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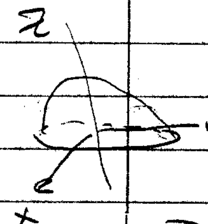
7701 Lecture 22

- Mathematica items
  - You can explicitly do 3-D integrations in other coordinate systems by including the measure.
  - E.g. spherical coordinates: volume_integral_spherical.nb on Mathematica example page.
    - convert integrand as needed

      $z = r\cos\theta, \; x = r\sin\theta\cos\phi, \; y = r\sin\theta\sin\phi$



  - Suppose $f(\vec{x}) = r^3 \sin^4\theta \cos^3\theta \sin^2\phi$ to be integrated over a hemisphere of radius 2 with base in xy plane (chose spherical because limits are easy)

$\Rightarrow$ Integrate[(r^2 * Sin[theta]) * r^3 * Sin[theta]^4 * Cos[theta]^3
measure → * Sin[phi]^2, {r, 0, 2}, {theta, 0, Pi/2}, {phi, 0, 2Pi}]

hemisphere $0 \le \theta \le \pi/2$ full range

- Mathematica also knows the coordinate systems.
  E.g.,

  Laplacian[f(r, theta, phi), {r, theta, phi}, "Spherical"]

  your names   also "Cylindrical" and more

  - Warning: it doesn't get the delta function in $\nabla^2 1/r$!

Recap: $$\vec{E}(\vec{x}) = \frac{1}{4\pi\epsilon_0}\int d^3x' \, \frac{\vec{x}-\vec{x}'}{|\vec{x}-\vec{x}'|^3} \cdot \rho(\vec{x}')$$

where $$\rho(\vec{x}') = \sum_{i=1} q_i \, \delta^3(\vec{x}' - \vec{x}_i)$$

compare later to boundary conditions

- Note that this is like our convolution example
  - If we know how to find the response (here the $\vec{E}$ field) from a point charge (we'll call this a Green's function) then the full response follows from a well-defined integral over the function of point charges (here $\rho(\vec{x}')$)
  - $\Rightarrow$ solution to electrostatics by "summation"
    - requires knowing $\rho(\vec{x}')$

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Let's return to calculate $\nabla^2 \frac{1}{|\vec{x}-\vec{x}'|}$:

- The problem happens with $\vec{x}' = \vec{x}$ and the denominator is zero. $\Rightarrow$ let's define it by a limit with a "regulator" ε (cf. an ultraviolet cutoff in field theory)

$$\nabla^2 \frac{1}{|\vec{x}-\vec{x}'|} = \lim_{\varepsilon\to 0} \nabla^2 \frac{1}{\sqrt{(\vec{x}-\vec{x}')^2+\varepsilon^2}} \quad \left(\lim_{\varepsilon\to 0} \text{ implicit from here on}\right)$$

$$= (\partial_x^2+\partial_y^2+\partial_z^2) \frac{1}{\sqrt{(x-x')^2+(y-y')^2+(z-z')^2+\varepsilon^2}}$$ just carry it out! 3 copies of each term

claim

$$= \frac{-3}{((x-x')^2+(y-y')^2+(z-z')^2+\varepsilon^2)^{3/2}} + \frac{3[(x-x')^2+(y-y')^2+(z-z')^2]}{[(x-x')^2+(y-y')^2+(z-z')^2+\varepsilon^2]^{5/2}}$$

$$= -3\frac{\varepsilon^2}{[(x-x')^2+(y-y')^2+(z-z')^2+\varepsilon^2]^{5/2}} \underset{\varepsilon\to 0}{=} \begin{cases} 0 & |\vec{x}-\vec{x}'|^2 \neq 0 \\ \infty & \vec{x}=\vec{x}' \end{cases}$$

$\Rightarrow$ looks like a delta function. We'll just check that it integrates to 1

$$\int_{\text{all space}} d^3x \, \frac{-3\varepsilon^2}{((\vec{x}-\vec{x}')^2+\varepsilon^2)^{5/2}} \underset{\vec{x}\to\vec{x}-\vec{x}'}{\overset{\text{change variables}}{=}} \int d^3x \, \frac{-3\varepsilon^2}{(\vec{x}^2+\varepsilon^2)^{5/2}} \overset{\text{spherical}}{=} -3\varepsilon^2 \underbrace{4\pi}_{\theta,\varphi \text{ integrals}} \int_0^\infty \frac{r^2\,dr}{(r^2+\varepsilon^2)^{5/2}}$$

