Warmsup: using div, grad, curl formulas on Jackson course.

- Evaluate \( \nabla \cdot \vec{F}(x) \)
  - First in spherical \( \Rightarrow \) find \( A_1, A_2, A_3 \) and use formula
  - Then in Cartesian \( \Rightarrow \) show that you get the same result
  - Details and answers on page 141.

\[ \vec{F}(x) = \nabla r = \frac{\partial}{\partial x} x + \frac{\partial}{\partial y} y + \frac{\partial}{\partial z} z = 3 \]

- Cylindrical: \( \vec{F} = \hat{r} + z \hat{z} \Rightarrow A_1 = 1, A_2 = 0, A_3 = z \)
  \[ \Rightarrow \nabla \cdot \vec{F} = \frac{\partial}{\partial r} (r \hat{r}) + \frac{1}{\rho} \frac{\partial}{\partial \theta} (\rho \hat{\theta}) + \frac{\partial}{\partial z} z \hat{z} = \frac{2}{\rho} + 1 = 3 \]

- Spherical: \( \vec{F} = \hat{r} \Rightarrow A_1 = r, A_2 = 0, A_3 = 0 \)
  \[ \Rightarrow \nabla \cdot \vec{F} = \frac{1}{r^2} \frac{\partial}{\partial r} (r^2 \hat{r}) = \frac{3r^2}{r^2} = 3 \]

- How does \( \nabla \times \vec{F} \) work out? All of the partial derivatives are zero.

Homework:

- \( \nabla \times \vec{F} \Rightarrow A_1 = 1, A_2 = 0, A_3 = 0 \)
  \[ \frac{\partial}{\partial \theta} (r \hat{r}) = \frac{\partial^2}{r^2} = \frac{2}{r} \text{ units} \]

- \( \nabla \times \vec{F} \) spherical: \( A_0, A_2 = 0, A_3 \) \Rightarrow two(?) non-zero terms

- Convolution revisited (147) \( \Rightarrow \) hint of Green's function
- Comments on Fourier transforms as matrix multiplication
- Electrostatics review
Convolution Revisited

A physical example of a convolution occurs in optics.

Suppose that a point source of light (e.g., from a distant star) is observed in our optical instrument (e.g., our eye) as a blob smeared out with a Gaussian shape. This corresponds to for any \( x_0 \) observing:

\[
g(x-x_0) = \frac{1}{\sqrt{2\pi \sigma}} e^{-\frac{(x-x_0)^2}{2\sigma^2}}
\]

Then what would we expect to see from this?

We assume linearity, then for \( f \) the superposition \( S(x-x_0) \) weighted by \( f(x) \):

\[
f(x) = \int S(x-x_0) f(x_0) \, dx_0
\]

of \( S(x-x_0) \) becomes a Gaussian

\[
h(x) = \int g(x-x_0) f(x_0) \, dx_0
\]

The result is the convolution of the input \( f(x) \) and the

Gaussian function \( g(x-x_0) \).

This is how and why a Green's function works \( \Rightarrow \) it tells you

the "response" due to a delta function, then superpose to get \( f(x) \) solution.

Check that \( \hat{h}(k) = \int_0^\infty e^{ikx} \hat{h}(x) \, dx \) or \( \hat{g}(k) \hat{f}(k) \):

\[
\hat{h}(k) = \int_0^\infty \left[ \int_0^\infty e^{ikx} \, dx \right] \hat{f}(k) \, \hat{g}(k)
\]

\[
= \int_0^\infty e^{ikx} \, dx \int_0^\infty \hat{f}(k) \, \hat{g}(x) \, dx
\]

\[
= \int_0^\infty \hat{f}(k) \, \hat{g}(x) \, dx
\]

\[
= \int_0^\infty \hat{f}(k) \, \hat{g}(x) \, dx
\]

\[
= \int_0^\infty \hat{g}(k) \hat{f}(k) \, dx
\]

\[
= \int_0^\infty \hat{f}(x) \, dx
\]

\[
= \int_0^\infty \hat{f}(x) \, dx
\]

[To avoid \( \hat{f} \) here we would have \( \hat{f} \) factor explicit in \( h(x) \) ]
Electrostatics (Jackson Ch. 1 & Zangwill Ch. 3)

- Time-independence (⇒ "static")

