10/16/2013 7701 Lecture 20: Vector Calculus Review

Div ($\vec{\nabla}\cdot\vec{A}$), Grad ($\vec{\nabla}\psi$), Curl ($\vec{\nabla}\times\vec{A}$) and all that...

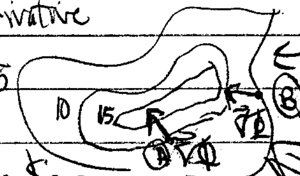
- Look at Jackson covers handout for explicit forms of vector operations
- Read Arfken or Lea or Cahill for more on physical interpretations

- gradient: $\vec{\nabla} = \hat{x}\frac{\partial}{\partial x} + \hat{y}\frac{\partial}{\partial y} + \hat{z}\frac{\partial}{\partial z}$ ⟸ derivatives give information about local change

vector to left of derivative

scalar field $\Phi(x,y,z)$ $\xrightarrow{\text{plot}}$ $\Phi(x,y)$

$\Rightarrow \vec{\nabla}\Phi = \hat{x}\frac{\partial\Phi}{\partial x} + \hat{y}\frac{\partial\Phi}{\partial y} + \hat{z}\frac{\partial\Phi}{\partial z}$



← Φ = constant lines

class draw $\vec{\nabla}\Phi$

[Think of numerical implementation] Take small step in Cartesian coordinates

$\Delta\vec{r} = \hat{x}\Delta x + \hat{y}\Delta y + \hat{z}\Delta z$

9 terms, 6 zero

How much does Φ change?

$\Phi(\vec{r}+\Delta\vec{r}) - \Phi(\vec{r}) = \Delta\Phi \doteq \frac{\partial\Phi}{\partial x}\Delta x + \frac{\partial\Phi}{\partial y}\Delta y + \frac{\partial\Phi}{\partial z}\Delta z = \vec{\nabla}\Phi\cdot\Delta\vec{r}$ $(\hat{x}\cdot\hat{x}=1, \hat{x}\cdot\hat{y}=0)$ etc.

Taylor expansion (what about Δx^2 or $\Delta x\Delta y$? negligible!)

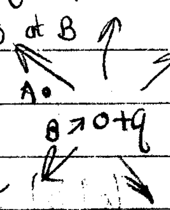
- If $\Delta\Phi = 0$, then "equipotential" traced by $\Delta\vec{r}$ steps, so $\vec{\nabla}\Phi \perp \Delta\vec{r}$
- Vector $\vec{\nabla}\Phi$ points in direction of most rapid positive change ("uphill")

* Can you justify this with a physics argument?

- divergence $\vec{V} = \hat{x}V_1 + \hat{y}V_2 + \hat{z}V_3$

$\Rightarrow \vec{\nabla}\cdot\vec{V} = \frac{\partial V_1}{\partial x} + \frac{\partial V_2}{\partial y} + \frac{\partial V_3}{\partial z} \equiv \frac{\partial}{\partial x_i}V_i$

$\vec{\nabla}\cdot\vec{E} = 0$ at A
$\vec{\nabla}\cdot\vec{E} \neq 0$ at B



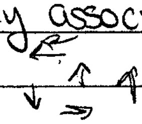
- measures spreading out of vector fields around a point

vocabulary: If $\vec{\nabla}\cdot\vec{V} = 0 \Rightarrow$ "solenoidal" ⟹ net outward flow through volume with that point.

- curl $\vec{\nabla}\times\vec{V} = \hat{x}\left(\frac{\partial}{\partial y}V_3 - \frac{\partial}{\partial z}V_2\right) + \hat{y}\left(\frac{\partial}{\partial z}V_1 - \frac{\partial}{\partial x}V_3\right) + \hat{z}\left(\frac{\partial}{\partial x}V_2 - \frac{\partial}{\partial y}V_1\right)$

$\equiv \varepsilon_{ijk}\frac{\partial}{\partial x_j}V_k$ (or determinant representation)

- physically associated with circulation – integral of vector around closed curve.



