· Comment on alternative solution to prob. 2
  · Use electric field only from other plate

$+Q$ — $\uparrow E_1 \downarrow E_2 \Rightarrow \vec{E}=0$

$\downarrow E_1 \downarrow E_2 \Rightarrow |\vec{E}| = |E_1 + E_2|$

$-Q$ — $\downarrow E_1 \uparrow E_2 \Rightarrow \vec{E}=0$

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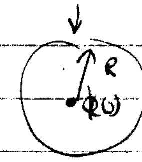
$\boxed{\text{7701 Lecture 25}}$

- Midterm will be Wednesday evening November 6 in Sm1138
  - contact me about conflicts
  - covers material on PS#5 – PS#8 (so just the start of Green functions)
  - emphasis on "checks" of answers

- See: What is $\frac{d}{dx}\int_0^x f(x')\,dx'$? etc. on 177 ← relevant for PS#8 4 b)

- Today: Green functions → set up and execution
  - boundary conditions and uniqueness 170
  - master formula 171



PS#8 Problem 3a.: Show $\phi(0) = \underbrace{\frac{1}{4\pi R^2}}_{\text{surface area of sphere}} \oint da\, \phi(\vec{y}_s)$ ← $\vec{y}_s$ on surface

surface of sphere of radius R

$\underbrace{\qquad\qquad}_{\text{average of } \phi(\vec{y}_s) \text{ on surface}}$

Charge free $\Rightarrow \rho(\vec{x}') = 0$ in V

· How would master formula apply?

$$\Phi(0) = \frac{1}{4\pi\epsilon_0}\int_V d^3x'\, G(0,\vec{x}')\overset{0}{\cancel{\rho(\vec{x}')}} + \frac{1}{4\pi}\oint_S \left[G(0,\vec{x}')\frac{\partial\Phi}{\partial n} - \Phi(\vec{x}')\frac{\partial G(0,\vec{x}')}{\partial n'}\right] da'$$

$r'^2 d\Omega$

$\frac{\partial}{\partial r'}\frac{1}{r'} = -\frac{1}{r'^2}$

We know $\nabla^2_{x'} G(0,\vec{x}') = -4\pi\delta^3(\vec{x}')$ but $G(0,\vec{x}') = \frac{1}{|\vec{x}'|}$ won't make the $G(0,\vec{x}')\frac{\partial\Phi}{\partial n}$ term vanish. (but the average of $\Phi(\vec{x}')$ works).

· But what about $G(0,\vec{x}') = \frac{1}{|\vec{x}'|} + \text{const.}$?

· How could you choose const. so that $G(0,\vec{x}') = 0$ on the surface?

· This is an example of a Dirichlet boundary condition.

Note: Here we use the general procedure with Dirichlet BC's taking $G_D(\vec{x},\vec{x}') = 0$ on the surface. That is not necessary in 3a, because Gauss's law makes the $\nabla\Phi$ integration vanish anyway (no charge contained)

· Now continue with 174, 175, and 178 #

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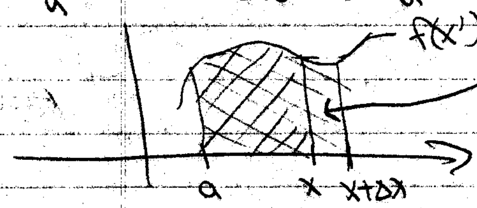
Side note (relevant to PS#8 problem 4):

① What is $\frac{d}{dx}\int_a^x f(x')\,dx'$?

Let $g(x) \equiv \int_a^x f(x')\,dx' \Rightarrow \frac{d}{dx}g(x) = \lim_{\Delta x \to 0} \frac{g(x+\Delta x) - g(x)}{\Delta x}$

$\Rightarrow \frac{d}{dx}\int_a^x f(x')\,dx' = \lim_{\Delta x\to 0}\frac{1}{\Delta x}\left[\int_a^{x+\Delta x} f(x')\,dx' - \int_a^x f(x')\,dx'\right]$



approximation to area

$= \lim_{\Delta x\to 0}\frac{1}{\Delta x}\left[f(x)\cdot\Delta x\right] = \boxed{f(x)}$

independent of a!

