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7701 Lecture 18

Lecture Plan

- Continue through Fourier transform discussion
  - key examples from last time if incomplete
  - FT as a linear operator → apply to PDE's
  - 3D FT and homework problem
  - Wave equation
  - Sine and Cosine transforms
  - [illegible] convolution [illegible] 77)
  - [illegible] ally solving S eqn (7_) (7_)

Before class:

- Start up 7701 page + Mathematica
- notebook: diffusion1.nb [illegible]

On board:

$$F(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\, e^{-ikx}\, dx \quad \text{cf.,} \quad G(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} g(t)\, e^{i\omega t}\, dt$$

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(k)\, e^{ikx}\, dk \quad \text{cf.,} \quad g(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} G(\omega)\, e^{-i\omega t}\, dt$$

Recap: Carrying out transforms are just exercises in doing integrals.

- often contour integrals because $\int_{-\infty}^{\infty}$ integrals
- Mathematica can do many using FourierTransform or InverseFourierTransform (Convention $e^{i\omega t}/e^{-i\omega t}$)
  - ⇒ be careful of convention → you may get complex conjugate
- Several examples you need to be familiar with (problem set!)
  - exponential 118, Gaussian (completing the square) 119, delta function 120. Also, use of Θ functions.

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Revisit application to partial differential equations (diffusion example)

Key: consider linear differential equations ⇒ taking a Fourier transform is a linear operation. [generalize: other integral transforms]

Denote it $F[\ ]$ ⇒ $F[f] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-ikx} dx$

Apply to diffusion equation from last time
(mass m of dye initially a $x=\ell$ in long pipe of water. Find $t \geq 0$ distribution)

$$F\left[\frac{\partial \rho(x,t)}{\partial t} - D\frac{\partial^2 \rho(x,t)}{\partial x^2} = 0\right]$$ ← cf. PS#6, diffusion equation with additional term

linear: $F[f+g] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} (f(x)+g(x)) e^{-ikx} dx$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x) e^{-ikx} dx + \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} g(x) e^{-ikx} dx$$

$$= F[f] + F[g]$$

Important: f and g here are just patterns to match; they can be anything.
This is how Mathematica works! (example notebook if time)

$$\Rightarrow F\left[\frac{\partial \rho}{\partial t}\right] - F\left[D\frac{\partial^2 \rho}{\partial x^2}\right] = 0 \qquad (F[0]=0)$$

Now $F[af(x)] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} (af(x)) e^{-ikx} dx = a\left(\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x) e^{-ikx} dx\right) = aF[f]$

$$\Rightarrow F\left[\frac{\partial \rho}{\partial t}\right] - DF\left[\frac{\partial^2 \rho}{\partial x^2}\right] = 0$$

still partial since dependence on k

But $F\left[\frac{\partial f}{\partial t}\right] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \frac{\partial f(x,t)}{\partial t} e^{-ikx} dx = \frac{\partial}{\partial t}\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x,t) e^{-ikx} dx = \frac{\partial}{\partial t}F[f]$

But $F\left[\frac{\partial f}{\partial x}\right] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \frac{\partial f}{\partial x} e^{-ikx} dx \overset{\text{by parts}}{=} -\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x,t)\frac{d}{dx}e^{-ikx} dx + \underbrace{\cancel{f(x,t)e^{-ikx}\Big|_{-\infty}^{\infty}}}_{0 \text{ by assumption}}$

$$= +ik\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x,t) e^{-ikx} dx = +ik F[f]$$

two minus signs

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So we have $\frac{\partial}{\partial t} F[\rho] - D(ik)^2 F[\rho] = 0$

or finally

$$\frac{\partial}{\partial t}\tilde{\rho}(k,t) + \underbrace{k^2 D}_{>0}\tilde{\rho}(k,t) = 0$$

