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$\boxed{770}$ Lecture 19

Lecture Plan

- Loose ends on Fourier transforms
- Comments on homework.
- Comments on Mathematica

Before class:

- Start up 770 page + Mathematica
- $\text{FourierTransform}[f[x], x, k] =' \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{+ikx}\, dx$ (plus)
  $\text{InverseFourierTransform}[f[x], x, k] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-ikx}\, dx$

} choose according to convention (e.g. $\{k, x\}$ vs. $\{\omega, t\}$)

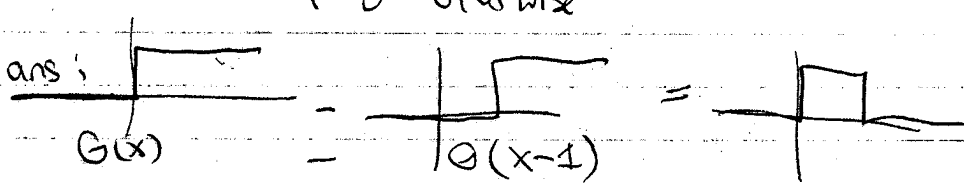
On board:

Warm-ups:

① a) Write $f(x) = \begin{cases} x & \text{if } 0 \le x \le 1 \\ 0 & \text{otherwise} \end{cases}$ in terms of Θ functions

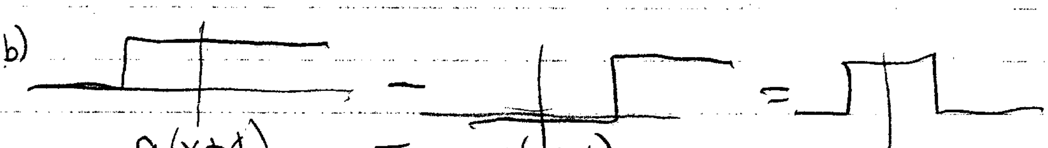
b) Write $f(x) = \begin{cases} e^x & \text{if } -1 \le x \le 1 \\ 0 & \text{otherwise} \end{cases}$ in terms of Θ functions

a) ans:



$\Theta(x) - \Theta(x-1)$

b)



$\Theta(x+1) - \Theta(x-1)$

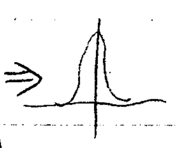
$= \Theta(x-(-1))$

② What is $\int_{-\infty}^{\infty} e^{ik(x-x_0)}\, dk$? ans: $2\pi\,\delta(x-x_0)$

$\int_{-\infty}^{\infty} e^{-ik(x-x_0)}\, dk$? ans: $2\pi\,\delta(x-x_0)$

Cf. discussion $\delta^3(\vec{r}) = \frac{1}{4\pi r^2}\delta(r)$ → $\int_0^{\infty} dr\, \delta(r) = \frac{1}{2}$

relevant for PS#6 prob. 4c ⇒ $\int_0^{\infty} \cos kx\, dk$? ans: $\pi\,\delta(k)$ (not $2\pi\,\delta(k)$) ⇒



symmetric: so only half included

$\int_0^{\infty} \delta(k)\, dk = \frac{1}{2}$ here

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Recap on Sine and Cosine Transforms (on board at beginning)

a) $F_c[f] = F_c(k) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \cos kx \, dx$

b) $F_s[f] = F_s(k) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \sin kx \, dx$

c) $F_c\left[\frac{df}{dx}\right] = -\sqrt{\frac{2}{\pi}} f(0) + k F_s(k)$

d) $F_s\left[\frac{df}{dx}\right] = -k F_c(k)$

e) $F_c\left[\frac{d^2f}{dx^2}\right] = -\sqrt{\frac{2}{\pi}} \left.\frac{df}{dx}\right|_{x=0} - k^2 F_c(k)$

f) $F_s\left[\frac{d^2f}{dx^2}\right] = k\sqrt{\frac{2}{\pi}} f(0) - k^2 F_s(k)$

- $f(x)$ for $x>0$ only (or $f(x)$ known to be even or odd)
- In diffusion problem: use a) + e) since 2nd order and $\left.\frac{d\rho(y,t)}{dy}\right|_{y=0}$ given
- Note: rate $r = \int_0^\infty dy \left.\frac{\partial\rho}{\partial t}\right|_{t=0}$ checks $m \cdot \frac{\text{atoms}/m^3}{s} = \text{atoms}/m^2 s$ ✓

  proportional to $\delta(y)$

Any questions?