- Coulomb's Law
  - Experiment on forces between two charges \( q_1 \) and \( q_2 \)
  - Coulomb observed that
    - i) force \( F = \frac{k q_1 q_2}{r^2} \), where \( q_1 \) is a scalar with a sign:
      - opposite sign charges attract \( (+ -) \)
      - same sign charges repel \( (+ + \text{ or } - -) \)
    - ii) strength is proportional to the product of charge magnitudes
    - iii) force decreases with separation \( r \)
  - by \( F \propto \frac{1}{r^2} \) ("inverse square law")

- The force has a definite direction: lined up with vector connecting charges
  \( \Rightarrow F \parallel \vec{r} \)

These characteristics can be summarized by referring to the vector positions \( \vec{x}_1 \) and \( \vec{x}_2 \) of \( q_1 \) and \( q_2 \), after specifying an origin by:

\[
F = \frac{k q_1 q_2}{|\vec{x}_1 - \vec{x}_2|^2}
\]

- The proportionality constant \( k \) has units and is different in different systems of units, e.g., SI vs. Gaussian.
  - We will use SI this semester.

- Check that \( F \) is in the correct direction for a force on \( q_2 \):
  - Can rewrite \( F = \frac{k q_1 q_2}{R^2} \left( \frac{x_2 - x_1}{|x_2 - x_1|} \right) \) as magnitude \( x \) (direction) \( \Rightarrow \) origin:

\[
\vec{F} = \frac{k q_1 q_2}{R^2} \left( \frac{x_2 - x_1}{|x_2 - x_1|} \right)
\]
10/18/2013

Comments on units
- electric charge unit: Coulomb (charge on $e^-$ = 1.6 x 10^{-9} C)
- force ~ Newtons, length ~ meters

\[ F = k \frac{q_1 q_2}{r^2} \Rightarrow 1 \text{N} = \frac{1 \text{C}^2}{1 \text{m}^2} \]

Value of $k$ in SI:

\[ k = 1 \parallel \varepsilon_0 \]

\[ \varepsilon_0 = 8.854 \times 10^{-12} \text{C}^2 \text{N}^{-1} \text{m}^{-2} \]

<br>

Summary: Coulomb's Law (SI) \[ F = \frac{1}{4 \pi \varepsilon_0} \frac{q_1 q_2}{r^2} \]

<br>

Electric Field
- We define electric field $E$ as force per unit charge:

\[ F = qE \]

Implication is that the vector $F/q_a$ goes to a limit as $q_a$ is made smaller ("test charge"), independent of $q_a$.

\[ E = \frac{1}{4 \pi \varepsilon_0} \frac{q_a}{r^2} \]

is $E$-field at $r$ due to charge $q_a$ at $r_a$.

Units:

\[ [E] = \frac{\text{N} \cdot \text{C}^2}{\text{m}^2} = \frac{\text{N} \cdot \text{m}}{\text{C}^2} = \frac{\text{V}}{\text{m}} 

\]

Check with $q_1 q_2 \frac{1}{4 \pi \varepsilon_0} \frac{q_2}{r^2} \text{C} \cdot \text{V} = \text{force} \cdot \text{distance} = \text{N} \cdot \text{m}/\text{V}$.
Note: Coulomb's law is action at a distance and $E$ as defined also has this character here.

- inconsistent with finite speed of light if we go beyond statics, move $x$, it looks like $E$ changes without time delay.
- Later: $E$ is a local (at $x$) field.
- For electrostatics, no difference in practice, but keep in mind the conceptual difference.

We assume the equations for electrostatics are linear in $E$ (true for Maxwell's equations as applied here).

$\Rightarrow$ we have the superposition principle:
- given many charges $q_1, q_2, \ldots$ at $x_1, x_2, \ldots x_n$, then combined electric field at $x$ is

$$E(x) = \frac{1}{4\pi\varepsilon_0} \sum_{i=1}^{n} q_i \frac{x-x_i}{|x-x_i|^3}$$

When does superposition not hold?

