10/18/2013

7701 Lecture 21

Warmups: using div, grad, curl formulas on Jackson covers.

- Evaluate $\vec{\nabla}\cdot[\hat{r}f(r)]$
  - First in spherical $\Rightarrow$ find A_1, A_2, A_3 and use formula
  - Then in Cartesian $\Rightarrow$ show that you get the same result
  - Details and answers on page 141.

compare to $\vec{\nabla}\cdot\hat{r}$

- Evaluate $\vec{\nabla}\cdot\vec{r}$ in all three coordinate systems
  - Cartesian: $\vec{r} = \hat{x}x + \hat{y}y + \hat{z}z \Rightarrow A_1 = x, A_2 = y, A_3 = z$
    $\Rightarrow \vec{\nabla}\cdot\vec{r} = \frac{\partial x}{\partial x} + \frac{\partial y}{\partial y} + \frac{\partial z}{\partial z} = 3$ ✓
  - Cylindrical: $\vec{r} = \hat{\rho}\rho + \hat{z}z \Rightarrow A_1 = \rho, A_2 = 0, A_3 = z$
    $\Rightarrow \vec{\nabla}\cdot\vec{r} = \frac{1}{\rho}\frac{\partial}{\partial\rho}(\rho\cdot\rho) + \frac{\partial}{\partial z}(z) = \frac{2\rho}{\rho} + 1 = 3$ ✓
  - Spherical: $\vec{r} = \hat{r}r \Rightarrow A_1 = r, A_2 = 0, A_3 = 0$
    $\Rightarrow \vec{\nabla}\cdot\vec{r} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2\cdot r) = \frac{3r^2}{r^2} = 3$ ✓
- How does $\vec{\nabla}\times\vec{r}$ work out? All of the partial derivatives are zero!
- Homework:
  - $\vec{\nabla}\cdot\hat{r} \Rightarrow A_1 = 1, A_2 = 0, A_3 = 0 \Rightarrow \frac{1}{r^2}\frac{\partial}{\partial r}(r^2) = \frac{2r}{r^2} = \frac{2}{r}$ units ✓
  - $(\nabla\times\hat{\phi})_{\text{spherical}}$ $A_1 = 0, A_2 = 0, A_3 = 1 \Rightarrow$ two(!) non-zero terms

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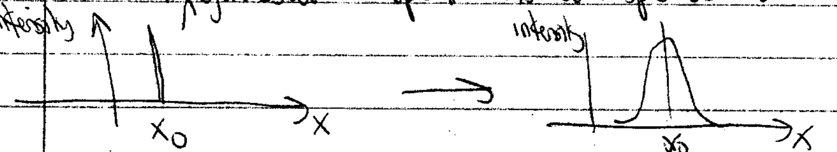
- Convolution revisited 147 $\rightarrow$ hint of Green's function
- Comments on Fourier transforms as matrix multiplication
- Electrostatics review

10/18/2013

Convolution Revisited

The actual shape depends on the instrument

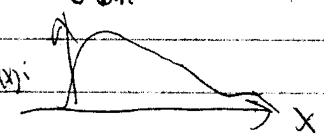
• A physical example of a convolution occurs in optics.
Suppose that a point source of light (e.g. from a distant star)
is observed in our optical instrument (e.g. our eye) as a blob (smeared out)
with a (say) Gaussian shape. This corresponds to for any x_0 observing:



$\delta(x-x_0)$

$$g(x-x_0) = \frac{1}{\sigma\sqrt{2\pi}}\, e^{-\frac{1}{2}\left(\frac{x-x_0}{\sigma}\right)^2}$$

Then what would we expect to see from $f(x)$:



We assume linearity, then $f(x)$ is the superposition of $\delta(x-x_0)$'s weighted by $f(x_0)$:

$$f(x) = \int_{-\infty}^{\infty} \delta(x-x_0)\, f(x_0)\, dx_0 \xrightarrow[\text{becomes a Gaussian}]{\text{each } \delta(x-x_0)} h(x) = \int_{-\infty}^{\infty} g(x-x_0)\, f(x_0)\, dx_0$$

$\Rightarrow$ the result is the convolution of the input $f(x)$ and the smearing function $g(x-x_0)$.

