Delta function property follow-ups: (open dirac_delta.nb)

What if there is a double (or higher) root of \( g(x) \)?

Then \( \int g(x) f(x) \, dx = \sum_{i=1}^{n} \frac{f(x_i)}{|g'(x_i)|} \) does not apply.

Quick check for class:

What is \( \int_{-\infty}^{\infty} f(x) \delta(x^2-9) \, dx \)?

\( g(x) = x^2 - 9 \Rightarrow x_0 = 3 \)

\( \Rightarrow \frac{f(3)}{6} + \frac{f(-3)}{6} = \frac{1}{6} \delta(x+3) + \frac{1}{6} \delta(x-3) \)

What is \( \int_{-\infty}^{\infty} f(x) \delta(x^2+9) \, dx \)?

\( \cos 0 \) \( \Rightarrow \) Only real roots matter here!

Can we do a Dirac delta function in Mathematica?

\( \text{Yes}, \ \delta(x) \Rightarrow \text{DiracDelta}[x] \)

Demonstrate sifting property on \( \delta(x-a) \)

Try \( \text{DiracDelta}[x^2-9] \) // \( \text{FunctionExpand} \) \( + \) \( \text{DiracDelta}[-3+x] \) \( + \) \( \text{DiracDelta}[3+x] \)

Predict \( \text{DiracDelta}[x^2+9] \) // \( \text{FunctionExpand} \)

\( \text{Ans: 0} \)

Try an Integrate with \( \text{DiracDelta}[x] \) to show sifting function.

Try doing a double root \( \text{Leg, } \delta(x^2) \) or just \( \delta(x^2) \)

Analytically in Mathematica (fails) then numerically (blows up)

Use \( \lim_{a \to \pm \infty} \int_{-\infty}^{\infty} f(x) \delta(x-a) \, dx = f(a) \)

Can you write this as a delta sequence? \( \text{Ex. 1/n} \)?

\( \Rightarrow \int_{-\infty}^{\infty} f(3x) \delta(x-a) \, dx = \lim_{n \to \infty} \int_{-\infty}^{\infty} \frac{1}{n} \delta(x-a) \, dx = \frac{1}{n} f(3a) \)

\( \Rightarrow f(0) \int_{-\infty}^{\infty} du = 0 \)
Comments on P5 problem 2: delta sequences

Part a. We can write \( \phi_n(x) \) two ways:

\[
\phi_n(x) = \frac{1}{\pi} \frac{4}{1 + 4x^2} = \frac{1}{4\pi} \frac{1}{x^2 + 1/4}
\]

Which form do you think will be most useful for doing a contour integral and taking \( n \to \infty \)? (And why?)

The general idea is:

\[
\lim_{n \to \infty} \left[ \int dx \phi_n(x) f(x) \right] = \lim_{n \to \infty} \left[ \oint dz \phi_n(z) f(z) \right]
\]

For a suitable contour. Do we close \( \infty \) (or something else?)

Plan: assume \( f(z) \) is analytic first and then relax that assumption as much as possible.

Does \( \phi_n(z) = \frac{1}{\pi} \frac{1}{z^2 + 1/4} \) case which half-plane is closed in?

- No, since \( 1/z \), either way is equally good to make \( \phi \) integral over \( C \) vanish if \( f(z) \) doesn't blow up.

- If \( f(z) \) has a \( e^{i\alpha z} \) factor, close according to phase sign.

- Check that \( \phi_n \) answer doesn't depend on sign.

- What other limitations are on \( f(z) \) as \( z \to \infty \)? (Cauchy's lemma, for example).

When you have your first result assuming \( f(z) \) analytic, then ask: what if \( f(z) \) has a pole in the upper half plane and \( \mu \) close there? Does it matter? (What will \( \mu \) weighting of the pole be as \( n \to \infty \)?)
Part b) Now we apply contour integration again but with

\[ f_n(z) = \frac{1 - \cos nz}{\pi z^2} \]

Does \( f_n(z) \) have a simple or double pole?
- Be careful: check for Laurent series near \( z = 0 \)
- Expand numerator in Taylor series and look for non-zero \( 1/2 \) coefficient in \( f_n(z) \) expansion.