Consider this for each $k \Rightarrow$ 1st order like $\frac{\partial f}{\partial t} = \alpha^2 f \Rightarrow f(t) = f_0 e^{\alpha^2 t}$

$$\Rightarrow \tilde{\rho}(k,t) = \tilde{\rho}(k,0)\, e^{-Dk^2 t}$$ here k^2 in exponent is just a constant

Find $\tilde{\rho}(k,0)$ from initial conditions

$$\tilde{\rho}(k,0) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \underbrace{\rho(x,0)}_{\frac{m}{A}\delta(x-l)} e^{-ikx}\,dx = \frac{1}{\sqrt{2\pi}}\frac{m}{A}e^{-ikl}$$

cross sectional area $\rightarrow \frac{m}{A}\delta(x-l)$

$$\Rightarrow \boxed{\tilde{\rho}(k,t) = \frac{m}{A}\frac{e^{-ikl}}{\sqrt{2\pi}}e^{-k^2 Dt}}$$

Inverse transform: $\rho(x,t) = \frac{m}{A}\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ikx-ikl-k^2 Dt}\,dk \underset{\text{square}}{\overset{\text{complete}}{=}} \frac{m}{A}\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-\left(\sqrt{Dt}\,k + \frac{i(l-x)}{2\sqrt{Dt}}\right)^2} e^{-\frac{1}{4}\frac{(x-l)^2}{Dt}}\,dk$

check $-\left(\sqrt{Dt}\,k + \frac{i(l-x)}{2\sqrt{Dt}}\right)^2 = -Dt\,k^2 - 2\sqrt{Dt}\,k\,\frac{i(l-x)}{2\sqrt{Dt}} - \frac{i^2(l-x)^2}{4Dt} - \frac{1}{4}\frac{(x-l)^2}{Dt}$ ✓

$$\Rightarrow \rho(x,t) = \frac{m}{A}\frac{1}{2\pi}e^{-\frac{1}{4}\frac{(x-l)^2}{Dt}}\frac{1}{\sqrt{Dt}}\cdot\sqrt{\pi} = \frac{m}{A}\frac{1}{2\sqrt{\pi Dt}}e^{-(x-l)^2/4Dt}$$

shift $\rightarrow$ integral; no poles

· How do we check?

① satisfies initial condition? $\delta_n(y) = \frac{n}{\sqrt{\pi}}e^{-n^2y^2} \xrightarrow[n\to\infty]{} \delta(y)$

$y = x-l,\ n^2 = 1/4Dt \Rightarrow n = \frac{1}{2\sqrt{Dt}} \Rightarrow \frac{m}{A}\frac{n}{\sqrt{\pi}}e^{-n(x-l)^2} \xrightarrow[t\to 0]{n\to\infty} \frac{m}{A}\delta(x-l)$ ✓

② satisfies differential equation? Not obviously! Only $\frac{1}{\sqrt{t}}e^{-\frac{(x-l)^2}{4Dt}}$ matters

$$\frac{\partial}{\partial t} t^{-1/2}e^{-\frac{(x-l)^2}{4Dt}} = \left[+\frac{1}{\sqrt{t}}\frac{(x-l)^2}{4D}\frac{1}{t^2} - \frac{1}{2}t^{-3/2}\right]e^{-\frac{(x-l)^2}{4Dt}} \overset{?}{=} D\left[\frac{-2}{4Dt^{3/2}} + \frac{4(x-l)^2}{16D^2t^2}\frac{1}{\sqrt{t}}\right]e^{-\frac{(x-l)^2}{4Dt}}$$ ✓ works

Show spreading of ~~m~~ initial distribution.

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Clear["Global`*"]
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Animate the solution to the diffusion example of a mass m of dye starting at x=l and diffusing in a pipe of cross sectional area A with diffusion constant Df.