- Do (124), cases i) and ii) simultaneously

Consider $\int_{-\infty}^\infty e^{ikx} dk = 2\pi\delta(x)$

$\int_{-\infty}^\infty e^{-ikx} dk = 2\pi\delta(x)$

but this $\delta(x)$ means that $\int_0^\infty \delta(x) dx = \frac{1}{2}\int_{-\infty}^\infty \delta(x) dx = \frac{1}{2}$

since even

sum $\int_{-\infty}^\infty (e^{ikx} + e^{-ikx}) dk = 2\int_{-\infty}^\infty \cos kx \, dk = 4\pi\delta(x)$

$\Rightarrow \int_{-\infty}^\infty \cos kx \, dk = 2\pi\delta(x)$ or $\int_0^\infty \cos kx \, dk = \pi\delta(x)$

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Initial conditions and Causality in Fourier Transform theory

Based on Lau, example 7.4

Key take-away point: causality (response only happens after a driving term starts) is reflected in location of poles of Fourier transform. (cf. homework PS#6 problem 3)

Electron at rest initially, then $\vec{E}(t) = \Theta(t)\vec{E}_0 e^{-\alpha t}$ acts (so $t>0$ only)
- Damping force $\vec{F}_d = -\gamma\vec{v}$, $\gamma > 0$
- Find motion of electron. Causality $\vec{v}(t) = 0$ for $t<0$!

$\vec{F}=m\vec{a}$: $\quad m\frac{d\vec{v}}{dt} = -\gamma\vec{v} - e\vec{E}(\vec{x},t)$

- looks 3-Dimensional but motion is only in one-dimension ⇒ from $\vec{E}_0$

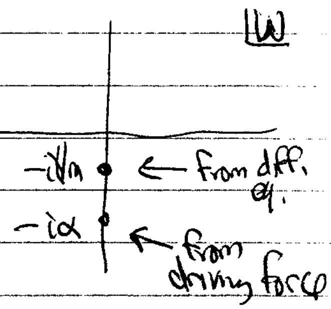
Take FT with $F(\omega) = \int_{-\infty}^{\infty} f(t)e^{i\omega t}dt$ ← note choice of sign for time FT ⇒ only pay attention to t dependence

$$\Rightarrow \tilde{\vec{E}}(\omega) = \frac{\vec{E}_0}{\sqrt{2\pi}}\int_0^\infty e^{-\alpha t}e^{i\omega t}dt = \frac{\vec{E}_0}{\sqrt{2\pi}}\frac{1}{i\omega-\alpha}e^{-\alpha t}e^{i\omega t}\Big|_0^\infty = \frac{\vec{E}_0}{\sqrt{2\pi}}\frac{1}{\alpha-i\omega} = \frac{\vec{E}_0}{\sqrt{2\pi}}\frac{i}{\omega+i\alpha}$$

key; pole in definite (lower) half plane

F equation ⇒ $-i\omega m\tilde{\vec{v}}(\omega) + \gamma\tilde{\vec{v}}(\omega) = -e\tilde{\vec{E}}(\omega)$

$$\Rightarrow \tilde{\vec{v}}(\omega) = \frac{1}{-i\omega m+\gamma}\frac{-e\vec{E}_0}{\sqrt{2\pi}}\frac{i}{\omega+i\alpha} = \frac{e}{m}\frac{\vec{E}_0}{\sqrt{2\pi}}\frac{1}{\omega+i\gamma/m}\frac{1}{\omega+i\alpha}$$



* Causality requires transform have no poles in the upper half plane.

$$\vec{v}(t) = \frac{1}{2\pi}\frac{e}{m}\vec{E}_0\int_{-\infty}^{\infty}\frac{1}{\omega+i\gamma/m}\frac{1}{\omega+i\alpha}e^{-i\omega t}d\omega$$

If $t<0$, close in upper half plane ⇒ $\vec{v}(t<0) = 0$!