- Can we close \( \oint f_n(z) \, dz \) in the upper half plane? lower half plane?
- How can we rewrite \( 1 - \cos nz \)?
  - Real part of exponential
  - The exponentials

- If we introduce a pole (or poles) as intermediate steps, we don't want to include that contribution in the integral \( \Rightarrow \) principal value

  \[ \alpha, \beta \]

  Omit this part and take \( \epsilon = \delta^+ \)

- Evaluate principal value as in past examples, problems.
Fourier representations:

\[ S(x) = \frac{1}{2L} \sum_{m=-\infty}^{\infty} C_m e^{im\pi x/L} \quad \text{for} \quad -L < x < L \]

Key feature: all of the \( C_m \) coefficients are the same constant!

\[ \Rightarrow \text{every harmonic contributes equally.} \]

Note that this sum is quite ill-defined, but we don't intend to use it in this form, but in an integral.

Leu has an extended discussion of how to make this well-defined, but we won't consider this now.

Where does this come from?

On \(-L < x < L\), use the Fourier series of a delta sequence:

\[ \delta_n(x) = \frac{1}{2L} \sum_{m=-\infty}^{\infty} C_m e^{im\pi x/L} \]

Plan: find \( C_m \) for given \( \{\delta_n\} \) and then take \( n \to \infty \Rightarrow \sum_{m=-\infty}^{\infty} C_m \delta_n \to \delta \)

**Try this out in Mathematica with explicit example and also build \( S(x) \) directly. \Rightarrow \text{dirac_delta.nb}**

Check the shifting property (in the limit \( n \to \infty \)):

\[ \int S(x) \delta_n(x) \, dx = \int \frac{1}{2L} \sum_{m=-\infty}^{\infty} C_m e^{im\pi x/L} \delta_n(x) \, dx \]

\[ = \frac{1}{2L} \sum_{m=-\infty}^{\infty} \int C_m e^{im\pi x/L} \delta_n(x) \, dx \]

But if \( \delta_n(x) = \sum_{m=-\infty}^{\infty} C_m e^{im\pi x/L} \), what coefficient is projected out? \( \text{Ans: } C_m \)

\[ \int S(x) \delta_n(x) = \sum_{m=-\infty}^{\infty} C_m = \sum_{m=-\infty}^{\infty} C_m = f(0) \]

\[ \sum_{m=-\infty}^{\infty} C_m e^{im\pi x/L} \]

\[ \sum_{m=-\infty}^{\infty} C_m e^{im\pi x/L} \]

\[ \sum_{m=-\infty}^{\infty} C_m e^{im\pi x/L} = \sum_{m=-\infty}^{\infty} C_m e^{im\pi x/L} \]
When we go from \(-L < x < L\) to infinite intervals, \(\mathcal{F}\) have the Fourier transform representation:

\[
\mathcal{F}[f(t)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t) e^{-i\omega t} \, dt = \frac{1}{\sqrt{2\pi}} \left( \int_{-\infty}^{0} f(t) e^{-i\omega t} \, dt + \int_{0}^{\infty} f(t) e^{-i\omega t} \, dt \right)
\]

(Why?)

We can find other representations, e.g.,

\[
\mathcal{F}[f(t)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t) e^{-i\omega t} \, dt = \frac{1}{\sqrt{2\pi}} \left( \int_{-\infty}^{0} e^{-i\omega t} f(t) \, dt + \int_{0}^{\infty} e^{-i\omega t} f(t) \, dt \right)
\]

\[
= \frac{1}{\sqrt{2\pi}} \left( \int_{-\infty}^{\infty} e^{-i(\omega t - i\omega t)} \, dt \right)
\]

\[
= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \cos(\omega t) \, dt
\]

Similarly, \(\mathcal{F}(x-a) = \frac{2}{\sqrt{\pi}} \int_{0}^{\infty} \cos(\omega x) \cos(\omega a) \, d\omega\)

More later!
Consider example 6.3 in Lea.

We have an initially stationary string that we hit with impulse here at \( x = \frac{1}{3} \) at \( t = 0 \) with impulse \( I_0 \).

\[ \text{So force} \quad \text{at} \quad x = \frac{1}{3} \]

\[ \text{\Delta} P = \mathbf{F} \Delta t = I_0 \quad \text{impulse!} \]

- We consider the limit \( \varepsilon \to 0 \) and also that the impulse is delivered just at \( x = \frac{1}{3} \).

The wave equation is \( \frac{\partial^2 y}{\partial x^2} = \frac{1}{v^2} \frac{\partial^2 y}{\partial t^2} \).

We solve by separation of variables with the boundary conditions in \( x \) given by \( y(x, 0) = y(L, t) = 0 \).