Mathematica $\Rightarrow$ $= -4\pi$

$$\Rightarrow \lim_{\varepsilon\to 0} \frac{-3\varepsilon^2}{(\vec{x}^2+\varepsilon^2)^{5/2}} = -4\pi\,\delta^3(\vec{x})$$

$$\ast\ast \Rightarrow \boxed{\nabla^2 \frac{1}{|\vec{x}-\vec{x}'|} = -4\pi\,\delta^3(\vec{x}-\vec{x}')}$$

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Aside: Could we have done this directly in spherical coordinates? Yes

Two ways, both with $\vec{x}'=0$ (do the easier problem first!)

i) $\nabla^2 \frac{1}{\sqrt{r^2+\varepsilon^2}}$ using Jackson cover $\nabla^2\psi = \frac{1}{r^2}\frac{\partial}{\partial r} r^2 \frac{\partial\psi}{\partial r} = \frac{1}{r}\frac{\partial^2}{\partial r^2}(r\psi)$
if angular derivatives are zero.

$$\Rightarrow \nabla^2 (r^2+\varepsilon^2)^{-1/2} = \frac{1}{r^2}\frac{\partial}{\partial r} r^2\left(-\frac{1}{2}(r^2+\varepsilon^2)^{-3/2}\cdot 2r\right) = -\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^3(r^2+\varepsilon^2)^{-3/2}\right)$$

$$= -\frac{1}{r^2}\left(\frac{3r^2}{(r^2+\varepsilon^2)^{3/2}} - \frac{3}{2}\frac{r^3}{(r^2+\varepsilon^2)^{5/2}}2r\right) = \frac{-3}{(r^2+\varepsilon^2)^{3/2}} + \frac{3r^2}{(r^2+\varepsilon^2)^{5/2}}$$

$$= \frac{-3}{(r^2+\varepsilon^2)^{5/2}}\left((r^2+\varepsilon^2)-r^2\right) = \frac{-3\varepsilon^2}{(r^2+\varepsilon^2)^{5/2}} \xrightarrow[\varepsilon\to 0]{} \begin{cases} 0 & \text{if } r>0 \\ \frac{1}{\varepsilon^3}\to\infty & \text{if } r=0 \end{cases}$$

Check $\int d^3x \frac{-3\varepsilon^3}{(r^2+\varepsilon^2)^{5/3}} = -3\cdot 4\pi\varepsilon^2\int_0^\infty r^2 dr \frac{1}{(r^2+\varepsilon^2)^{5/3}} = -4\pi$ as before

$$\Rightarrow \boxed{\nabla^2\frac{1}{r} = -4\pi\,\delta^3(\vec{x})}$$

ii) $\nabla^2\frac{1}{r+\varepsilon} = \frac{1}{r}\frac{\partial^2}{\partial r^2}\left(\frac{r}{r+\varepsilon}\right) = -\left(\frac{2r}{(r+\varepsilon)^2}-\frac{2r^2}{(r+\varepsilon)^3}\right) = -2r\left(\frac{r+\varepsilon}{(r+\varepsilon)^3}-\frac{r}{(r+\varepsilon)^3}\right) = \frac{-2\varepsilon/r}{(r+\varepsilon)^3}$

Check $\int d^3x \frac{-2\varepsilon/r}{(r+\varepsilon)^3} = -2\varepsilon\, 4\pi\int_0^\infty r^2 dr\frac{1/r}{(r+\varepsilon)^3} = -2\varepsilon\cdot 4\pi\int_0^\infty\frac{r\,dr}{(r+\varepsilon)^3} = -4\pi$ ✓

Does it work to use the divergence theorem?