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rho[x_, t_, Df_] := (m / A) * 1 / (2 Sqrt[Pi Df t]) Exp[- (x - l) ^2 / (4 Df t)]
```

Pick some values:

```
m = 1; A = 1; l = 1 / 2;

xmin = -10; xmax = +10;
tmin = 0.001; tmax = 10;

Df1 = 0.5;

Animate[Plot[rho[x, t, Df1], {x, xmin, xmax}, PlotRange → Full],
 {t, tmin, tmax}, AnimationRunning → False, AnimationDirection → ForwardBackward]
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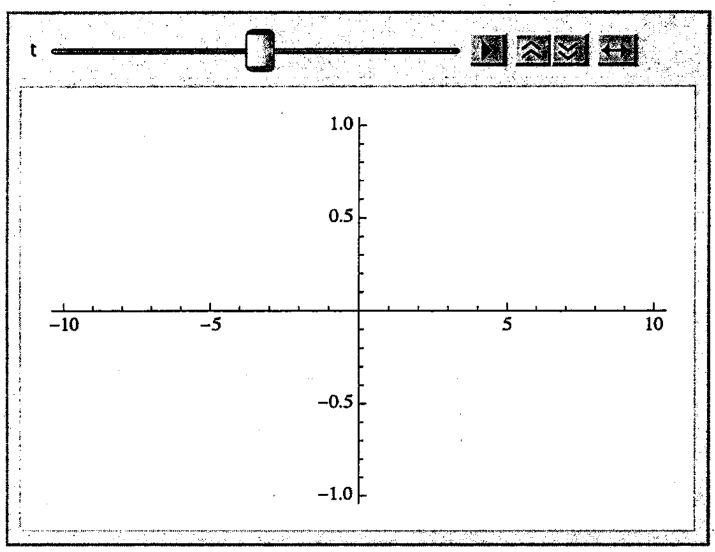


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Df2 = 0.1;
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Fourier Transform of Wave Equation

Let's revisit the wave equation $\frac{\partial^2 y}{\partial t^2} = v^2 \frac{\partial^2 y}{\partial x^2}$. Should we Fourier transform with respect to t or x? Answer: both!

$$\tilde{y}(k,\omega) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} dx\, e^{-ikx} \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} dt\, e^{i\omega t}\, y(x,t)$$ note $e^{-ikx} e^{i\omega t}$ convention

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} dx\,dt\, e^{-ikx+i\omega t}\, y(x,t)$$

inverse: $$y(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} dk \int_{-\infty}^{\infty} d\omega\, \tilde{y}(k,\omega)\, e^{ikx-i\omega t} \, dk\, d\omega$$

$$\Rightarrow (-i\omega)^2 \tilde{y} = v^2 (ik)^2 \tilde{y} \Rightarrow \tilde{y}(\omega^2 + v^2 k^2) = 0 \Rightarrow \boxed{\tilde{y}(\omega + vk)(\omega - vk) = 0}$$

Now what? $\tilde{y} = 0$ is a solution, but not the one we want. Looks like $\omega = \pm vk$, but what does that imply?

When is a Fourier transform zero except for special values of ω? Answer: when it has delta functions!

$$\Rightarrow \boxed{\tilde{y}(k,\omega) = A(k)\,\delta(\omega - vk) + B(k)\,\delta(\omega + vk)}$$

check: $(\omega+vk)(\omega-vk)\tilde{y}(k,\omega) = A(k)(\omega+vk)\cancel{(\omega-vk)\delta(\omega-vk)}^{0} + B(k)(\omega-vk)\cancel{(\omega+vk)\delta(\omega+vk)}^{0}$

$$\Rightarrow y(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} dk \int_{-\infty}^{\infty} d\omega\, e^{ikx-i\omega t}\left[A(k)\,\delta(\omega - kt) + B(k)\,\delta(\omega + vk)\right]$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} dk \left[A(k)\, e^{ik(x-vt)} + B(k)\, e^{ik(x+vt)}\right]$$

moves to right   moves to left

What if we had just transformed wrt x and then solved the equation wrt t?