So \( y(x, t) = \sum_{n=1}^{\infty} \sin \frac{n\pi x}{L} \left( a_n \sin \frac{n\pi vt}{L} + b_n \cos \frac{n\pi vt}{L} \right) \) (by form)

In this case, \( y(x, 0) = 0 \) for all \( x \leq b_0 \equiv 0 \). (by projecting or uniqueness of expansion)

\[ y(x, t) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{L} \sin \frac{n\pi vt}{L} \]

\[ \frac{\partial y}{\partial t}(x, t) = \sum_{n=1}^{\infty} b_n \frac{n\pi \cos \frac{n\pi vt}{L}}{L} \sin \frac{n\pi x}{L} \sin \frac{n\pi vt}{L} \]

What is the initial condition on \( \frac{\partial y}{\partial t} \)?

We apply an impulse to length \( dx \), which has mass \( \mu dx \) and so the change in momentum is \( (I = \Delta P) \)

\[ (\mu \, dx) \left( \frac{\partial y(x, t)}{\partial t} - \frac{\partial y(0, t)}{\partial t} \right) = c_0 \delta(x - \frac{L}{3}) \, dx \]

When we integrate over \( x \), the total impulse is \( I_0 \) \( \implies C = I_0. \)
\[ y(x, t) = \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{\sin \frac{m\pi}{L}}{m^2 \sin \frac{m\pi}{L}} \cos \left( \frac{m\pi}{L} x \right) \sin \left( \frac{m\pi}{L} t \right) \]

\[ u_x(x, t) = \frac{2}{\mu} \int_0^L \mu \phi(x - \frac{L}{3}) \sin \left( \frac{m\pi}{L} x \right) \sin \left( \frac{m\pi}{L} t \right) \, dx \]

\[ a_n = \frac{2}{L} \int_0^L \phi(x - \frac{L}{3}) \sin \left( \frac{m\pi}{L} x \right) \sin \left( \frac{m\pi}{L} t \right) \, dx \]

\[ a_n = \frac{2}{m^2 \sin \frac{m\pi}{L}} \int_0^L \phi(x - \frac{L}{3}) \sin \left( \frac{m\pi}{L} x \right) \sin \left( \frac{m\pi}{L} t \right) \, dx \]

\[ a_n = \frac{2}{m \sin \frac{m\pi}{3}} \frac{1}{m^2 \sin \frac{m\pi}{3}} \]

\[ y(x, t) = \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{\sin \frac{m\pi}{3}}{m^2 \sin \frac{m\pi}{3}} \cos \left( \frac{m\pi}{L} x \right) \sin \left( \frac{m\pi}{L} t \right) \]

If \( n = 3m \), this is zero \( \Rightarrow \) third harmonics are missing. 

Try this out in Mathematica!
Problem 6.17 in text

A line of charge with uniform line charge density \( \lambda \) (units: charge/length) lies along the z-axis. Find the
volume charge density a) in cylindrical coordinates
and b) in spherical coordinates.

In Cartesian, \( \rho_{\text{ch}}(x) = \rho_{\text{ch}}(x, y, z) = C \equiv \lambda \) constant.

What if it only went from \( 0 \leq z < \infty \)?

\[ \Rightarrow \rho_{\text{ch}}(y) = C \Theta(z) \]

\( \Theta \) is a step or delta function.

What if from \( -1 \leq z \leq 1 \)?

\[ \Rightarrow \rho_{\text{ch}}(x) = C \Theta(z+1) \Theta(1-z) \Theta(x) \]

Recall \( \Theta(x) = \int_{-\infty}^{x} \delta(z) \mathrm{d}z = \left\{ \begin{array}{ll} 1 & \text{if } x \geq 0 \\ 0 & \text{if } x < 0 \end{array} \right. \)

and \( \frac{\mathrm{d}\Theta(z)}{\mathrm{d}z} = \delta(z) \)

Find \( C \) by the condition that length \( L \) has charge \( \lambda L \)

\[ \Rightarrow \lambda L = \int_{-1}^{1} \int_{0}^{1} \int_{0}^{1} \rho_{\text{ch}}(x, y, z) \mathrm{d}x \mathrm{d}y \mathrm{d}z \]

\( \rho_{\text{ch}}(x, y, z) = C \Theta(z+1) \Theta(1-z) \Theta(x) \)

units: \( \rho = \frac{1}{\text{length}^3} \) so \( C \propto 1/\text{length}^3 \)

\( \checkmark \)
What about our coordinates?

