- See immediate_vs_delayed_1.nb for a resolution to the Mathematica puzzle raised last time.

10/4/13 **Physics 7701 Lecture 17**

Fourier Transforms: Core Competencies

① definition and basic Fourier transforms ← completing the square ← Gaussian, exponential, delta function

② Using Mathematica for complicated transforms

③ Use Fourier transforms to solve partial differential equations

Definition

$$F(k) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)\, e^{-ikx}\, dx \qquad \text{transform}$$

$$f(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} F(k)\, e^{ikx}\, dk \qquad \text{inverse}$$

but symmetric (with − sign in exponent)

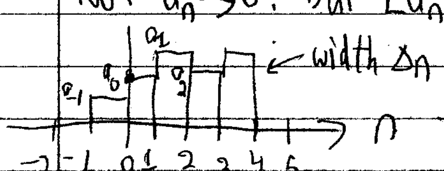
conventions:

- notation for $F(k)$: sometimes $\tilde{f}(k)$, sometimes just $f(k)$
- product of factors out front is $1/2\pi$ but different fields distribute differently (eg. $\int d^3x \Leftrightarrow \int \frac{d^3k}{(2\pi)^3}$)
- which exponential has the − sign (x,k) vs. (t,ω), for example

Comparison to Fourier Series: $f(x) = \sum_{n=-\infty}^{\infty} a_n e^{in\pi x/L}$ for $-L \le x \le +L$

Can we just take $L \to \infty$? $\qquad a_n = \frac{1}{2L}\int_{-L}^{L} f(x)\, e^{-in\pi x/L}\, dx$

No: $a_n \to 0$. But $L a_n$ would be ok.



$$\Rightarrow f(x) = \sum_{n=-\infty}^{\infty} a_n e^{in\pi x/L}\Delta n \underset{k=\frac{n\pi}{L}}{\overset{\text{rescale } \Delta n \text{ to}}{=}} \sum_n \left(\frac{L}{\pi} a_n\right) e^{i\left(\frac{n\pi}{L}\right)x} \frac{\Delta n\, \pi}{L}$$

sum is like an integral if we could take $\Delta n \to 0$.

$$\Rightarrow \Delta k = \Delta n \frac{\pi}{L} = \sum_{k(n)} \left(\frac{L}{\pi} a_n\right) e^{ikx} \Delta k \xrightarrow[L\to\infty]{} \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} F(k)\, e^{ikx}\, dk$$

where $\frac{1}{\sqrt{2\pi}}F(k) = \frac{L}{\pi}\, a\!\left(k = \frac{n\pi}{L}\right)$ [here $a(k)$ becomes continuous]

$$\Rightarrow F(k) = L\sqrt{\frac{2}{\pi}}\, a(k) \Rightarrow L\sqrt{\frac{2}{\pi}}\, a_n \xrightarrow[L\to\infty]{} F(k) = \sqrt{\frac{2}{\pi}}\,\frac{1}{2}\int_{-\infty}^{\infty} f(x)\, e^{-ikx}\, dx \quad ✓$$

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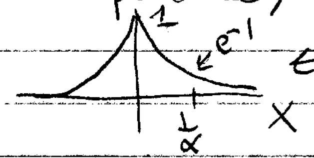
- Key examples: check with Mathematica FourierTransform

- Note: You should be able to do the manipulations here, not just using Mathematica (core competencies)

(A) $f(x) = e^{-\alpha|x|}$ exponential



"width" ⇒ standard deviation $\Delta x \sim 1/\alpha$

$$F(k) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-\alpha|x|}e^{-ikx}\,dx$$ ← just combine exponentials and integrate as usual

split so $|x|$ can be separated (but not even or odd ⇒ spot the error)

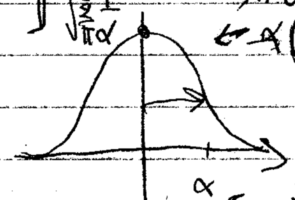
$$= \frac{1}{\sqrt{2\pi}}\left[\int_{-\infty}^{0} e^{(\alpha-ik)x}\,dx + \int_{0}^{\infty} e^{-(\alpha+ik)x}\,dx\right]$$

$x<0 \Rightarrow |x| = -x$ $\qquad$ $x>0 \Rightarrow |x| = +x$

$$= \frac{1}{\sqrt{2\pi}}\left[\frac{1}{\alpha-ik}e^{(\alpha-ik)x}\Big|_{-\infty}^{0} + \frac{e^{-(\alpha+ik)x}}{-(\alpha+ik)}\Big|_{0}^{\infty}\right]$$

$$= \frac{1}{\sqrt{2\pi}}\left(\frac{1}{\alpha-ik} - \left(\frac{-1}{\alpha+ik}\right)\right) = \boxed{\sqrt{\frac{2}{\pi}}\frac{\alpha}{\alpha^2+k^2}}$$