BC's: $\vec{v}=0$ at $t=0$, $\vec{v}\to 0$ as $t\to\infty$

$$\Rightarrow \vec{v}(t) = \Theta(t)\frac{1}{2\pi}\frac{e}{m}\vec{E}_0(-2\pi i)\left(\frac{e^{-i(-i\alpha)t}}{(-i\alpha+i\gamma/m)} + \frac{e^{-i(-i\gamma/m)t}}{(-i\gamma/m+i\alpha)}\right) = \boxed{\frac{e\vec{E}_0}{\alpha m-\gamma}\left(e^{-\alpha t}-e^{-\frac{\gamma}{m}t}\right)}$$

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Removable singularities in contour integrals

Compare

a) $\displaystyle\int_{-\infty}^{\infty} \frac{\sin kx}{x}\,dx = \pi\left[\theta(k)-\theta(-k)\right]$

b) $\displaystyle P\int_{-\infty}^{\infty} \frac{\sin kx}{x}\,dx = \pi\left[\theta(k)-\theta(-k)\right]$

$\varepsilon \to 0^+$ implied

c) $\displaystyle\int_{-\infty}^{\infty} \frac{\sin kx}{x-i\varepsilon}\,dx = \pi\left[\theta(k)-\theta(-k)\right]$

d) $\displaystyle\int_{-\infty}^{\infty} \frac{\sin kx}{x+i\varepsilon}\,dx = \pi\left[\theta(k)-\theta(-k)\right]$

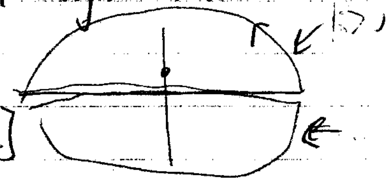
Analysis: Apparent pole at $x=0$, but $\lim_{x\to 0} \frac{\sin kx}{x} \to \frac{kx}{x} = k$
so not a real pole ⇒ "removable"

This means that a) = b) = c) = d) because $\varepsilon\to 0$ limits are all the same (nothing funny happens at $x=0$!)

⇒ use whichever is most convenient (e.g. fewest integrals to calculate)

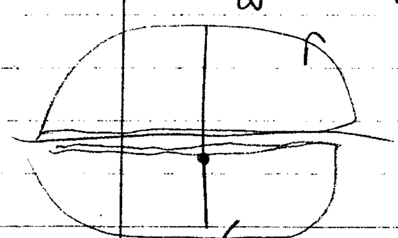
c) $\displaystyle \frac{1}{2i}\int_{-\infty}^{\infty} \frac{\overset{i)}{e^{ikx}} - \overset{ii)}{e^{-ikx}}}{x-i\varepsilon} = \frac{1}{2i}\Big[\theta(k)\big[(2\pi i)\cdot 1 - (-2\pi i)\cdot 0\big] + \theta(-k)\big[(-2\pi i)\cdot 0 - (+2\pi i)\cdot 1\big]\Big]$

$= \pi\left[\theta(k) - \theta(-k)\right]$



d) $\displaystyle \frac{1}{2i}\int_{-\infty}^{\infty} \frac{e^{ikx} - e^{-ikx}}{x+i\varepsilon} = \frac{1}{2i}\Big[\theta(k)\big[(2\pi i)\cdot 0 - (-2\pi i)\cdot 1\big] + \theta(-k)\big[(-2\pi i)\cdot 1 - (2\pi i)\cdot 0\big]\Big]$

$= \pi\left[\theta(k) - \theta(-k)\right]$ ✓



cf. 4b) $\frac{\partial}{\partial t}\rho_c(k,t) = (-Dk^2-\lambda)\rho_c(k,t) + \sqrt{\frac{2}{\pi}}\alpha D$

$$\Rightarrow \frac{df}{dx} = -af(x)+b \Rightarrow f(x) = Ce^{-ax} - \frac{b}{a}$$

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• Undetermined coefficients for diff. eqs. and integrals

Suppose $\frac{df(x)}{dx} + af(x) = b_0 + b_1x + b_2x^2$. What is $f(x)$?