What is \( S(x-x_0) \) in spherical and cylindrical?

\[
S(x-x_0) = S(r-r_0)S(\theta-\theta_0)S(\phi-\phi_0)
\]
\[
= S(r-r_0)S(\phi-\phi_0)S(z-z_0)
\]

Doesn't work that \( \int S(x-x_0)\,d^3x = 1 \)

due to

\[
\int_{\text{all space}} d\phi = 2\pi
\]

\[
d\phi = 2\pi \sin \alpha \, d\alpha \
\]

\[
\Rightarrow S(x-x_0) = \frac{1}{2\pi} S(r-r_0)S(\theta-\theta_0)S(\phi-\phi_0)
\]

\[
S(x-x_0) = \frac{4}{\pi} S(r-r_0)S(\phi-\phi_0)S(z-z_0)
\]

How do we deal with \( X_0 \)?

Then \( X_0 \) and \( \phi_0 \) don't matter \( \Rightarrow \) average over \( X_0 \) is one way

\[
S(x) \Rightarrow \frac{1}{4\pi^2} S(r) \Rightarrow S(r) \Rightarrow \frac{1}{2\pi} \int 0 \, dz
\]

Check it!

\[
[\text{requires } \int_S = 1 \text{ (not 0)} \because \text{this is } S(r,\theta,\phi)]
\]

So in problem 17

\[
\text{a) } S(\bar{x}) = \lambda S(r_0 \bar{y}) = \lambda \left( \frac{r_0}{\bar{r}} \right) \frac{1}{2\pi} = \lambda \int_0^{2\pi} S(r) \, d\phi \text{ check it!}
\]

\[
\text{b) } \text{check it}
\]

\[
\text{Next time!}
\]

(assumes \( \int_0^{2\pi} d\phi = 1 \))
9/27/13

Theta function follow up

\[ \theta(t-t_0) = \begin{cases} 1 & t \geq t_0 \\ 0 & t < t_0 \end{cases} \]

What is \( t = t_0 \)?

Note \( \theta(t-t_0) + \theta(t_0-t) = 1 \) for any \( t, t_0 \) suggests \( \theta(0) = \frac{1}{2} \).

What is \( \frac{d}{dt} \theta(t-t_0) \)? (Is \( \theta(t) \) a well-defined function?)

Consider \( \int f(t) \frac{d}{dt} \theta(t-t_0) \ dt \) when \( f(t) \to 0 \) as \( t \to \infty \).

Then partially integrate \( f(t) \theta(t-t_0) \) (with \( \theta(t-t_0) = 0 \) for \( t > t_0 \), \( \theta(t-t_0) = 1 \) for \( t < t_0 \)).

\[
\int_0^\infty f(t) \frac{d}{dt} \theta(t-t_0) \ dt = \left. f(t) \theta(t-t_0) \right|_0^\infty - \int_0^\infty \frac{df}{dt} \theta(t-t_0) \ dt
\]

\[
= \left[ f(t) \right]_0^\infty - \int_0^\infty \frac{df}{dt} \ dt
\]

Arbitrary \( f(t) \) \( \Rightarrow \) \( \frac{d}{dt} \theta(t-t_0) = \delta(t-t_0) \).

See Mathematica examples:

\[
\text{HeavisideTheta} \quad \left\langle \begin{array}{c}
\text{use most} \\
\text{UnitStep}
\end{array} \right. \\
\text{only this one knows}
\]

\[
\text{N[HeavisideTheta[1],10]} = \text{DiracDelta[1]}
\]
The charge on x-axis: \( q(x) = \delta(y) S(z) \) just stays the same.

Cylindrical: \( y = r \sin \phi \), \( x = r \cos \phi \).

That about \( \phi = \pi \) is on negative x-axis.

So: \( S(y) = S(r \sin \phi) \) because no \( S(x) \).

Spherical: \( \rho \), \( \theta \).

\[
S(\rho \sin \theta) = \frac{1}{\rho} S(r \sin \phi) \quad \text{because} \quad S(\rho \sin \theta) = S(r \sin \phi)
\]

Back to the issue of \( S(x, z) \) in spherical and cylindrical.

Getting to \( r \to 0 \) limit.