• This is the same way a Green's function works $\rightarrow$ it tells you the "response" due to a delta function, then superpose to get the solution.

cf. electrostatics and E-field from point charge

Check that $\hat{h}(k) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-ikx}\, h(x)\, dx \propto \tilde{g}(k)\hat{f}(k)$:

$$\hat{h}(k) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dx_0 \underbrace{\left[\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} dk'' e^{+ik''(x-x_0)}\, \tilde{g}(k'')\right]}_{g(x-x_0)} \underbrace{\left[\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} dk'\, e^{+ik'x_0}\, \hat{f}(k')\right]}_{f(x_0)} e^{-ikx}$$

$$= \frac{1}{(2\pi)^{3/2}}\int_{-\infty}^{\infty} dk'' \int_{-\infty}^{\infty} dk'\, \tilde{g}(k'')\hat{f}(k') \underbrace{\int_{-\infty}^{\infty} dx_0\, e^{-i(k''-k')x_0}}_{2\pi\,\delta(k''-k')} \underbrace{\int_{-\infty}^{\infty} dx\, e^{i(k''-k)x}}_{2\pi\,\delta(k''-k)}$$

$\leftarrow$ do k' integral; $\leftarrow$ do k'' integral

$$= \sqrt{2\pi}\, \tilde{g}(k)\hat{f}(k) \checkmark$$

[to avoid the $\sqrt{2\pi}$ here we would have $\frac{1}{\sqrt{2\pi}}$ factor explicit in $h(x)$]

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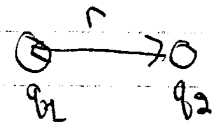
Electrostatics (Jackson Ch.1 or Zangwill Ch.3)

• time-independent ($\Rightarrow$ "static")

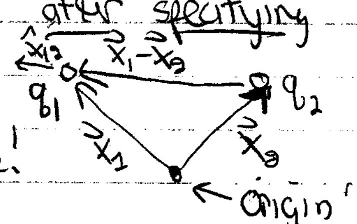
• Coulomb's Law
  • Experiment on forces between two charges



  • Coulomb observed that
    i) force $F \propto q_1 \cdot q_2$ where q_i is a scalar with a sign
      • opposite sign charges attract (+ −)
      • same sign charges repel (++ or −−)
      • strength is proportional to the product of charge magnitudes
    ii) strength of force decreases with separation of r by $F \propto 1/r^2$ ("inverse square law")

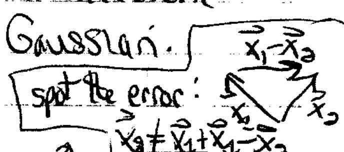
    iii) the force has a definite direction: lined up with vector connecting charges
    $\Rightarrow \vec{F} \parallel \vec{r}$
    ← "parallel to"

• These characteristics can be summarized by referring to the vector positions $\vec{x}_1$ and $\vec{x}_2$ of q_1 and q_2 after specifying an origin by



$\boxed{\text{force on } q_1 \text{ due to } q_2}$; $\vec{F} = k\, q_1 q_2 \dfrac{\vec{x}_1 - \vec{x}_2}{|\vec{x}_1 - \vec{x}_2|^3}$ ← note!

• The proportionality constant k has units and is different in different systems of units, eg. SI vs. Gaussian.
* • We will use SI this semester



• Check that $\vec{F}$ is in the correct direction for a force on q_1

• Can rewrite $\vec{F} = \dfrac{k q_1 q_2}{|\vec{x}_1 - \vec{x}_2|^2} \dfrac{\vec{x}_1 - \vec{x}_2}{|\vec{x}_1 - \vec{x}_2|}$ ← unit vector from q_2 to q_1
(magnitude) × (direction)

$\hat{x}_{12} = \dfrac{\vec{x}_1 - \vec{x}_2}{|\vec{x}_1 - \vec{x}_2|}$

What if origin is moved (translated)?