If $\vec{\nabla}\times\vec{V} = 0 \Rightarrow$ "irrotational"


← infinitesimal (very small!) loop

(Think about $\vec{\nabla}\times\vec{B}$ around a loop: circulation or vorticity)

General: derivatives tell you about changes nearby

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Practical use with formulas

$$\vec{\nabla}\cdot\vec{x} = \left(\hat{x}\frac{\partial}{\partial x}+\hat{y}\frac{\partial}{\partial y}+\hat{z}\frac{\partial}{\partial z}\right)\cdot\left(\hat{x}x+\hat{y}y+\hat{z}z\right)$$

use $\hat{x}\cdot\hat{x}=1$, $\hat{x}\cdot\hat{y}=0$

$$\overset{3\text{ terms}}{=} \frac{\partial x}{\partial x}+\frac{\partial y}{\partial y}+\frac{\partial z}{\partial z}=3$$

or $A_1=x$, $A_2=y$, $A_3=z$ on Jackson covers

$$\vec{\nabla}\times\vec{x} = \hat{x}\left(\cancel{\frac{\partial x}{\partial y}}^{\,0}-\cancel{\frac{\partial y}{\partial x}}\right)+\hat{y}\left(\cancel{\;}^{\,0}\right)+\hat{z}\left(\cancel{\;}^{\,0}\right)=0$$

- Curvilinear coordinates $\rightarrow$ take advantage of geometry (eg. page 29 from Lea notes chap. 1 for cylindrical)

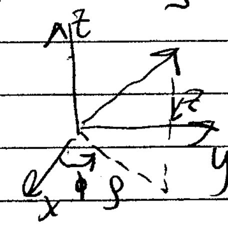
- General expressions in texts such as (eg. front cover & Arfken)

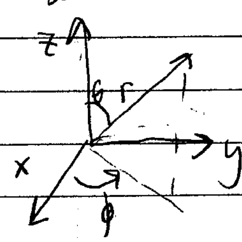
$$\vec{\nabla}\Phi = \sum_i \hat{q}_i\frac{1}{h_i}\frac{\partial\Phi}{\partial q_i} \qquad h_i \Rightarrow \text{scale factors}$$

[remember: it's a vector — common error]

$$\vec{\nabla}\cdot\vec{A} = \frac{1}{h_1h_2h_3}\left[\frac{\partial}{\partial q_1}(A_1h_2h_3)+\frac{\partial}{\partial q_2}(A_2h_1h_3)+\frac{\partial}{\partial q_3}(A_3h_1h_2)\right]$$

[Jackson $\hat{q}_i\rightarrow\hat{e}_i$]	$\hat{q}_1$	h_1	$\hat{q}_2$	h_2	$\hat{q}_3$	h_3
Cartesian	$\hat{x}$	1	$\hat{y}$	1	$\hat{z}$	1
cylindrical	$\hat{\rho}$	1	$\hat{\phi}$	ρ	$\hat{z}$	1
spherical	$\hat{r}$	1	$\hat{\theta}$	r	$\hat{\phi}$	$r\sin\theta$





* • Need to carry h_i's in formulas (units, at least!) for $\vec{\nabla}\Phi$, $\vec{\nabla}\cdot\vec{V}$, $\vec{\nabla}\times\vec{V}$

- Many other coordinate systems are defined! (eg. parabolic)

What are A_1, A_2, A_3 for $\vec{x}$ (or $\vec{r}$) in cylindrical and spherical coordinates? (Answer on bottom of next page)

What can go wrong?

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Consider cylindrical coordinates

$$\vec{\nabla} = \hat{\rho}\frac{\partial}{\partial \rho} + \hat{\phi}\frac{1}{\rho}\frac{\partial}{\partial \phi} + \hat{z}\frac{\partial}{\partial z}$$ note that each term has the correct dimensions

Take $\vec{A} = \hat{\rho} A_1 + \hat{\phi} A_2 + \hat{z} A_3$ and scalar function ψ

Now $\vec{\nabla}\psi = \hat{\rho}\frac{\partial \psi}{\partial \rho} + \hat{\phi}\frac{1}{\rho}\frac{\partial \psi}{\partial \phi} + \hat{z}\frac{\partial \psi}{\partial z}$

"Spot the Error!"

