{\infty} \langle x|m\rangle\langle m|x'\rangle$

$\langle x|m\rangle = \frac{1}{\sqrt{2L}} e^{im\pi x/L}$ $\quad \langle m|x'\rangle = \frac{1}{\sqrt{2L}} e^{-im\pi x'/L}$

(are $\frac{1}{\sqrt{2}}$'s correct?)

④ Differential equations with impulse terms

next time!: wave equation (string!) with impulse excitation