- If the (continuous) charge density is defined as the electric charge per unit volume,
  $$\delta q = \delta(x) dy = \delta(x, y, z) \Delta x \Delta y \Delta z \quad (\text{Cartesian})$$

$\Rightarrow$

$$E(x) = \frac{1}{4\pi\varepsilon_0} \int d^3 x' \frac{x-x'}{|x-x'|^3} \cdot \delta(x')$$

Clearly we relate these by

$$\delta(x') = \frac{1}{|x-x'|^3}$$

Of our example of a convolution, if we know how to find the response (electric field here) from a point charge (this is what we will call a Green's function), then the full response follows from a well-defined integral. This solution to electrostatics by summation requires knowing $\delta(x')$. 
Gauss's Law

Suppose point charge $q$ at the origin and consider a surrounding surface $S$. The electric field at $r$ is

$$\mathbf{E}(r) = \frac{q}{4\pi\varepsilon_0 r^2} = \frac{q}{4\pi\varepsilon_0 r^2} \hat{r}$$

To find the flux of $\mathbf{E}$ through a small surface area $d\mathbf{a}$, we need to account for $\mathbf{E}$ being at a net angle to $\mathbf{a}$ (not parallel).

Project $\mathbf{E}$ on $\hat{\mathbf{n}}$ (unit normal vector to surface):

$$\mathbf{E} \cdot \hat{\mathbf{n}} = \frac{q}{4\pi\varepsilon_0} \frac{\cos \theta}{r^2} = \frac{q}{4\pi\varepsilon_0} \cos \theta$$

The surface area element $d\mathbf{a}$ subtends a solid angle $d\Omega$ and these are related by $d\mathbf{a} \cdot \cos \theta = r^2 d\Omega$

(eg. on a sphere, the area is just $r^2 d\Omega$, as $\alpha = 0$).

$$\Rightarrow \quad \mathbf{E} \cdot \hat{\mathbf{n}} d\mathbf{a} = \frac{q}{4\pi\varepsilon_0} \frac{r^2 d\Omega}{r^2} = \frac{q}{4\pi\varepsilon_0} d\Omega$$

But integrating over all angles:

$$\int d\Omega = \int_0^{2\pi} d\phi \int_0^\pi \sin \theta d\theta = 4\pi$$

Integrate over closed surface $S$:

$$\int_S \mathbf{E} \cdot d\mathbf{a} = \frac{q}{4\pi\varepsilon_0} \int_S d\Omega = \frac{q}{4\pi\varepsilon_0} \times 4\pi = \frac{q}{\varepsilon_0}$$

For many point charges, use superposition (with appropriate origins):

$$\sum_s \mathbf{E} \cdot \hat{n} d\mathbf{a} = \frac{1}{\varepsilon_0} \sum q_i$$
For a continuous charge density \( \rho_i \), so the total charge enclosed by surface \( S \) is

\[
\oint_S \vec{E} \cdot \hat{n} \, d\mathbf{a} = \frac{1}{\varepsilon_0} \int_V d\mathbf{x} \, \rho(x) = \frac{1}{\varepsilon_0} Q,
\]

Gauss's Law.

You are familiar from previous E&M courses how to exploit this law in the case of symmetric situations, e.g., a spherical charge distribution. One of the homework problems is a review of this.

We can choose \( S \) to reflect the symmetry and then evaluate \( \int_S \vec{E} \cdot \hat{n} \, d\mathbf{a} \) is easy to evaluate in terms of the magnitude \( \vec{E} \) on the surface, which is then directly found by an integral over the charge distribution (assumed to be given).

The surface integral over \( \vec{E} \) has the form on one side of the divergence theorem:

\[
\oint_S \vec{A} \cdot \hat{n} \, d\mathbf{a} = \int_V \nabla \cdot \vec{A} \, d\mathbf{x} \quad \text{for any "well-behaved" vector field } \vec{A}
\]

\[
\Rightarrow \oint_S \vec{E} \cdot \hat{n} \, d\mathbf{a} = \int_V \nabla \cdot \vec{E} \, d\mathbf{x} = \frac{1}{\varepsilon_0} \int_V d\mathbf{x} \, \rho(x)
\]

**because this is true for any volume \( V \), the integrands must be equal**

\[
\Rightarrow \nabla \cdot \vec{E}(x) = \frac{1}{\varepsilon_0} \rho(x)
\]

- Differential form of Gauss's Law
- A local relation: \( \nabla \cdot \vec{E} \) involves \( \vec{E} \) near \( x \) (derivative) only.
Scalar Potential

Let's see how the result that \( E \) can always be written as a gradient of a scalar field, \( E = -\nabla \phi \), arises.