10/18/2013

Comments on units

· electric charge unit: Coulomb (charge on $\overset{\text{electron}}{e^-} \doteq 1.6\times10^{-19}\,C$)

· force ~ Newtons (N), length ~ meters (m)

$[A]$ "units of A"

$\Rightarrow |\vec{F}| = \frac{k q_1 q_2}{|\vec{x}_1-\vec{x}_2|} \Rightarrow N \sim \frac{[k]\,C^2}{m}$

$\Rightarrow k$ has dimensions of $\frac{\text{force}\cdot(\text{length})^2}{\text{charge}^2}$ or $[k] = \frac{N\cdot m^2}{C^2}$ in SI.

Value of k in SI: $\boxed{k = \frac{1}{4\pi\varepsilon_0}}$ $\varepsilon_0 = 8.854\times10^{-12}\,\frac{C^2}{N\cdot m^2}$

$= 8.854\times10^{-12}\,\frac{\text{Farad}}{m}$ $\quad 1\text{ Farad} = \frac{1C^2}{N m}$

* k is different in other units ⇒ see textbooks for conversions

Summary: Coulombs Law (SI) ⇒ $\boxed{\vec{F} = \frac{1}{4\pi\varepsilon_0} q_1 q_2 \frac{\vec{x}_1-\vec{x}_2}{|\vec{x}_1-\vec{x}_2|^3}}$

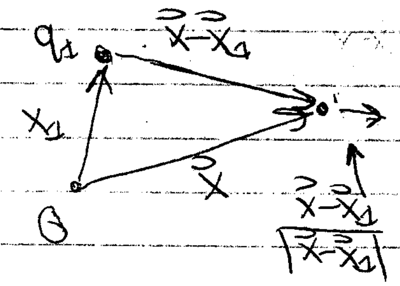
Electric Field

· We define electric field $\vec{E}$ as, vector!, force per unit charge:

$$\vec{F} = q\vec{E}$$

· implication is that the vector $\vec{F}/q$ goes to a uniform limit as q is made smaller ("test charge"), independent of q

$\Rightarrow \vec{E} = \frac{1}{4\pi\varepsilon_0} q_1 \frac{\vec{x}-\vec{x}_1}{|\vec{x}-\vec{x}_1|^3}$ is E-field at $\vec{x}$ due to charge q_1 at $\vec{x}_1$



units: $[E] = \frac{[F]}{[q]} = \frac{\text{Newton}}{\text{Coulomb}} = \frac{N}{C} = \frac{\text{Volts}}{\text{meter}} = \frac{V}{m}$

(check Work ~ $qV \Rightarrow [W] = C\cdot V$ = force×distance $= N\cdot m$ ✓)

10/18/2013

- Note: Coulomb's law is action at a distance and $\vec{E}$ as defined also has this character here.
  - inconsistent with finite speed of light if we go beyond statics: move $\vec{x}_1$, it looks like $\vec{E}$ changes without time delay.
  - later: $\vec{E}$ is a local (at $\vec{x}$) field.
  - For electrostatics, no difference in practice but keep in mind the conceptual difference.

- We assume the equations for electrostatics are linear in $\vec{E}$ (true for Maxwell's equations as applied here)
  $\Rightarrow$ we have the superposition principle
  - given many charges $q_1, q_2, \ldots, q_n$ at $\vec{x}_1, \vec{x}_2, \ldots \vec{x}_n$, then combined electric field at $\vec{x}$ is

$$\boxed{\vec{E}(\vec{x}) = \frac{1}{4\pi\varepsilon_0} \sum_{i=1}^{n} q_i \frac{\vec{x}-\vec{x}_i}{|\vec{x}-\vec{x}_i|^3}}$$

When does superposition not hold?