If we take the actual limit with \( \mathbf{r}_0 = (r_0, \theta_0, \phi_0) \) in spherical coordinates, then we could keep \( \theta_0 \) and \( \phi_0 \) fixed and take \( r \to 0^+ \).

\[
S(x) \to \frac{1}{2} S(r) S(\cos \theta_0 - \cos \phi_0) S(\phi_0 - \phi_0)
\]

So averaging over angles is just as good: \( S(x) = \frac{1}{4 \pi} S(r) \to \frac{1}{\rho} S(r) \to \frac{1}{\rho} S(r) \).

Because \( r \to 0^+ \), \( \int_0^\infty S(x) dx = \int_0^\infty S(r) dr = 1 \) (and not \( \frac{1}{3} \)).

So back to problem 6.17, where \( g(x) = \frac{1}{2} S(x) \).

Take the limit in cylindrical and spherical coordinates from it not being at the origin.

Cylindrical: \( x = r \cos \phi, y = r \sin \phi \) \( \Rightarrow S(x) \to \frac{1}{2} \int_0^\infty S(r) \).

Spherical: \( \rho = \frac{1}{2} \int_0^\infty S(r) \).

\[
\text{note: This is sometimes taken to be } \frac{1}{2}.
\]

Find data functions in coordinates that take on single values.

1. Determine function that multiplies appropriate region.

\[
S(x) \text{ and } S(y)
\]

This may be considered a mnemonic rather than a formal analysis.

\[
\int_0^\infty S(\rho) d\rho = 1
\]

\[
\int_0^\infty S(y) dy = 1
\]
Consider the comparison of a line of charge along
the x-axis, y-axis, and z-axis, in turn.
• You have the x-axis problem for homework.
• So we'll consider y-axis and z-axis here.
• Also consider on a sheet: \( y = \frac{1}{x} \)

In general, you want to follow a procedure something like this:
• Identify the range of each variable. May be dependent on each other (line of charge on slant)
• Use delta functions for constraints: combination that are fixed
• If a variable is ignorable, it can be replaced by a constant (example from 114)
• Simplify delta functions (and introduce measure factors)
• If not already included – e.g., FROM converting Cartesian expressions)
• Use \( \theta \) functions for finite endpoints.

Cylindrical \( \int_{r=a}^{r=b} dx \Rightarrow \theta = \frac{\theta}{\theta} \]
\( x = 0, y = 0, -L < z < L \)
\( \Rightarrow \phi(\theta) = \phi(x, y) \) \[ \theta(z+L) - \theta(z-L) \]

Additive is often easier, but not always.

To go to lines of charge on other axes, just switch labels \( \Rightarrow \) same form.
• Check normalization \( \int_{-L}^{L} e^{-\lambda} = e^{-\lambda} \) \( \Rightarrow e^{-\lambda} \)
$S(g(x)) = \frac{2}{\pi} \frac{g(x)}{\sin(x)} \quad \text{when } x_0 \text{ are zeros of } g(x)$

What if $\mu_2$ is in the $z=0$ plane?

Then $\phi(r) \neq S(2)$ for sure.

Range of $x$: $-\frac{1}{3} \leq x \leq \frac{1}{3}$

Constraint: $y = x$ or $y - x = 0 \Rightarrow S(y - x)$

$\Rightarrow S(\alpha(x)) = C \cdot S(y - x) \cdot S(z) \cdot g(x + \frac{1}{3}) - g(x - \frac{1}{3}) \cdot [6(y - x) - 6(y + x)]$

Check $S(\alpha(x)) = C \int_{\frac{1}{3}}^{\frac{1}{3}} dy \int dz \cdot S(y - x) \cdot S(z) = C \int_{\frac{1}{3}}^{\frac{1}{3}} dy \int dz = C \int_{\frac{1}{3}}^{\frac{1}{3}} = C \frac{1}{2}$

$\Rightarrow C = S(2)$

Or: $y'$

$x' = \frac{1}{2} x + \frac{1}{2} y \quad \Rightarrow S(y') = S(-\frac{1}{3} x + \frac{1}{3} y) = S(\frac{1}{3} y - x) = \frac{4}{\pi} S(y - x) = \frac{4}{\pi} S(\phi)$

$\Rightarrow S(\phi) = \frac{4}{\pi} [S(y - x)] \cdot [y - x] - y' = \text{as before}$

Now cylindrical on $z$-axis: $\rho = 0, \phi = \text{anything} \leq \pi$ replace by $\rho = \text{average over } z = \text{limit}$

