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770l Lecture 24

Recap and comments on PS#8

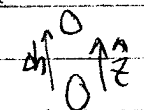
1. Gauss's Law and capacitance ⇒ 169 for conducting sphere and parallel plates
   - general: $Q_i = \sum_{j=1} C_{ij} V_j$ for n conductors.
   - For two conductors with equal and opposite Q, then $C_{11} = C_{22} = C$.
   - for parallel plates $C = \varepsilon_0 \frac{A}{d}$. In problem: concentric spheres and concentric cylinders. In what limit do these look like parallel plates? Note: only depend on geometry: A and d. (no Q, V, etc.)

- Recall electrostatic energy
  - work $= -q \int_A^B \vec{E} \cdot d\vec{l} = \phi_B - \phi_A$ on conductors

energy: $W = \frac{1}{8\pi\varepsilon_0} \int d^3x \int d^3x' \frac{\rho(\vec{x})\rho(\vec{x}')}{|\vec{x}-\vec{x}'|} = \frac{1}{2}\int d^3x\, \rho(\vec{x})\phi(\vec{x}) = \frac{\varepsilon_0}{2}\int d^3x\, |\vec{E}|^2$

⇒ go through perfect conductors 167 - 168

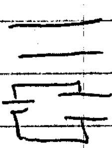
2. Comment on forces on capacitors: imagine pulling apart plates from d to d+Δd ⇒ requires work FΔd and this must equal the change in energy.
   - recall the idea. If we lift a ball against gravity by Δh,



force · Δh = $-\Delta W_{gravity}$ $F\Delta h = -mg\Delta h \Rightarrow F = -mg$ ✓

Make sure the signs agree with physics.

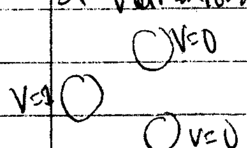
- Be careful:



a) constant Q means $W = W_{capacitor}$ relate to ΔQV

b) constant V mean ⇒ $W = W_{capacitor} + W_{battery}$

5. Variational principle for capacitance.



guess for scalar field including boundary conditions

$C[\Psi(\vec{x})] = \varepsilon_0 \int_V |\nabla\Psi|^2 d^3x \leq C$ ⇒ parametrize $\Psi(\vec{x})$ and minimize $C[\Psi(\vec{x})]$ to get best estimate of C.

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4. Comment: for Coulomb, $\phi(\vec{x}) = \int d^3x' \underbrace{\frac{1}{4\pi\varepsilon_0}\frac{1}{|\vec{x}-\vec{x}'|}}_{\phi_{\text{point charge}}(\vec{x}) \text{ from point charge at } \vec{x}'} \rho(\vec{x}')$

What would this be for $\phi_{\text{point charge}}(\vec{x}) = \frac{q}{4\pi\varepsilon_0} f(|\vec{x}|)$?

If you find $\phi(\vec{x})$, does $\vec{E}(\vec{x}) = -\vec{\nabla}\phi(\vec{x})$ still hold?

Now return to general solutions to Poisson's equation

- 170 : boundary conditions and uniqueness
- 171 : Green's function idea and master formula

3a. Show $\phi(0) = \frac{1}{4\pi R^2}\int da\, \phi(\vec{y})$ for $\vec{y}$ on surface

surface of sphere of radius R

average of $\phi(\vec{y})$ on surface

- From master formula, $G(\vec{x},\vec{x}') = \frac{1}{|\vec{x}-\vec{x}'|}$ is insufficient to make surface terms vanish.
  - But what about $\frac{1}{|\vec{x}-\vec{x}'|}$ + const?

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Return to master formula:

for $\vec{x} \in V$ $\quad \Phi(\vec{x}) = \frac{1}{4\pi\epsilon_0}\int_V d^3x' \, G(\vec{x},\vec{x}')\rho(\vec{x}') + \frac{1}{4\pi}\oint_S \left[G(\vec{x},\vec{x}')\frac{\partial \Phi}{\partial n'} - \Phi(\vec{x}')\frac{\partial G(\vec{x},\vec{x}')}{\partial n'} \right] da'$

General idea of a Green function $G(\vec{x},\vec{x}')$ for a linear differential equation $\hat{L}_x$

$$\hat{L}_x \psi(\vec{x}) = J(\vec{x})$$

where $J(\vec{x})$ is known and $\psi(\vec{x})$ is to be found.