- If the (continuous) charge density is defined as the electric charge per unit volume:

$$\Delta q \equiv \rho(\vec{x})\Delta V = \rho(x,y,z)\,\Delta x \Delta y \Delta z \quad \text{(Cartesian)}$$

$$\Rightarrow \boxed{\vec{E}(\vec{x}) = \frac{1}{4\pi\varepsilon_0} \int d^3x' \, \frac{\vec{x}-\vec{x}'}{|\vec{x}-\vec{x}'|^3} \cdot \rho(\vec{x}')}$$

- Clearly we relate these by $\boxed{\rho(\vec{x}') = \sum_{i=1}^{n} q_i \delta^3(\vec{x}'-\vec{x}_i)}$

- Cf. our example of a convolution. If we know how to find the response (electric field here) from a point charge (this is what we will call a Green's function), then the full response follows from a well defined integral. $\Rightarrow$ solution to electrostatics by "summation"
  $\Rightarrow$ requires knowing $\rho(\vec{x}')$.

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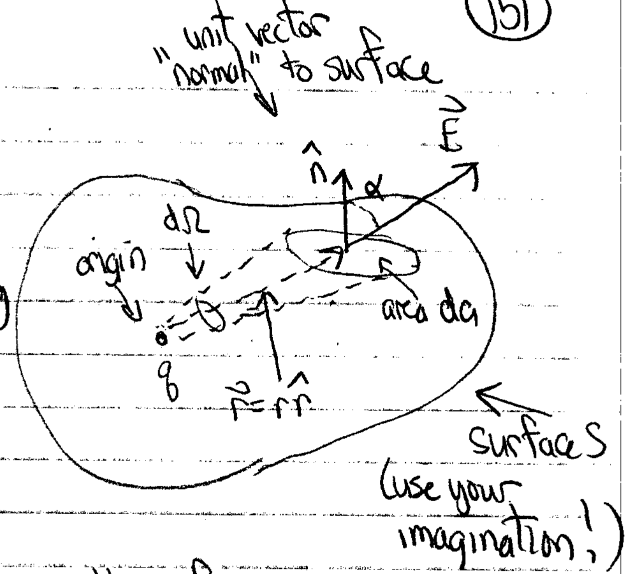
Gauss's Law



- Suppose point charge q at the origin and consider a surrounding surface S. The electric field at $\vec{r}$ is

$$\vec{E}(\vec{r}) = \frac{q}{4\pi\varepsilon_0}\frac{\vec{r}}{r^3} = \frac{q}{4\pi\varepsilon_0}\frac{\hat{r}}{r^2}$$

To find the flux of $\vec{E}$ through small surface area da we need to account for $\vec{E}$ being at a angle to da ($\hat{n}$ and $\vec{E}$ are not parallel)

$\Rightarrow$ Project $\vec{E}$ on $\hat{n}$ (unit normal vector to surface)

$$\vec{E}\cdot\hat{n} = \frac{q}{4\pi\varepsilon_0}\frac{\hat{r}\cdot\hat{n}}{r^2} = \frac{q}{4\pi\varepsilon_0}\frac{\cos\alpha}{r^2}$$

The surface area element da subtends a solid angle $d\Omega$ and these are related by $da\cdot\cos\alpha = r^2 d\Omega$ ($\sin\theta\, d\theta\cdot d\phi$)
(eg, on a sphere, the area is just $r^2 d\Omega$, as $\alpha = 0$)

$$\Rightarrow \vec{E}\cdot\hat{n}\, da = \frac{q}{4\pi\varepsilon_0}\frac{da\cdot\cos\alpha}{r^2} = \frac{q}{4\pi\varepsilon_0}d\Omega$$