First solve $\frac{df_1}{dx} + af_1 = 0 \Rightarrow f_1(x) = Ce^{-ax}$ $\quad C = \text{constant} = f_1(0)$

Now take $f(x) = f_1(x) + d_0 + d_1x + d_2x^2$ $\quad d_i$ are "undetermined coefficients"

substitute $\Rightarrow \frac{df}{dx} + af = \Big(\underbrace{\cancel{\frac{df_1}{dx} + af_1}}_{0}\Big) + d_1 + 2d_2x + a(d_0 + d_1x + d_2x^2)$

$$= b_0 + b_1x + b_2x^2$$

equate coefficients of each x^n ; $\quad b_0 = d_1 + ad_0$

$b_1 = 2d_2 + ad_1$

$b_2 = ad_2$

Solve : $d_2 = \frac{b_2}{a}$; $d_1 = (b_1 - 2d_2)/a = \frac{b_1}{a} - \frac{2b_2}{a}$;

$d_0 = (b_0 - d_1)/a = b_0/a - b_1/a^2 + 2b_2/a^2$

Diffusion case : $b_1 = b_2 = 0 \Rightarrow f(x) = Ce^{-ax} + b_0/a$ ✓

$f(0) = C + b_0/a$ so $C \neq f(0)$ here!

• Same idea used by Mathematica for many integrals.

special case of Risch algorithm

• Basic principle: The form of many indefinite integrals can be known with undetermined coefficients (rational functions of polynomials, trig, exp.)

• derivatives are easy to program $\Rightarrow$ work backwards

example $I = \int x^2e^x\,dx = ax^2e^x + bxe^x + ce^x$ $\quad$ find a, b, c

$$x^2e^x = \frac{d}{dx}(ax^2e^x + bxe^x + ce^x) = 2axe^x + ax^2e^x + be^x + bxe^x + ce^x$$

$x^2e^x \quad xe^x \quad e^x$

$\Rightarrow 1 = a$, $0 = 2a + b$, $0 = b + c$ $\Rightarrow a = 1, b = -2, c = 2$

$\Rightarrow I = x^2e^x - 2xe^x + 2e^x$ ✓

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Fourier Transform: convolution

Suppose $F(k) = \mathcal{F}[f(x)]$ and $G(k) = \mathcal{F}[g(x)]$

Then $H(k) = F(k)\,G(k) \xrightarrow[\text{transform}]{\text{inverse}} h(x)$

Is $h(x) = f(x)\,g(x)$? No! What is it?

$$h(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} H(k)\, e^{ikx}\, dk$$ note different x variables

substitute $$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\left[\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x_1)\, e^{-ikx_1}\, dx_1\right]\left[\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} g(x_2)\, e^{-ikx_2}\, dx_2\right] e^{ikx}\, dk$$

Where is k dependence? All in exponents: $e^{-ik(x_1+x_2-x)}$

$\Rightarrow$ do k integral first $\displaystyle\int_{-\infty}^{\infty} e^{-ik(x_1+x_2-x)}\, dk = 2\pi\,\delta(x_1+x_2-x)$

$$\Rightarrow h(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} dx_1 \int_{-\infty}^{\infty} dx_2\, f(x_1)\, g(x_2)\, \delta(x_1+x_2-x) \quad \Rightarrow x_2 = x - x_1$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} dx_1\, f(x_1)\, g(x-x_1)$$ convolution!