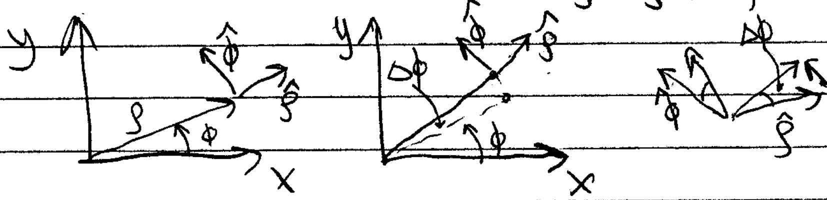
but $\vec{\nabla}\cdot\vec{A} \overset{?}{=} \frac{\partial A_1}{\partial \rho} + \frac{1}{\rho}\frac{\partial A_2}{\partial \phi} + \frac{\partial A_3}{\partial z}$ Wrong! disagrees with Jackson (has $\frac{1}{\rho}\frac{\partial}{\partial \rho}(\rho A_1) = \frac{\partial A_1}{\partial \rho} + \frac{A_1}{\rho}$)

2 terms with A_1

Where did we fail?

$\Rightarrow$ Units vectors move! $\hat{\rho} = \hat{\rho}(\phi)$, $\hat{\phi} = \hat{\phi}(\phi)$, $\hat{z} = \text{const}$.

(x-y plane)



$\Delta\hat{\rho} = \Delta\phi\,\hat{\phi}$ or $\frac{\partial \hat{\rho}}{\partial \phi} = \hat{\phi}$!

looks at A_1 term $\Rightarrow \hat{\phi}\cdot\frac{1}{\rho}\frac{\partial}{\partial \phi}\left[\hat{\rho} A_1\right] = \hat{\phi}\cdot\frac{1}{\rho}\hat{\rho}\frac{\partial A_1}{\partial \phi} + \hat{\phi}\cdot\frac{1}{\rho}\hat{\phi} A_1 = \frac{1}{\rho}A_1$ ✓

extra A_1 term

all other terms zero.

- Try it with spherical coordinates!
- Moral: It can be dangerous to get too used to Cartesian coordinates. (Or, Cartesian is safest! :)

Back to $\vec{\nabla}\cdot\vec{x}$ $\quad \vec{x} = \hat{\rho}\rho + \hat{z}z \Rightarrow$ Jackson $A_1 = \rho, A_2 = 0, A_3 = z$

$$\vec{\nabla}\cdot\vec{x} = \frac{1}{\rho}\frac{\partial}{\partial \rho}(\rho\cdot\rho) + \frac{1}{\rho}\frac{\partial}{\partial \phi}\cdot 0 + \frac{\partial}{\partial z}z = \frac{1}{\rho}2\rho + 1 = 3 \checkmark$$

You try $\vec{\nabla}\times\vec{x}$!

Jackson $\swarrow$ $A_1 = r, A_2 = 0, A_3 = 0$

Spherical $\vec{x} = \hat{r}r$ that's it! Simple applications often (see HW).

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What about other formulas on Jackson covers?

$$\vec{\nabla}\cdot\left[\hat{r}f(r)\right] \underset{\substack{A_3=0 \\ \text{spherical}}}{\overset{\substack{A_1=f(r) \\ A_2=0}}{=}} \frac{1}{r^2}\frac{\partial}{\partial r}\Big(\underbrace{r^2 f(r)}_{r^2 A_1}\Big) = \frac{1}{r^2}2r\,f(r) + \frac{1}{r^2}r^2\frac{df}{dr}$$

$$= \frac{2}{r}f(r) + \frac{df}{dr} \quad \checkmark$$

What about Cartesian? $\vec{r} = \hat{r}r = \hat{x}x + \hat{y}y + \hat{z}z \Rightarrow (\vec{r})_i = x_i$