But integrating over all angles: $\int d\Omega = \int_0^{\pi}\sin\theta\, d\theta\int_0^{2\pi}d\phi = \int_{-1}^{1}dx\int_0^{2\pi}d\phi = 4\pi$

$$\underbrace{\oint_S}_{\text{integrate over closed surface } S} \vec{E}\cdot\hat{n}\, da = \frac{q}{4\pi\varepsilon_0}\oint_S d\Omega = \frac{q}{\cancel{4\pi}\varepsilon_0}\cancel{4\pi} = \frac{q}{\varepsilon_0}$$

net outward flux

- For many point charges, use superposition (with appropriate origins):

$$\oint_S \vec{E}\cdot\hat{n}\, da = \frac{1}{\varepsilon_0}\sum_{i=1}^{n} q_i$$

10/18/2013

For a continuous charge density $\sum_{i=1}^{n} q_i \to \int_V d^3x\, \rho(\vec{x})$, so

$$\boxed{\oint_S \vec{E}\cdot\hat{n}\, da = \frac{1}{\varepsilon_0}\int_V d^3x\, \rho(\vec{x}) \equiv \frac{1}{\varepsilon_0} Q}$$ Gauss's Law

so $Q \equiv \int_V d^3x\, \rho(\vec{x})$ is the total charge enclosed by surface S.

- You are familiar from previous E/M course(s) how to exploit this law in the case of symmetric situations, eg. a spherical charge distribution.
  - one of the homework problems is a review of this.
  - we can choose S to reflect the symmetry and then $\oint_S \vec{E}\cdot\hat{n}\, da$ is easy to evaluate in terms of the magnitude $|\vec{E}|$ on the surface, which is then directly found by an integral over the charge distribution (assumed to be given).

- The surface integral over $\vec{E}$ has the form on one side of the divergence theorem

$$\oint_S \vec{A}\cdot\hat{n}\, da = \int_V \vec{\nabla}\cdot\vec{A}\, d^3x \quad \text{for any "well-behaved" vector field } \vec{A}$$

$V \leftarrow$ (volume inside S)

$\vec{A}\to\vec{E}$ $$\Rightarrow \oint_S \vec{E}\cdot\hat{n}\, da = \int_V \vec{\nabla}\cdot\vec{E}\, d^3x = \frac{1}{\varepsilon}\int_V d^3x\, \rho(\vec{x})$$

(important to understand this argument)

** because this is true for any volume V, the integrands must be equal

$$\Rightarrow \boxed{\vec{\nabla}\cdot\vec{E}(\vec{x}) = \frac{1}{\varepsilon_0}\rho(\vec{x})}$$

- Differential Form of Gauss's Law
- A local relation: $\vec{\nabla}\cdot\vec{E}$ involves $\vec{E}$ near $\vec{x}$ (derivative) only.

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Scalar Potential

- Let's see how the result that $\vec{E}$ can always be written as the gradient of a scalar field, $\vec{E} = -\vec{\nabla}\phi$, arises.

- The key starting point is the identity

$$\frac{\vec{x}-\vec{x}'}{|\vec{x}-\vec{x}'|^3} = -\vec{\nabla}\frac{1}{|\vec{x}-\vec{x}'|}$$

How to prove this? Cartesian coordinates is direct

$$\vec{x} = (x,y,z) \quad , \quad \vec{x}' = (x',y',z')$$

$$\Rightarrow \underbrace{\left(\vec{\nabla}\frac{1}{|\vec{x}-\vec{x}'|}\right)_x}_{x \text{ component}} = \frac{\partial}{\partial x}\frac{1}{\sqrt{(x-x')^2+(y-y')^2+(z-z')^2}} = -\frac{1}{\cancel{2}}\frac{\cancel{2}(x-x')}{\left[(x-x')^2+(y-y')^2+(z-z')^2\right]^{3/2}}$$

$$= \left(-\frac{(\vec{x}-\vec{x}')}{|\vec{x}-\vec{x}'|^3}\right)_x$$

and similarly for the y and z components. QED

$$\Rightarrow \vec{E}(\vec{x}) = \frac{1}{4\pi\epsilon_0}\int d^3x' \frac{\vec{x}-\vec{x}'}{|\vec{x}-\vec{x}'|^3}\rho(\vec{x}') = -\frac{1}{4\pi\epsilon_0}\int d^3x' \vec{\nabla}_{\underset{\text{not } \vec{x}'!}{\vec{x}}}\frac{1}{|\vec{x}-\vec{x}'|}$$