For k small, $F(k) = \sqrt{\frac{2}{\pi}}\frac{1}{\alpha}\frac{1}{1+k^2/\alpha^2}$ $\leftarrow \propto \left(1-\frac{k^2}{\alpha^2}\right)$



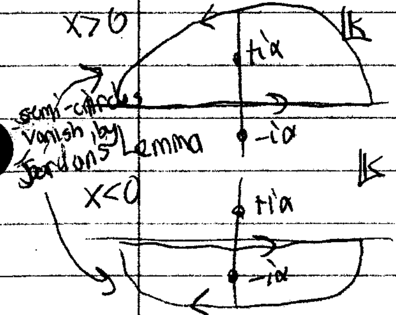
← $\Delta k \propto \alpha$

- product of widths, $\Delta x\,\Delta k \sim \frac{1}{\alpha}\cdot\alpha \sim 1$! (up to numerical factors)

More precise: $\Delta x \equiv \sqrt{\langle x^2\rangle - \langle x\rangle^2}$ (use normalized distribution for easiest calculation of $\langle x^2\rangle$, $\langle x\rangle$)

- Now the inverse: this is a familiar contour integral

$$f(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\sqrt{\frac{2}{\pi}}\frac{\alpha}{\alpha^2+k^2}e^{ikx}\,dk = \frac{\alpha}{\pi}\int_{-\infty}^{\infty}\frac{e^{ikx}}{(k+i\alpha)(k-i\alpha)}$$



$$x>0: f(x) = \frac{\alpha}{\pi}2\pi i\left(\frac{e^{-\alpha x}}{2i\alpha}\right) = e^{-\alpha x}$$

$$x<0: f(x) = \frac{\alpha}{\pi}(-2\pi i)\frac{e^{i(-i\alpha)x}}{-2i\alpha} = e^{\alpha x}$$

↖ clockwise

$\Big\}\ e^{-\alpha|x|}$ ✓

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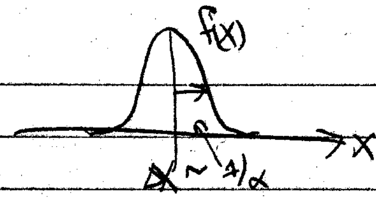
(B) Gaussian $f(x) = N e^{-\alpha^2 x^2}$



$$F(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} N e^{-\alpha^2 x^2} e^{-ikx} dx$$

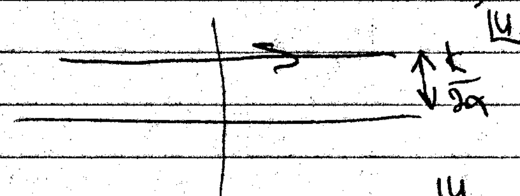
observe $e^{-\alpha^2(x+\beta)^2} = e^{-\alpha^2 x^2} e^{-2\alpha^2 \beta x} e^{-\alpha^2 \beta^2}$

$$\Rightarrow \underbrace{e^{-\alpha^2 x^2 - 2\alpha^2 \beta x}}_{\text{simplify terms}} = e^{-\alpha^2(x+\beta)^2} e^{+\alpha^2\beta^2} \Rightarrow 2\alpha^2\beta = ik \Rightarrow \beta = \frac{ik}{2\alpha^2}$$

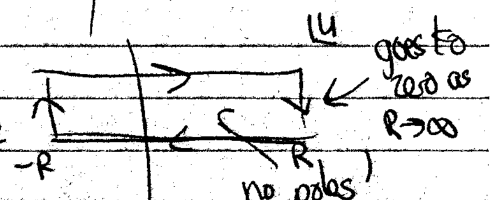
$$\alpha^2 \beta^2 = \alpha^2 \left(-k^2/4\alpha^4\right)$$

$$\Rightarrow F(k) = \frac{1}{\sqrt{2\pi}} e^{-\frac{k^2}{4\alpha^2}} N \int_{-\infty}^{\infty} e^{-\alpha^2\left(x + \frac{ik}{2\alpha^2}\right)^2} dx \quad \text{change to } u = \alpha\left(x + \frac{ik}{2\alpha^2}\right)$$

$$= \frac{N}{\sqrt{2\pi}} e^{-\frac{k^2}{4\alpha^2}} \int_{-\infty + \frac{ik}{2\alpha}}^{\infty + \frac{ik}{2\alpha}} e^{-u^2} \frac{du}{\alpha}$$



$$= \frac{N}{\sqrt{2\pi}} e^{-\frac{k^2}{4\alpha^2}} \frac{1}{\alpha} \underbrace{\int_{-\infty}^{\infty} e^{-u^2} du}_{= \sqrt{\pi}} \quad \text{because}$$