$(\hat{r})_i = \frac{x_i}{r}$

$$\vec{\nabla}\cdot\hat{r}f(r) \underset{\text{convention}}{\overset{\text{summation}}{=}} \frac{\partial}{\partial x_i}\left(\frac{x_i}{r}f(r)\right)$$

$$= \frac{\partial x_i}{\partial x_i}\frac{f(r)}{r} + x_i\frac{\partial \frac{1}{r}}{\partial x_i}f(r) + \frac{x_i}{r}\frac{\partial}{\partial x_i}f(r)$$

$$= \delta_{ii}\frac{f(r)}{r} + x_i\left(-\frac{1}{r^2}\right)\frac{\partial r}{\partial x_i}f(r) + \frac{x_i}{r}\frac{\partial f}{\partial r}\frac{\partial r}{\partial x_i}$$

find equation with r and x_i!

$r^2 = x_i x_i$

$\Rightarrow 2r\frac{\partial r}{\partial x_i} = 2x_i$

$\Rightarrow \frac{\partial r}{\partial x_i} = \frac{x_i}{r}$

$$\Rightarrow \quad = 3\frac{f(r)}{r} - \frac{x_i x_i}{r^3}f(r) + \frac{x_i x_i}{r^2}\frac{df}{dr} = \frac{2f(r)}{r} + \frac{df}{dr} \quad \checkmark$$

units!

<u>Homework!</u> $\vec{\nabla}\cdot\hat{r} \Rightarrow f(r)=1 \Rightarrow \vec{\nabla}\cdot\hat{r} = \frac{2}{r}$

$\vec{\nabla}\times\hat{\theta} \Rightarrow A_1=0,\ A_2=1,\ A_3=0$

$\Rightarrow \vec{\nabla}\times\hat{\theta} = \hat{r}\cdot 0 + \hat{\theta}\cdot 0 + \hat{\phi}\frac{1}{r}\frac{\partial}{\partial r}(r\cdot 1)$

$= \frac{1}{r}\hat{\phi}$

spherical

and so on.

$\vec{\nabla}\times\hat{\phi} \overset{?}{=}$

cylindrical! $A_1=0,\ A_2=1,\ A_3=0 \Rightarrow \hat{z}\frac{1}{\rho}\left(\frac{\partial}{\partial\rho}\rho\cdot 1\right) = \hat{z}\frac{1}{\rho}$

spherical! $A_1=0,\ A_2=0,\ A_3=1 \Rightarrow$ two terms

(units?)

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Vector Calculus Theorems → "Jackson covers" again

Divergence theorem
$$\int_V \vec{\nabla}\cdot\vec{A}\, d^3x = \int_S \vec{A}\cdot\hat{n}\, da$$
"flux" of A through surface S; $\hat{n}$ is outward normal

$\vec{\nabla}\Rightarrow\hat{n}$:
$$\int_V \vec{\nabla}\psi\, d^3x = \int_S \psi\,\hat{n}\, da$$
$\hat{n}$ is outward normal
$$\int_V (\vec{\nabla}\times\vec{A})\, d^3x = \int_S \hat{n}\times\vec{A}\, da$$

Stokes's Theorem:
$$\int_S (\vec{\nabla}\times\vec{A})\cdot\hat{n}\, da = \oint_C \vec{A}\cdot d\vec{r}$$
$\hat{n}$ by right-hand-rule for direction of C
$$\int_S \hat{n}\times\vec{\nabla}\psi\, da = \oint_C \psi\, d\vec{r}$$

and more.

✗ What you need to know: a) intuitive idea (including physics applications)
PS#4 examples: b) how to interpret (V, S, $\hat{n}$, orientation)
c) how to apply in specific cases

Common features:
- relates local quantities, with derivatives, like $\vec{\nabla}\cdot\vec{A}$ to global quantities like flux through surface $\int_S \vec{A}\cdot\hat{n}\, da$
- sum of derivatives in interior is related to value on boundary (integrals are sums!)
- eliminate one dimension with derivative; leaving integral over remaining surface

Mantra: "Do the simple problems first!"