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Extension to 3-dimensions

e.g. homework problem $\rho(\vec{r}) = \rho_0 \frac{e^{-r/a}}{4\pi r}$

If we think of this as $\rho(x,y,z)$, the extension is trivial $\Rightarrow F_x F_y F_z [\rho]$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dx \, e^{-ik_x x} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dy \, e^{-ik_y y} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dz \, e^{-ik_z z} \rho(x,y,z)$$

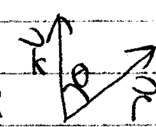
$$= \frac{1}{(2\pi)^{3/2}} \int d^3r \, e^{-i\vec{k}\cdot\vec{r}} \rho(\vec{r})$$

Linear properties carry through. Gradients and the like become:

$$F[\vec{\nabla} f] = i\vec{k} F[f]$$
$$F[\nabla^2 f] = -k^2 F[f]$$
$$F[\vec{\nabla} \times \vec{f}] = i\vec{k} \times F[\vec{f}]$$

But what if $\rho(\vec{r}) = \rho(r)$?
Then spherical coordinates makes sense:

$\vec{k}\cdot\vec{r} = kr\cos\theta$ so choose $\hat{z}$-axis aligned with $\vec{k}$:



$$\Rightarrow \int d^3r = \int_0^{\infty} r^2 dr \int_0^{\pi} \underbrace{\sin\theta \, d\theta}_{-d\cos\theta} \int_0^{2\pi} d\varphi = \int_0^{\infty} r^2 dr \int_{-1}^{1} dx \int_0^{2\pi} d\varphi \qquad x = \cos\theta$$

Homework problem is typical example of what you'll encounter.

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Sine and Cosine Fourier Transforms

Just as with Fourier series, we can also have Sine and Cosine Fourier transforms, which add an assumption about $f(x)$ compared to $f(-x)$. Two general cases:

1. $f(x)$ is known to be even or odd about $x=0$.
2. $f(x)$ is defined only for $x>0$.

Cosine

$$F_c(k) = \sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\cos kx \, dx = F_c(-k)$$

for cosine →

$$f(x) = \sqrt{\frac{2}{\pi}}\int_0^\infty F_c(k)\cos kx \, dk = f(-x) \quad \text{[even extension to } x<0\text{]}$$

Sine

$$F_s(k) = \sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\sin kx \, dx$$

$$f(x) = \sqrt{\frac{2}{\pi}}\int_0^\infty F_s(k)\sin kx \, dk$$

What is different? We have integrals from $\int_0^\infty$ ⇒ surface terms can survive.

remember [ ]'s → mean the linear operator

$$\mathcal{F}_c\left[\frac{df}{dx}\right] = -\sqrt{\frac{2}{\pi}}f(0) + k\mathcal{F}_s[f] \qquad \text{check!}$$

$$\mathcal{F}_s\left[\frac{df}{dx}\right] = -k\mathcal{F}_c[f]$$

$$\mathcal{F}_c\left[\frac{d^2f}{dx^2}\right] = -\sqrt{\frac{2}{\pi}}f'(0) + k\mathcal{F}_s\left[\frac{df}{dx}\right] = -\sqrt{\frac{2}{\pi}}f'(0) - k^2\mathcal{F}_c[f]$$
$$= -\sqrt{\frac{2}{\pi}}f'(0) - k^2F_c(k)$$

$$\mathcal{F}_s\left[\frac{d^2f}{dx^2}\right] = -k\mathcal{F}_c\left[\frac{df}{dx}\right] = k\sqrt{\frac{2}{\pi}}f(0) - k^2F_s(k)$$

⇒ · Most useful when only even or odd derivatives (so that $\mathcal{F}_s \Rightarrow \mathcal{F}_s$ only, etc.) and choose according to initial conditions (eg. whether $f(0)$ or $f'(0)$ is known)