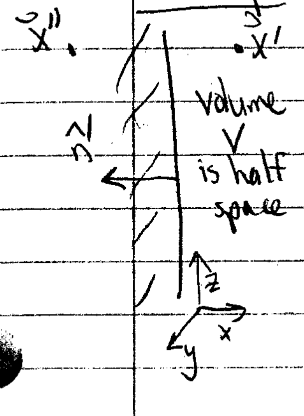
- e.g. $\hat{L}_x = \nabla_x^2$, $J(\vec{x}) = -\rho(\vec{x})/\epsilon_0$, $\psi(\vec{x}) \to \phi(\vec{x})$

Then if $\hat{L}_x G(\vec{x},\vec{x}') = \delta^3(\vec{x}-\vec{x}') \Rightarrow \psi(\vec{x}) = \int d^3x' \, G(\vec{x},\vec{x}') J(\vec{x}')$
with $G(\vec{x},\vec{x}')$ including the boundary conditions for $\psi(\vec{x})$.

We will define the G for electrostatics by $\nabla^2 G(\vec{x},\vec{x}') = -4\pi\delta^3(\vec{x}-\vec{x}')$, (particularly convenient)
with a lot of freedom:

$$G(\vec{x},\vec{x}') = \frac{1}{|\vec{x}-\vec{x}'|} + F(\vec{x},\vec{x}') \quad \text{for any } F \text{ with } \nabla_x^2 F(\vec{x},\vec{x}') = 0 \text{ in } V.$$

** We use this freedom for F to fix boundary conditions for G on S.



Example: We have an infinite flat conductor, which corresponds to Dirichlet boundary conditions.

- The Dirichlet Green's function $G_D(\vec{x},\vec{x}')$ satisfies
  $\nabla^2 G_D(\vec{x},\vec{x}') = -4\pi\delta(\vec{x}-\vec{x}')$ for $\vec{x},\vec{x}' \in V$ (other constant besides 4π can be chosen, like $\frac{1}{\epsilon_0}$)

choose $G_D(\vec{x},\vec{x}_S) = 0$ for $\vec{x}_S \in S$, $\vec{x} \in V$ ($\vec{x}_S$ on surface)

Then master formula reduces to

$$\Phi(\vec{x}) = \frac{1}{4\pi\epsilon_0}\int_V G_D(\vec{x},\vec{x}')\rho(\vec{x}')\,dV' - \frac{1}{4\pi}\int_S \Phi \vec{\nabla} G_D \cdot \hat{n} \, dS'$$

Chose $F(\vec{x},\vec{x}') = -\frac{1}{|\vec{x}-\vec{x}''|}$ where $\vec{x}'' = (-x', y', z')$

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So $\vec{x}''$ is the mirror of $\vec{x}'$. We know

$$\nabla^2 F(\vec{x},\vec{x}') = 4\pi\delta^3(\vec{x}-\vec{x}'')$$ but $\vec{x}''$ is not in V!

$$\Rightarrow \nabla^2 F(\vec{x},\vec{x}') = 0 \text{ in } V$$

$$\Rightarrow G_D(\vec{x},\vec{x}') = \frac{1}{|\vec{x}-\vec{x}'|} - \frac{1}{|\vec{x}-\vec{x}''|}$$

on the surface, $\vec{x}'' = \vec{x}' \Rightarrow G_D(\vec{x},\vec{x}_s) = 0$

- If $\Phi(\vec{x}_s) = 0$ on the surface, then we are done.
- Otherwise include contribution of $-\frac{1}{4\pi}\int_S \Phi \vec{\nabla}_{x'} G_D \cdot \hat{n}\, dS'$

- We'll consider specific examples of Dirichlet and discuss Neumann boundary conditions next time.