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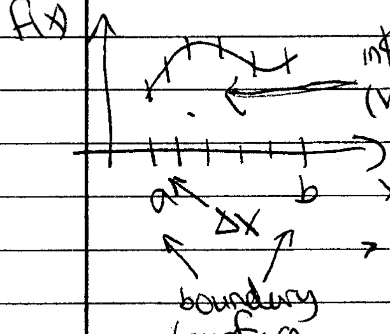
Simplest example: 1-d integrals of f(x)



$$\int_a^b \frac{df}{dx}\,dx = f(x)\Big|_{x=a}^{x=b} = f(b)-f(a)$$

$$\doteq \frac{f(a+\Delta x)-f(a)}{\Delta x}\cdot\Delta x + \frac{f(a+2\Delta x)-f(a+\Delta x)}{\Delta x}\cdot\Delta x$$

$$+\frac{f(a+3\Delta x)-f(a+2\Delta x)}{\Delta x}\Delta x + \ldots$$

$$\boxed{\text{approximation to derivative}}$$

• numerical approximations for derivative and integral (bad choices here for practical applications!)

$$+\frac{f(b-\Delta x)-f(b-2\Delta x)}{\Delta x}\Delta x + \frac{f(b)-f(b-\Delta x)}{\Delta x}\Delta x$$

$$\boxed{\text{sum of integrand} * \Delta x \text{ gives approximation to integral}}$$

only the end points survive $= f(b)-f(a)$

• everything cancels with adjacent contribution, leaving only boundaries uncanceled

$$\Rightarrow \int_V \vec{\nabla}\psi\, d^3x = \int_S \psi\,\hat{n}\,da$$ component by component (at least obvious for regular shape ⇒ build up from small volumes)

generic! E.g. $\int_V \vec{\nabla}\cdot\vec{A}\,d^3x = \int_S \vec{A}\cdot\hat{n}\,da$

derivative in interior — function at boundary

see text: break into cubes

$\vec{\nabla}\cdot\vec{A}$ in small cube ∝ outward flux.

But what goes out one cube goes in the next ⇒ cancel except for ends

Similar for Stokes's Theorem. (see text)

* Aside: vector dot product $\vec{\Delta x} = (\Delta x, \Delta x, \ldots, \Delta x) \Rightarrow \Delta x_i$, $i=1, \#$ of divisions

$\vec{x} = (a, a+\Delta x, a+2\Delta x, \ldots b-\Delta x, b) \Rightarrow x_i$

integral is dot product! $\Rightarrow \int_a^b g(x)\,dx \doteq \sum_i g(x_i)\Delta x_i \equiv g_i\,\Delta x_i$

Do in parallel on computer. Continuous equations become finite matrix equations!

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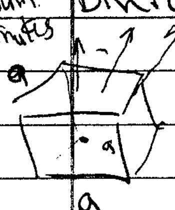
Example applications 1.2 and 1.3 from Lea text. (just setup)

1.2 Compute divergence of vector field $\vec{v} = k\vec{r}$, k = constant, $\vec{r}$ the position vector. (Draw picture of vector field!)
Find flux of $\vec{v}$ through cube of side a centered at origin and show it equals $\int_V \vec{\nabla}\cdot\vec{v}\, d^3x$ over volume.

cube ⇒ cartesian coordinates

Divergence: $\vec{\nabla}\cdot\vec{v} = k\vec{\nabla}\cdot(\hat{x}x + \hat{y}y + \hat{z}z) = k\left(\frac{\partial x}{\partial x} + \frac{\partial y}{\partial y} + \frac{\partial z}{\partial z}\right) = 3k$



⇒ volume integral is $3k \cdot \int_V d^3x = 3ka^3$

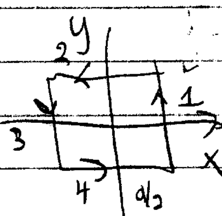
Flux: $\oint_{\text{surface}} k(x\hat{x} + y\hat{y} + z\hat{z})\cdot\hat{n}\, dA$ normals are $\hat{n} = \pm\hat{x}, \pm\hat{y}, \pm\hat{z}$

at $x = \frac{a}{2}$, $\hat{n} = +\hat{x}$ ⇒ $\int_{\text{face}} k\underbrace{x}_{\frac{a}{2}}\, \hat{x}\cdot\hat{x}\, dA = k\frac{a}{2}\int_{\text{face}} dA = k\frac{a}{2}a^2 = \frac{ka^3}{2}$

