Recap and comments on P5.8

1. Gauss's Law and capacitance \( \Rightarrow (169) \), for conducting sphere;
   - general: \( Q_i = \frac{1}{\varepsilon_0} E_i V_i \) for \( n \) conductors; for parallel plates
   - for two conductors with equal and opposite \( Q \), then
     \( C_m = C_p = C \).
   - for parallel plates \( C = \frac{\varepsilon_0 A}{d} \). In problems concentric spheres
     and concentric cylinders. In what limit do these look like
     parallel plates? Note: only depend on geometry: \( A \) and \( d \) (\( \varepsilon_0, V, \text{etc.} \)).

2. Recall electrostatic energy
   - work: \( -q \int \vec{E} \cdot d\vec{l} = q\phi_B - q\phi_A \) on conductors
   - energy: \( W = \frac{1}{2 \varepsilon_0} \int \vec{E} \cdot \vec{D} \) \( \Rightarrow \int \left( \frac{\partial \phi}{\partial x} \right) \left( \frac{\partial \phi}{\partial x} \right) ds = \frac{\varepsilon_0}{2} \int \vec{E} \cdot \vec{D} \)

3. Comment on forces on capacitors: imagine pulling apart
   plates from \( d \) to \( d + d\delta \) requires work \( F \delta d \) and this must
   agree the change in energy.
   - recall the idea. If we lift a ball against gravity by \( \Delta h \),
     \( \Delta U = m g \Delta h \)
     force: \( F \Delta h = -mg \Delta h \)
     \( F = -mg \)

     \( \text{be careful:} \)
     a) constant \( G \) means \( W = W_{\text{capacitor}} \)
     b) constant \( V \) mean \( \Rightarrow W = W_{\text{capacitor}} + W_{\text{battery}} \)

4. Variational principle for capacitance
   \( \nabla V = 0 \)
   \( \varepsilon_0 \int \vec{E} \cdot d\vec{A} = \frac{Q}{\varepsilon_0} \int \vec{D} \cdot d\vec{A} \leq C \) \( \Rightarrow \frac{1}{2} \sigma(x)^2 \leq \frac{1}{2} \varepsilon_0 \int \vec{E} \cdot d\vec{A} \)
   \( \Rightarrow \) minimize \( \int \vec{E} \cdot d\vec{A} \) to get best estimate of \( C \).
10/28/13

4. Comment: For Coulomb, $\phi(x) = \frac{q}{4\pi \epsilon_0} \frac{1}{r_{x'x}}$.

What would this be for $\phi(x) = \frac{q}{4\pi \epsilon_0} f(|x|)$? Is it from point charge $q$ at $x$?

If you find $\phi(x)$, does $E(x) = -\nabla \phi(x)$ still hold?

Now return to general solutions to Poisson's equation.

(170) Boundary conditions and uniqueness.

(191) Green's function idea and master formula.

3a. Show $\phi(x) = \frac{1}{4\pi \epsilon_0} \int \Sigma d\Omega \phi(x')$ for $x$ on surface.

Surface integrals of $1/|x-x'|$ is average of $\phi(x')$ on surface.

From master formula, $G(x,x') = \frac{1}{|x-x'|}$ is insufficient to make surface terms vanish.

But what about $\frac{\partial}{\partial x'}$, and so on?
return to master formula:

\[
\Phi(x) = \frac{1}{4\pi}\int_{\mathbb{R}^3} \frac{G(x, x') \Phi(x')}{|x-x'|} \, dx' + \frac{1}{4\pi} \int_{S^2} (\nabla G(x, x')) \cdot \hat{n} \, dS',
\]

General idea of a Green function \( G(x, x') \) for a linear differential equation \( \mathcal{L}_x \Phi(x) = f(x) \)

\( \mathcal{L}_x \Phi(x) = f(x) \)

where \( f(x) \) is known and \( \Phi(x) \) is to be found.

\( e.g. \quad \mathcal{L}_x \Phi = \nabla^2 \Phi, \quad f(x) = -\frac{\Phi(x)}{\epsilon_0} \Rightarrow \Phi(x) \to \phi(x) \)

Then if \( \mathcal{L}_x \Phi(x, x') = \delta(x-x') \Rightarrow \Phi(x) = \int G(x, x') f(x') \, dx' \)

with \( G(x, x') \) including the boundary conditions for \( \Phi(x) \).

We will define \( G \) for electrostatics by \( \nabla^2 G(x, x') = -\frac{\delta(x-x')}{\epsilon_0} \),

with a lot of freedom:

\( G(x, x') = \frac{1}{4\pi|x-x'|} + F(x, x') \) for any \( F \) with \( \nabla^2 F(x, x') = 0 \) in \( V \).

\( \star \star \) We use this freedom for \( F \) to fix boundary conditions for \( G \) on \( S \).

Example: We have an infinite flat conductor, which corresponds to Neumann boundary conditions:

\( \nabla \cdot \mathbf{E} = 0 \) on \( S \)

The Neumann Green's function \( G_N(x, x') \) satisfies

\( \nabla^2 G_N(x, x') = -\frac{\delta(x-x')}{\epsilon_0} \) for \( x, x' \in V \) (other constant terms)

\( \frac{\partial G_N}{\partial n} \) can be chosen, like \( \frac{1}{\epsilon_0} \nabla \Phi \)

Choose \( G_N(x, x') = 0 \) for \( x \notin S \), \( x' \in V \) (\( \Phi \) on surface).

Then master formula reduces to

\( \Phi(x) = \frac{1}{4\pi} \int_{\mathbb{R}^3} \frac{G_N(x, x') \Phi(x')}{|x-x'|} \, dx' + \frac{1}{4\pi} \int_{S^2} (\nabla G_N(x, x')) \cdot \hat{n} \, dS' \)

Choose \( F(x, x') = -\frac{1}{4\pi|x-x'|} \) where \( x'' = (-x', y', z') \).
So $\bar{x}'$ is the mirror of $\bar{x}'$. We know
$\nabla^2 F(x, \bar{x}') = 4\pi \delta'(x-\bar{x})$, but $\bar{x}''$ is not in $V$,

$\Rightarrow \nabla^2 F(x, \bar{x}') = 0 \text{ in } V$

$\Rightarrow G_D(x, \bar{x}') = \frac{\pi}{x-\bar{x}}$

on the surface, $\bar{x}'' = \bar{x}' \Rightarrow G_D(x, \bar{x}'') = 0$

If $\bar{\delta}(x)$ is 0 on the surface, then we are done.

Otherwise, include contribution of $-\frac{1}{4\pi} \int_{\frac{1}{2}} ds'$

Well consider specific examples of Dirichlet and discuss Neumann boundary conditions next time.