$S(x) \cdot S(y) = \frac{4}{\pi} S(y - x) \exp \left[ i \sin(\phi) - \theta(z) \right]$

Note that $\int_{-\infty}^{\infty} S(\phi) d\phi = 0$

Cylindrical on $y$-axis: $z = 0, x = 0, -1 \leq y \leq 1 \Rightarrow \beta = 0, 0 \leq \phi \leq \pi$

$\Rightarrow \text{expect } S \{ \phi(x) \} \left[ S(\phi - \frac{\pi}{2}) + S(\phi + \frac{\pi}{2}) \right]$

Try translating: $S(x) = S(\cos \phi) = \frac{1}{2} \sin(\phi) \left[ S(\phi - \frac{\pi}{2}) + S(\phi + \frac{\pi}{2}) \right]$

Since $\sin^2 \phi = 1, \sin^2 \phi = -1$ works.

So check both ways!

Similar for spherical: $2$-axis $\theta = 0 \text{ or } \pi; \phi = \text{any angle}, 0 \leq \theta \leq \pi \text{ or } \pi \leq \theta \leq 2 \pi$

$y$-axis: $\theta = \frac{\pi}{2}, \phi = \frac{3\pi}{2}$

$0 \leq \theta \leq \pi$
Important equations we'll see again:

\[ \nabla \cdot \frac{\mathbf{A}}{r^3} = 4\pi S(\mathbf{r}) \]

or

\[ \nabla^2 \left( \frac{1}{r} \right) = -4\pi S(\mathbf{r}) \quad \text{(note } \nabla^2 \left( \frac{1}{r} \right) = -\frac{\delta}{r^2} \text{)} \]

On homework, \[ \nabla^2 \left( \ln \frac{r}{a} \right) = 2\pi S(\mathbf{r}) \], in the xy plane

where \[ \nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}, \quad a \text{ is any constant}, \quad r = \sqrt{x^2 + y^2} \]

Prove these by usual steps:

i) Easy to show \[ \nabla^2 \left( \frac{1}{r} \right) = 0 \text{ for } r \neq 0 \text{ (we've done this!)} \]

ii) Show satisfying properly

\[ \int \nabla^2 \left( \frac{1}{r} \right) f(\mathbf{r}) \, d^3r = -4\pi f(0) \]

and other vector calculus.

Proofs in Leo, Boos, etc. \Rightarrow \text{ use divergence theorem, but it's non-trivial } \Rightarrow \text{ some terms cancel}

Extension:

\[ \nabla \cdot \frac{\mathbf{A}}{r^3} = -4\pi S(\mathbf{r} - \mathbf{r}_0) \]

we'll return to this!
An E&M example (see example 6.9)

Find the solution for the electric field due to a sheet of charge with charge density \( \rho(x) = \sigma_0 S(x-a) \).

Usual method: invoke a Gaussian cylinder.

Instead, integrate from \( a-x \) to \( a+x \).

\[
\int_{a-x}^{a+x} \frac{dE_x}{dx} \cdot dx = E_x(a+x) - E_x(a-x) = \int_{a-x}^{a+x} \frac{\sigma_0 S(x-a)}{\varepsilon_0} \, dx = \frac{\sigma_0}{\varepsilon_0} S(a-x)
\]

By symmetry, \( E_x(a+x) = -E_x(a-x) \) [can we get this any other way?]

\[
\Rightarrow 2E_x(a+x) = \frac{\sigma_0}{\varepsilon_0} \quad \text{or} \quad E_x(a+x) = \frac{\sigma_0}{2\varepsilon_0}
\]

\[
E_x(x) = \frac{\sigma_0}{2\varepsilon_0} \left[ \theta(x-a) + \theta(a-x) \right]
\]

Plug in to check the equation

\[
\frac{dE_x}{dx} = \frac{\sigma_0}{\varepsilon_0} \left( \frac{d\theta(x)}{dx} - \frac{d\theta(x-a)}{dx} \right) \Rightarrow \frac{d\theta(x-a)}{dx} = -\theta(x-a)
\]

\[
= \frac{\sigma_0}{\varepsilon_0} \left[ \theta(x-a) - (-\theta(x-a)) \right] \Rightarrow \frac{\sigma_0}{\varepsilon_0} \theta(x-a) \quad \text{important to get the signs!}
\]

\[
= \frac{\sigma_0}{\varepsilon_0} S(x-a)
\]