at $x = -\frac{a}{2}$, $\hat{n} = -\hat{x}$ but $\int_{\text{face}} k(-\frac{a}{2})\hat{x}\cdot(-\hat{x})\, dA = \frac{ka^3}{2} \xrightarrow[\text{same}]{\text{all}} 6\cdot\frac{ka^3}{2} = 3ka^3$

units? ✓

1.3 $\vec{u} = x^2y^3(\hat{x} + \hat{y})$
Find circulation of $\vec{u}$ around square of side a in x-y plane centered at origin. Compare $\int(\vec{\nabla}\times\vec{u})\cdot\hat{n}\, dA$



identify $d\vec{r}$ on each side

$d\vec{r}_1 = dy\,\hat{y}$ $\quad d\vec{r}_3 = -dy\,\hat{y}$

$\oint \vec{u}\cdot d\vec{r} = I_1 + I_2 + I_3 + I_4 = 0 - \frac{2}{3}\left(\frac{a}{2}\right)^6 + 0 - \frac{2}{3}\left(\frac{a}{2}\right)^6 = -\frac{4}{3}\left(\frac{a}{2}\right)^6$ note sign

$d\vec{r}_2 = -dx\,\hat{x}$ $\quad d\vec{r}_4 = dx\,\hat{x}$ $\quad y<0$

$(\vec{\nabla}\times\vec{u})\cdot\hat{n} = (\vec{\nabla}\times\vec{u})_3 = \frac{\partial u_y}{\partial x} - \frac{\partial u_x}{\partial y} = 2xy^3 - 3x^2y^2$

$\hat{z}$

Surface integral $\int(\vec{\nabla}\times\vec{u})\cdot\hat{n}\, dA = \int_{-a/2}^{a/2}\int_{-a/2}^{a/2}(2xy^3 - 3x^2y^2)\, dy\, dx = \int_{-a/2}^{a/2}\left(\frac{xy^4}{2} - x^2y^3\right)\Big|_{-a/2}^{a/2} dx$

$= 2\int_{-a/2}^{a/2} -x^2\left(\frac{a}{2}\right)^3 dx = -2\left(\frac{a}{2}\right)^3\frac{x^3}{3}\Big|_{-a/2}^{a/2} = -\frac{4}{3}\left(\frac{a}{2}\right)^6$ ✓

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Partial integration in vector calculus

recall $\int_a^b u\,dv = uv\Big|_a^b - \int_a^b v\,du$

or $\int_a^b (u\,dv + v\,du) = \int_a^b d(uv) = uv\Big|_a^b$

volume surface

or $$\int_a^b u(x)\,\frac{dv(x)}{dx}\,dx = -\int_a^b v(x)\,\frac{du(x)}{dx}\,dx + u(x)\,v(x)\Big|_a^b$$

$\Rightarrow$ switch $\frac{d}{dx}$ from $v(x)$ to $u(x)$ costs minus sign plus surface term (which often vanishes)

cf. Green's identity and theorem

$$\int_V \phi\,\vec\nabla^2\psi\,d^3x \overset{\text{move }\vec\nabla}{=} -\int_V \vec\nabla\phi\cdot\vec\nabla\psi\,d^3x + \int_S \phi(\vec\nabla\psi)\cdot\hat{n}\,da$$

$$-\int_V \vec\nabla\phi\cdot\vec\nabla\psi\,d^3x \overset{\text{move }\vec\nabla}{=} +\int_V (\vec\nabla^2\phi)\,\psi\,d^3x - \int_S (\vec\nabla\phi)\,\psi\cdot\hat{n}\,da$$

$$\Rightarrow \int_V \phi\,\vec\nabla^2\psi\,d^3x = \int_V (\vec\nabla^2\phi)\,\psi\,d^3x + \int_S \left[\phi(\vec\nabla\psi) - (\vec\nabla\phi)\,\psi\right]\cdot\hat{n}\,da$$

as on the cover.