② What about: $\frac{d}{dx}\int_x^a f(x')\,dx' = \frac{d}{dx}\left(-\int_a^x f(x')\,dx'\right) = -f(x)$.

③ What about: $\frac{d}{dx}\int_a^{g(x)} f(x')\,dx' = \frac{d}{dg}\left(\int_a^g f(x')\,dx'\right)\cdot\frac{dg}{dx} = f(g(x))\,\frac{dg(x)}{dx}$

④ What if the integrand is $f(x,y)$?

$\frac{d}{dx}\int_a^x f(x,y)\,dy$?

Repeat our trick: $= \lim_{\Delta x\to 0}\left[\int_a^{x+\Delta x} \underbrace{f(x+\Delta x, y)}_{=f(x)+\Delta x\frac{\partial f}{\partial x}(x,y)}\,dy - \int_a^x f(x,y)\,dy\right]/\Delta x$

$= \lim_{\Delta x\to 0}\left\{\frac{1}{\Delta x}\left[\int_a^{x+\Delta x} - \int_a^x f(x,y)\,dy\right] + \frac{1}{\cancel{\Delta x}}\int_a^{x+\Delta x}\cancel{\Delta x}\,\frac{\partial f}{\partial x}(x,y)\,dy\right.$

$$\boxed{= f(x,y) + \int_a^x \frac{\partial f}{\partial x}(x,y)\,dy}$$

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More general comments on Green Functions...

- We are interested in the specific case of the Coulomb interaction for electrostatics with different BC's.
- We will solve this for many different geometries.
- But the Green function concept is much more general
  $\Rightarrow$ we want to look at other choices for $\mathcal{L}_x$!

- Problem 4. on PS#8 has one type of non-Coulomb generalization

Given: electrostatic potential at $\vec{x}$ of a point charge located at the origin is

$$\phi_{pc}(\vec{x}) = \frac{q}{4\pi\varepsilon_0} f(|\vec{x}|) = \frac{q}{4\pi\varepsilon_0} f(r)$$

- So $\rho_{pc}(\vec{x}) = q\,\delta^3(\vec{x})$

Did you recognize that this was a Green's Function problem?

- first, if the point charge is at $\vec{x}'$, then $\phi_{pc} \Rightarrow \phi_{pc}(\vec{x}-\vec{x}')$
- second, given $\rho(\vec{x}')$, superposition says

$$\phi(\vec{x}) = \int d^3x' \, \frac{\phi_{pc}(\vec{x}-\vec{x}')}{q} \, \rho(\vec{x}')$$

Green's function version $\Rightarrow$ $= \int d^3x' \, G(\vec{x},\vec{x}') \, \rho(\vec{x}')$ with $G(\vec{x},\vec{x}') = \frac{1}{4\pi\varepsilon_0} f(|\vec{x}-\vec{x}'|)$

Check: put $\underline{\rho(\vec{x}') = q\,\delta^3(\vec{x}')} \Rightarrow \phi(\vec{x}) = \frac{q}{4\pi\varepsilon_0} f(|\vec{x}|)$ ✓

- One-dimensional generalization: solve

$$\frac{d^2}{dx^2}\psi(x) + k^2\psi(x) = f(x) \quad \text{or} \quad \frac{d^2}{dx^2}\psi(x) - m^2\psi(x) = f(x)$$

$$\Rightarrow \frac{d^2}{dx^2}G(x,x') + k^2 G(x,x') = \delta(x-x') \Rightarrow \psi(x) = \int dx' \, G(x,x') f(x')$$

check: $\left(\frac{d^2}{dx^2}+k^2\right)\psi(x) = \int dx' \left(\frac{d^2}{dx^2}+k^2\right) G(x,x') f(x') = \int dx' \, \delta(x-x') f(x') = f(x)$ ✓

Here $(\hat{L}_x + k^2)\psi(x) = f(x)$, inverse matrix, vector, vector

Formally, think of matrices: $(\hat{L}_x + k^2)^{-1}(\hat{L}_x + k^2)\psi(x) = (\hat{L}_x + k^2)^{-1} f(x)$

$$\Rightarrow \psi(x) = (\hat{L}_x + k^2)^{-1} f(x)$$

$\Rightarrow$ Green's function is the inverse (with BC's needed) of the operator $\hat{L}_x + k$.