The key starting point is the identity

\[
\frac{x-x'}{x-x'^2} = -\frac{1}{x-x'}
\]

How to prove this? Cartesian coordinates is direct

\[
\vec{r} = (x, y, z), \quad \vec{r}' = (x', y', z')
\]

\[
E = \frac{1}{\sqrt{x-x'^2}} \frac{d}{dx} \left( \frac{1}{\sqrt{x-x'^2}} \right) = \frac{1}{x-x'} \frac{d}{dx} \left( \frac{1}{x-x'} \right) = -\frac{1}{x-x'} \frac{d}{dx} \left( \frac{x-x'}{x-x'} \right) = -\frac{1}{(x-x')^2} \frac{d}{dx} (x-x')
\]

and similarly for the y and z components. QED

\[
E(x) = \frac{1}{4\pi\varepsilon_0} \int \frac{\vec{E} \cdot \vec{r}'}{|\vec{r}'|^3} \cdot \vec{r}' \left( \frac{1}{|\vec{r}'|} \right) = \frac{1}{4\pi\varepsilon_0} \int \frac{dE}{dx} \left( \frac{1}{x-x'} \right) \frac{d}{dx} \left( \frac{1}{x-x'} \right)
\]

\[
= -\nabla_x \left[ \frac{1}{4\pi\varepsilon_0} \int \frac{dE}{dx} \left( \frac{1}{x-x'} \right) \right]
\]

\[
= -\nabla \phi(x)
\]

where \( \phi(x) = \frac{1}{4\pi\varepsilon_0} \int \frac{dE}{dx} \left( \frac{1}{x-x'} \right) \) is electrostatic (scalar) potential

"Gauge freedom": \( \phi(x) = \phi(x') + \text{(constant)} \) gives same \( E \)

If we know \( \phi(x) \) everywhere, we can find \( \phi(x') \). How is this modified if we have conductors that give boundary conditions?
\( \nabla \times E = \frac{1}{\varepsilon_0} \varepsilon \phi \), what is \( \nabla \times E \)?

\[
\nabla \times E = \nabla \times (-\nabla \phi) \Rightarrow (\nabla \times \phi)_{j_k} = \varepsilon_{ijk} \partial_j \phi_{k} = 0
\]

\Rightarrow \nabla \times E = 0

We'll come back to specific implications next time. For now, let's look at equations satisfied by \( \phi(x) \):

\[
\nabla \cdot \mathbf{E} = \nabla \cdot (-\nabla \phi) = -\nabla^2 \phi = \frac{\phi}{\varepsilon_0}
\]

\Rightarrow Poisson equation:

\[
\nabla^2 \phi(x) = -\frac{\phi(x)}{\varepsilon_0}
\]

So if you are given \( \phi \), you can find \( \phi(x) \) by taking Poisson's.

⇒ homework problem.

But be careful. Let's check whether this works with:

\[
\nabla^2 \phi(x) = \nabla^2 \left[ \frac{1}{4\pi \varepsilon_0} \int \frac{\varepsilon \phi(y)}{|x-x'|} \right] = -\frac{\phi(x)}{\varepsilon_0}
\]

We need \( \nabla^2 \frac{1}{|x-x'|} \) to do this.

But if we follow the procedure we did with \( \frac{1}{|x-x'|} \):

\[
\left( \partial_x^2 + \partial_y^2 + \partial_z^2 \right) \frac{1}{\sqrt{x-x'^2 + y^2 + z^2}} = -3 \frac{3 \left( x-x'^2 \right) \left( y-y'^2 \right) \left( z-z'^2 \right)}{\left( x-x'^2 \right)^2 \left( y-y'^2 \right)^2 \left( z-z'^2 \right)^2} = 0 ! !
\]

So we don't get \(-\frac{\phi(x)}{\varepsilon_0} \). What went wrong?

\[
\int_{\partial x'} \left( \frac{1}{|x-x'|} \right) |\phi(x)| = -4\pi \frac{\phi(x)}{\varepsilon_0}
\]

implies \( \frac{1}{|x-x'|} = -4\pi \frac{\phi(x)}{\varepsilon_0} \), verify.