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So suppose we have $\frac{d^2y}{dx^2} - \alpha^2 y = 0$ [so we know the solutions]

for two cases: i) $y(0) = y_0$ and $y \to 0$ as $x \to \infty$

ii) $y'(0) = y'_0$ and $y \to 0$ as $x \to \infty$

} boundary (initial) conditions

Note: Why don't we need two initial conditions?
Because using FT implies that $f(x) \to 0$ as $x \to \infty$.
Otherwise use Laplace Transform (more later).

i) $y(0)$ is known, so use F_s

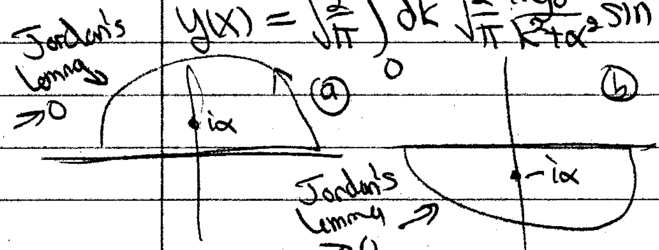
$$\Rightarrow F_s\left[\frac{d^2y}{dx^2} - \alpha^2 y = 0\right] \Rightarrow -k^2 F_s(y) + k\sqrt{\frac{2}{\pi}}\, y_0 - \alpha^2 F_s(y) = 0$$

$$\Rightarrow F_s(y) = \sqrt{\frac{2}{\pi}}\,\frac{k y_0}{k^2+\alpha^2}$$

$k \sin kx$ is even (factor of $\frac{1}{2}$)

Now we just find the inverse

$$y(x) = \sqrt{\frac{2}{\pi}}\int_0^\infty dk\, \sqrt{\frac{2}{\pi}}\,\frac{k y_0}{k^2+\alpha^2}\sin kx = \frac{y_0}{\pi}\int_{-\infty}^{\infty} dk\, \frac{k}{k^2+\alpha^2}\,\frac{\overset{(a)}{e^{ikx}} - \overset{(b)}{e^{-ikx}}}{2i}\, dk$$



$$= \frac{y_0}{2\pi i}\left[(+2\pi i)\frac{i\alpha}{2i\alpha}e^{i(i\alpha)x} - \underset{\text{clockwise}}{(-2\pi i)}\frac{-i\alpha}{2(-i\alpha)}e^{-i(-i\alpha)x}\right]$$

$$= y_0 e^{-\alpha x}$$ which is what we would have written down.

checks: $y(0) = y_0$ ✓, $y \to 0$ as $x \to \infty$, satisfies diff eq.

ii) Now $y'(0)$ is known, so use F_c

$$\Rightarrow F_c\left[\frac{d^2y}{dx^2} - \alpha^2 y = 0\right] \Rightarrow -k^2 F_c(y) - \sqrt{\frac{2}{\pi}}\, y'_0 - \alpha^2 F_c(y) = 0$$

$$\Rightarrow F_c(y) = -\sqrt{\frac{2}{\pi}}\,\frac{y'_0}{k^2+\alpha^2}$$ ⇒ Find inverse ⇒ same contours

$$y(x) = \sqrt{\frac{2}{\pi}}\int_0^\infty dk\, -\sqrt{\frac{2}{\pi}}\,\frac{y'_0}{k^2+\alpha^2}\cos kx = -\frac{y'_0}{\pi}\int_{-\infty}^{\infty} dk\, \frac{1}{k^2+\alpha^2}\,\frac{e^{ikx}+e^{-ikx}}{2}\, dk$$

$$= -\frac{y'_0}{2\pi}\left[(+2\pi i)\frac{1}{2i\alpha}e^{i(i\alpha)x} + (-2\pi i)\frac{1}{2(-i\alpha)}e^{-i(-i\alpha)x}\right] = -\frac{y'_0}{\alpha}e^{-\alpha x}$$, Check $\frac{dy}{dx} = y'_0 e^{-\alpha x}\Big|_{x=0} = y'_0$ ✓