$$= -\vec{\nabla}_{\vec{x}}\left[\frac{1}{4\pi\epsilon_0}\int d^3x' \frac{\rho(\vec{x}')}{|\vec{x}-\vec{x}'|}\right]$$

$$\equiv -\vec{\nabla}\phi(\vec{x})$$

$\vec{\nabla}_{\vec{x}}$ means gradient with respect to $\vec{x}$

where $\boxed{\phi(\vec{x}) \equiv \frac{1}{4\pi\epsilon_0}\int d^3x' \frac{\rho(\vec{x}')}{|\vec{x}-\vec{x}'|}}$ "electrostatic (scalar) potential"

- "Gauge freedom": $\phi'(\vec{x}) = \phi(\vec{x}) + (\text{constant})$ ← could be time dependent

gives the same $\vec{E}$

- If we know $\rho(\vec{x}')$ everywhere, we can find $\phi(\vec{x})$. How is this modified if we have conductors that give boundary conditions?

10/18/2013

- We have $\vec{\nabla}\cdot\vec{E} = \frac{1}{\varepsilon_0}\rho(\vec{x})$, what is $\vec{\nabla}\times\vec{E}$?

$$\vec{\nabla}\times\vec{E} = \vec{\nabla}\times(-\vec{\nabla}\phi) \Rightarrow (\underset{j}{\vec{\nabla}}\times\underset{k}{\vec{\nabla}}\phi)_i = \underbrace{\varepsilon_{ijk}}_{\substack{\text{antisymmetric} \\ j\leftrightarrow k}} \overbrace{\partial_j\partial_k}^{\text{symmetric } j\leftrightarrow k}\phi = 0$$

$$\Rightarrow \boxed{\vec{\nabla}\times\vec{E} = 0}$$

- We'll come back to specific implications next time. For now, let's look at equations satisfied by $\phi(\vec{x})$

$$\vec{\nabla}\cdot\vec{E} = \vec{\nabla}\cdot(-\vec{\nabla}\phi) = -\nabla^2\phi = \frac{\rho}{\varepsilon_0}$$

$\Rightarrow$ Poisson equation: $\boxed{\nabla^2\phi(\vec{x}) = -\frac{\rho(\vec{x})}{\varepsilon_0}}$

- So if you are given $\phi(\vec{x})$, you can find $\rho(\vec{x})$ by taking the Laplacian.
  $\Rightarrow$ homework problem.
- But be careful! Let's check whether this works with

$$\nabla^2\phi(\vec{x}) = \nabla^2_{\vec{x}}\left[\frac{1}{4\pi\varepsilon_0}\int d^3x' \frac{\rho(\vec{x}')}{|\vec{x}-\vec{x}'|}\right] \overset{?}{=} -\frac{\rho(\vec{x})}{\varepsilon_0}$$

- We need $\nabla^2_{\vec{x}}\frac{1}{|\vec{x}-\vec{x}'|}$ to do this.
- But if we follow the procedure we did with $\vec{\nabla}_{\vec{x}}\frac{1}{|\vec{x}-\vec{x}'|}$:

$$(\partial_x^2+\partial_y^2+\partial_z^2)\frac{1}{\sqrt{(x-x')^2+(y-y')^2+(z-z')^2}} = \frac{-3}{[\underbrace{(x-x')^2+(y-y')^2+(z-z')^2}]^{3/2}} + \frac{3[(x-x')^2+(y-y')^2+(z-z')^2]}{[\text{ same }]^{5/2}}$$

$$= 0 \; !!$$

So we don't get $-\frac{\rho(\vec{x})}{\varepsilon_0}$. What went wrong?

note minus sign

$\int d^3x' \left(\nabla^2_{\vec{x}}\frac{1}{|\vec{x}-\vec{x}'|}\right)\rho(\vec{x}') = -4\pi\rho(\vec{x})$ implies $\boxed{\nabla^2_{\vec{x}}\frac{1}{|\vec{x}-\vec{x}'|} = -4\pi\delta^3(\vec{x}-\vec{x}')}$! Next time we verify.