$\int d^3x\, \nabla^2\frac{1}{|\vec{x}|} = \int d^3x\, \vec{\nabla}\cdot\left(\vec{\nabla}\frac{1}{|\vec{x}|}\right)$ integrate over a sphere of radius R centered at origin

$$= \oint_{\substack{\text{surface}\\ \text{of}\\ \text{sphere}}} \underbrace{\left(\vec{\nabla}\frac{1}{r}\right)}_{\frac{\partial}{\partial r}\left(\frac{1}{r}\right)}\cdot\underbrace{\hat{n}}_{\hat{r}}\, da = \int R^2 d\Omega\left(-\frac{1}{R^2}\right) = -4\pi \checkmark$$

[Jackson: $\vec{\nabla}\psi = \hat{r}\frac{\partial\psi}{\partial r} + \hat{\theta}\frac{1}{r}\frac{\partial\psi}{\partial\theta} + \hat{\varphi}\frac{1}{r\sin\theta}\frac{\partial\psi}{\partial\varphi}$]

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So back to Poisson's equation:

$$\nabla^2 \phi(\vec{x}) = \frac{1}{4\pi\varepsilon_0}\int d^3x' \, \nabla^2 \frac{1}{|\vec{x}-\vec{x}'|}\rho(\vec{x}')$$

$$= -\frac{1}{\varepsilon_0}\int d^3x' \, \delta^3(\vec{x}-\vec{x}')\rho(\vec{x}') = -\frac{1}{\varepsilon_0}\rho(\vec{x}) \checkmark$$

So Poisson's equation works!

What about delta functions in Gauss's Law?

$$\vec{\nabla}_x \cdot \vec{E}(\vec{x}) = \frac{1}{4\pi\varepsilon_0}\int d^3x' \, \vec{\nabla}_x \cdot \left(\frac{\vec{x}-\vec{x}'}{|\vec{x}-\vec{x}'|^3}\right)\rho(\vec{x}') = ?$$

- Specialize again to $x'=0$ and use regulator again

$$\vec{\nabla}\cdot\frac{\vec{x}}{|\vec{x}|^3} = \frac{\partial}{\partial x_i}\frac{x_i}{(x_1^2+x_2^2+x_3^2+\varepsilon^2)^{3/2}} \overset{\frac{\partial}{\partial x_i}x_j=\delta_{ij}}{=} \frac{3}{(x_1^2+x_2^2+x_3^2+\varepsilon^2)^{3/2}} - \frac{3}{2}\frac{\partial(x_1^2+x_2^2+x_3^2)}{(x_1^2+x_2^2+x_3^2+\varepsilon^2)^{5/2}}$$

$$= \frac{3\varepsilon^2}{(\vec{x}^2+\varepsilon^2)^{5/2}} \xrightarrow[\varepsilon\to 0]{} \begin{cases} 0 & \text{if } |\vec{x}|\neq 0 \\ -\frac{3}{\varepsilon^3}\to\infty & \text{if } |\vec{x}|=0 \end{cases}$$

Again,

$$\int d^3x \, \vec{\nabla}\cdot\left(\frac{\vec{x}}{|\vec{x}|^3}\right) = 4\pi\int_0^\infty dr\, r^2 \frac{3\varepsilon^2}{(r^2+\varepsilon^2)^{5/2}} = 4\pi \Rightarrow \vec{\nabla}\cdot\left(\frac{\vec{x}}{|\vec{x}|^3}\right) = 4\pi\delta^3(\vec{x})$$

$$\Rightarrow \nabla\cdot\left(\frac{\vec{x}-\vec{x}'}{|\vec{x}-\vec{x}'|^3}\right) = 4\pi\delta^3(\vec{x}-\vec{x}')$$

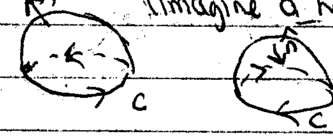
$$\Rightarrow \vec{\nabla}_x\cdot\vec{E}(\vec{x}) = \frac{1}{4\pi\varepsilon_0}\int d^3x' \, 4\pi\delta^3(\vec{x}-\vec{x}')\rho(\vec{x}') = \frac{\rho(\vec{x})}{\varepsilon_0} \checkmark$$

Return to $\vec{\nabla}\times\vec{E}=0$ ...