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- More general form in one-d:

$$\hat{\mathcal{L}}_x \equiv \frac{d}{dx} f(x) \frac{d}{dx} - g(x)$$

some function

and $\hat{\mathcal{L}}_x \psi(x) + \lambda w(x) \psi(x) = h(x)$

some parameter

{ we'll see this from cylindrical and spherical coordinates after separating variables

Then find Green's function satisfying (or whatever constant you want)

$$\hat{\mathcal{L}}_x G(x,x') + \lambda w(x) G(x,x') = -4\pi\delta(x-x')$$

to solve the problem: $\psi(x) = \int dx' \, G(x,x') h(x')$.

What possible methods of there to find $G(x,x')$ [or $G(\vec{x},\vec{x}')$]?

- In >1 dimension, typically use separation of variables first.

i) Guess (or look up/construct from known answers) to satisfy BC's ⇒ method of images

ii) eigenfunction expansion.
eg. for $\hat{L}_x \psi(x) - \lambda \psi(x) = h(x)$ given $\hat{L}_x \varphi_n(x) = \lambda_n \varphi_n(x)$
and $\varphi_n(x)$ complete set ⇒ $\sum_n \varphi_n(x') \varphi_n(x) = \delta(x-x')$

[This would be $\varphi_n^*(x')$ more generally]

$$\Rightarrow G(x,x') = \sum_n \frac{\varphi_n(x')\varphi_n(x)}{\lambda - \lambda_n} \text{ satisfies } (\hat{L}_x - \lambda) G(x,x') = -\delta(x-x')$$

[try it!]

iii) transform methods ⇒ eg. Fourier transform/series

iv) division of region method

- find general solutions (with given BC's) from $x<x'$ and $x>x'$ where $\delta(x-x')=0$, then match at $x=x'$.

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Aside: What if our equation has a nonlinear part? Can we use the Green's function approach?

E.g., we want to solve

$$\frac{d^2}{dx^2}\phi(x) + \lambda\phi^3(x) = f(x)$$

- One way
  - Find $G(x,x')$ for $\frac{d^2}{dx^2}$ alone: $\frac{d}{dx^2}G(x,x') = \delta(x-x')$
  - Then write $\frac{d^2}{dx^2}\phi(x) = f(x) - \lambda\phi^3(x) \equiv \tilde{f}(x)$

$$\Rightarrow \phi(x) = \int dx'\, G(x,x')\tilde{f}(x) = \int dx'\, G(x,x')\left(f(x) - \lambda\phi^3(x)\right)$$

- This is an integral equation for $\phi(x)$.

- One solution strategy that usually works when $\lambda\phi^3(x)$ is a perturbation (and sometimes when it is not a small correction) is by iteration: find $\phi^{(0)}(x), \phi^{(1)}(x), \ldots \phi^{(n)}(x)$ by

first $\phi^{(0)}(x) = \int dx'\, G(x,x') f(x')$

$$\rightarrow \phi^{(1)}(x) = \int dx'\, G(x,x')\left[f(x') - \lambda\left(\phi^{(0)}(x')\right)^3\right]$$

$$\vdots$$

$n=2,3,\ldots$ $\phi^{(n)}(x) = \int dx'\, G(x,x')\left[f(x') - \lambda\left(\phi^{(n-1)}(x')\right)^3\right]$

Keep going until $\phi^{(n)}(x)$ stops changing to the needed accuracy.

- Other approach: matrix method after discretizing x.