$\Rightarrow$ integral on top = integral on bottom

$$\boxed{F(k) = \frac{N}{\alpha\sqrt{2}} e^{-\frac{k^2}{4\alpha^2}}}$$

Gaussian $\xleftrightarrow[\text{Transform}]{\text{Fourier}}$ Gaussian

width $\Delta k \sim 2\alpha$ so $\Delta x \Delta k \sim \frac{1}{\alpha} 2\alpha \sim 1$ again

[More precise with $\Delta x = \sqrt{\langle x^2 \rangle - \langle x \rangle^2}$

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(c) $f(x) = 1 \Rightarrow$ delta function

$$F(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} 1 \cdot e^{-ikx} dx = \sqrt{2\pi}\, \delta(k)$$

$$\text{or} \quad \delta(k) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-ikx} dx \qquad \delta(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{+ikx} dk$$

Properties of Fourier transform $F[\ \ ]$ ← notation for applying the Fourier tran

- can you prove these?

- $F[f+g] = F[f] + F[g] \Leftarrow \int (f(x)+g(x)) e^{-ikx} dx = \int f(x) e^{-ikx} dx + \int g(x) e^{-ikx} dx$
- $F[af] = aF[f] \Leftarrow \int a f(x) e^{-ikx} dx = a \int f(x) e^{-ikx} dx$
- $F\left[\frac{df}{dx}\right] = ikF(k)$ ← take the derivative inside ← the transformed function, not the operator!)
- $F^*(k) = F(-k)$

Parseval: $$\int_{-\infty}^{\infty} |f(x)|^2 dx = \int_{-\infty}^{\infty} |F(k)|^2 dk$$

so normalized distribution in x is also normalized in k!

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Application to a PDE (partial differential equation): Diffusion

- physical situation: at $t=0$, total mass m at $x=\ell$ in long pipe of cross section A, full of water. Find $t>0$ distribution.

Or, temperature of rod initially at T_0, heated at one end to T_1 at $t=0$.

equation: $$\frac{\partial\rho(x,t)}{\partial t} = \underbrace{D}_{\text{diffusion constant}} \frac{\partial^2\rho(x,t)}{\partial x^2}$$ ρ is mass distribution (integrate over x to get m)

(cf. time-dependent Schrödinger's equation)

$\rho(x, t<0) = 0$ and assume $\rho(x,t) \xrightarrow[x\to\infty]{} 0$ [physically reasonable]

Initial condition: $\rho(x,0) = \frac{m}{A}\delta(x-L)$ [check: $\int_{\text{pipe}} d^3x\, \rho(x,0) = \frac{m}{A}\cdot A = m$ ✓]

Let $$\tilde{\rho}(k,t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \rho(x,t)e^{-ikx}\,dx$$

Plan: "apply" $\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-ikx}dx\,[\;]$ to both sides of diffusion equation

On left, pull $\frac{\partial}{\partial t}$ from integral $\Rightarrow \frac{\partial\tilde{\rho}(k,t)}{\partial t}$ (still a partial derivative!)

On right, $\frac{D}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-ikx}\frac{\partial^2\rho(x,t)}{\partial x^2}dx \Rightarrow$ partially integrate twice (no surface terms because $\rho(\pm\infty)=0$

$$= \frac{D}{\sqrt{2\pi}}(-ik)^2\int_{-\infty}^{\infty} e^{-ikx}\rho(x,t)\,dx = -k^2 D\tilde{\rho}(k,t) \;!$$

$$\Rightarrow \boxed{\frac{\partial\tilde{\rho}(k,t)}{\partial t} = -k^2 D\tilde{\rho}(k,t)}$$ but this is simple for any given value of k!

Solve by inspection: $\tilde{\rho}(k,t) = \tilde{\rho}_0(k)e^{-k^2Dt}$ and find $\tilde{\rho}_0$ from initial condition.

$$\tilde{\rho}_0(k) = \tilde{\rho}(k,0) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\frac{m}{A}\delta(x-L)e^{-ikx}dx = \frac{1}{\sqrt{2\pi}}\frac{m}{A}e^{-ikl} \Rightarrow \boxed{\tilde{\rho}(k,t) = \frac{m}{A}\frac{e^{-ikl}}{\sqrt{2\pi}}e^{-k^2Dt}}$$

Find $$\rho(x,t) = \frac{m}{A}\frac{1}{2\pi}\underbrace{\int_{-\infty}^{\infty} e^{(ikx-ikl-k^2Dt)}dk}_{\text{Complete square!}} = \frac{m}{A}\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-\left(\sqrt{Dt}\,k+i\frac{(l-x)}{2\sqrt{Dt}}\right)^2}e^{-\frac{1}{4}\frac{(x-l)^2}{Dt}}dk = \boxed{\frac{m}{A}\frac{1}{2\sqrt{\pi Dt}}e^{-\frac{(x-l)^2}{4Dt}}}.$$

note $1/\sqrt{t}$

← Gaussian spread