· We have another vector calculus theorem: Stoke's theorem, that says if we have an open surface S and C is the contour bounding it with line element $d\vec{l}$, then

$$\int_S (\vec{\nabla}\times\vec{A})\cdot\hat{n}\,da = \oint_C \vec{A}\cdot d\vec{l}$$

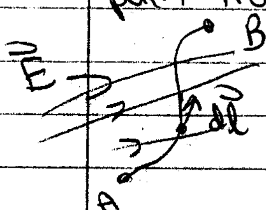
· $\hat{n}$ is defined by right-hand-rule given the direction of the line integral around C. e.g., (imagine a hemisphere)



[Also: $\int_S (\hat{n}\times\vec{\nabla}\psi)\,da = \oint_C \psi\, d\vec{l}$]

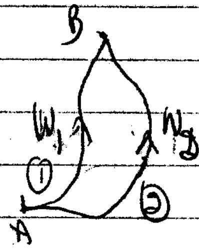
· So if $\vec{\nabla}\times\vec{E}$ is zero, then the right side is zero for any closed contour $\Rightarrow \oint_C \vec{E}\cdot d\vec{l}=0$.

· But what is $\vec{E}\cdot d\vec{l}$ physically? $\vec{F}=q\vec{E}$, so if we have a path from A to B, then the work we do moving a point charge q from A to B is



$$W=-\int_A^B \vec{F}\cdot d\vec{l} = -q\int_A^B \vec{E}\cdot d\vec{l}$$

But since we can close the path a different way



$$W_1-W_2 = -q\int_{①}\vec{E}\cdot d\vec{l} + q\int_{②}\vec{E}\cdot d\vec{l} = q\oint_{C=①-②}\vec{E}\cdot d\vec{l}$$

backwards

Stoke's Theorem $= \oint_S (\vec{\nabla}\times\vec{E})\cdot\hat{n}\,da = 0 \Rightarrow W_1=W_2$ for any 2 contours.

Check $W=-q\int_A^B \vec{E}\cdot d\vec{l} = q\int_A^B \vec{\nabla}\phi\cdot d\vec{l} = q\int_A^B d\phi = q(\phi_B-\phi_A) \Rightarrow$ Work is independent of path! (for electrostatics)

only depends on end points

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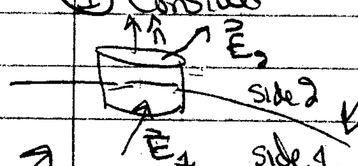
Application of Gauss's Law: Discontinuities of $\vec{E}$ at a surface.

- Given $\vec{\nabla}\cdot\vec{E} = \frac{1}{\varepsilon_0}\rho$ and $\vec{\nabla}\times\vec{E} = 0$

(1) Consider



surface charge density σ

Gauss's law applied to a "Gaussian cylinder" with sides that go to zero $\Rightarrow$ only surviving contributions from end caps!

make cylinder small enough so $\vec{E}_1, \vec{E}_2$ can be taken as constants

area of cap

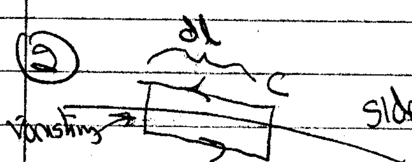
charge inside

$$\left(\vec{E}_2\cdot\hat{n} - \vec{E}_1\cdot\hat{n}\right) S = \frac{1}{\varepsilon_0}\sigma S$$

$$\Rightarrow \boxed{\left(\vec{E}_2 - \vec{E}_1\right)\cdot\hat{n} = \frac{\sigma}{\varepsilon_0}}$$

$\Rightarrow$ normal component of $\vec{E}$ has discontinuity given by surface charge.

(2)



$\vec{\nabla}\times\vec{E} = 0 \Rightarrow \oint_C d\vec{\ell}\cdot\vec{E} = 0$

loop with length $d\ell$ small enough so $\vec{E}_1, \vec{E}_2 \approx$ constant

$$\Rightarrow \vec{E}_2\cdot d\vec{\ell} - \vec{E}_1\cdot d\vec{\ell} = 0$$

sides vanishes in limit

$$\Rightarrow \boxed{E_{2t} = E_{1t}}$$

$\Rightarrow$ tangential component is always continuous (even if $\sigma \neq 0$)