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7701 Lecture 27

- Emphasis points for 2nd midterm
  - go over problem sets #5–#8 together with lecture notes
  - formula sheet plus Jackson covers are available
    $\Rightarrow$ extra copies at the exam, but you can bring your own.
  - Focus on the core competencies for each topic
    a. delta and theta functions
    b. Fourier transforms
    c. Vector calculus theorems and div, grad, curl, ∇^2
    d. Electrostatics Gauss's Law, scalar potential
    e. Poisson/Laplace equations, capacitance, energy

    } • look through emphasis points in lecture notes

a. delta and theta functions

- Spot the Error! problems
- charge (or other) densities in other coordinates
- derivatives of θ-functions
- representations as Fourier transforms
- delta functions of functions:

$$I = \int_{-\infty}^{\infty} e^{-x^2}\delta(x^2+x-6)\,dx = \int_{-\infty}^{\infty} e^{-x^2}\delta(\underset{x_0=2}{(x-2)}\underset{x_0=-3}{(x+3)}) = \frac{1}{5}\left(e^{-9}+e^{-4}\right)$$

b. Fourier transforms

- carrying out a transform and inverse (often contour integration)
- theta functions from closing in one half plane or the other and getting different results (often zero)
- solving $\ddot{x}+2\alpha\dot{x}+\omega_0^2 x = f(t)$ by applying $\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} dt\, e^{i\omega t}$ to both sides, finding the resulting $\hat{x}(\omega)$, then inverting to find $x(t)$.
- Focus on setting up integrals mostly, but some features in carrying them out, like θ functions
- 3-dimensional transform in spherical coordinates

e.g.,) $\rho(\vec{x}) = \rho_0 \frac{e^{-r/a}}{4\pi} \Rightarrow \rho(\vec{k}) = \frac{1}{(2\pi)^{3/2}}\int d^3r\, \rho(\vec{r})\underset{\text{choosing } \hat{z} \text{ axis along } \vec{k}}{e^{-i\vec{k}\cdot\vec{r}}} = \left(\frac{1}{2\pi}\right)^{3/2}\frac{\rho_0}{k^2+1/a^2}$

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c. Vector calculus theorems and div, grad, curl, ∇^2
- carrying out operators using Jackson covers (A_1, A_2, A_3)
- applying Stokes's and Green's and divergence theorems

d. Electrostatics I
- Coulomb potential in vector form
- Gauss's law in symmetric cases; $\vec{E}$-field BC's
- scalar potential $\vec{E} = -\vec{\nabla}\Phi$, point charge and $\rho(\vec{x})$ formulas

e. Electrostatics II
- $\vec{\nabla}\cdot\vec{E} = -\nabla^2\Phi = \frac{1}{\varepsilon_0}\rho(\vec{x})$, $\vec{\nabla}\times\vec{E} = 0$, $\vec{\nabla}^2\frac{1}{|\vec{x}-\vec{x}'|} = -4\pi\delta^3(\vec{x}-\vec{x}')$
- electrostatic energy in terms of $\rho\cdot\rho$, $\rho\Phi$, $|\vec{E}(\vec{x})|^2$
- capacitance

- Recap of Neumann BC's and master formula: 183
- Dirichlet Green function for region outside of sphere
  - use method of images 184 - 185
  - also force and surface charge density
- Return to general discussion of solving for Green function 178 - 179
  - Aside: nonlinear equation 180
  - conducting sphere in uniform field $\vec{E}_0$
- Expansions $\Rightarrow$ separation of variables 191 +

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- Let's apply our knowledge of Fourier series (and transforms) to find solutions in Cartesian coordinates, to Poisson/Laplace equations.
  - use separation of variables
  - apply to problems with a rectangular box

- Laplace's equation:

$$\nabla^2\phi = \frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + \frac{\partial^2\phi}{\partial z^2} = 0$$

$\Rightarrow$ look for $\phi(x,y,z) = X(x)\,Y(y)\,Z(z)$

$$\Rightarrow X''YZ + XY''Z + XYZ'' = 0$$

Isolate dependence on x or y or z by dividing by XYZ

$$\Rightarrow \frac{X''}{X}(x) + \frac{Y''}{Y}(y) + \frac{Z''}{Z}(z) = 0$$

only on x — only on y — only on z

As we vary x, y, z, each term must stay the same, so each is a constant. Positive or negative constants? Can't all be the same sign!
- This will lead to multiple representations of ϕ or G_D.
- Here choose x, y separation constants negative and z constant positive:

$$\frac{X''}{X} = -\alpha^2 \quad \Rightarrow X(x) = C_1 e^{i\alpha x} + C_2 e^{-i\alpha x} \quad \text{(or sin's and cos's)}$$

$$\frac{Y''}{Y} = -\beta^2 \quad \Rightarrow Y(y) = \tilde{C}_1 e^{i\beta y} + \tilde{C}_2 e^{-i\beta y} \quad \text{(or sin's and cos's)}$$

$$\frac{Z''}{Z} = \alpha^2 + \beta^2 \equiv \gamma^2 \quad \Rightarrow Z(z) = \hat{\tilde{C}}_1 e^{\gamma z} + \hat{\tilde{C}}_2 e^{-\gamma z} \quad \text{(or sinh's and cosh's)}$$

- At this stage a general solution to Laplace's equation in a rectangular system.

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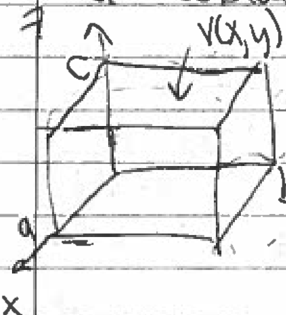
- Let's consider a definite example to see how it plays out.



- Rectangular box $a \times b \times c$
  - All surfaces grounded except top, which has $V(x,y)$
  - Note: $V(0,y) = V(a,y) = V(x,0) = V(x,b) = 0$

Plan: Use the boundary conditions at $x=0$, $y=0$, $z=0$ to determine the correct combination of exponentials, then $X(a) = Y(b) = 0$ to constrain α and β. Finally, fix the expansion coefficients by requiring $V(x,y)$ at $z=c$.

$X(0) = 0 \Rightarrow X(x) \propto \sin \alpha x$; $X(a) = 0 \Rightarrow \sin(\alpha a) = 0 \Rightarrow \alpha_n = \frac{n\pi}{a}$

$Y(0) = 0 \Rightarrow Y(y) \propto \sin \beta y$, $Y(b) = 0 \Rightarrow \sin(\beta b) = 0 \Rightarrow \beta_m = \frac{m\pi}{b}$

$Z(0) = 0 \Rightarrow Z(z) \propto \sinh(\gamma z)$ and $\gamma_{nm} = \sqrt{\alpha_n^2 + \beta_m^2} = \sqrt{\left(\frac{n\pi}{a}\right)^2 + \left(\frac{m\pi}{b}\right)^2} \equiv \gamma_{nm}$

So Φ is a combination of $\Phi_{nm}(x,y,z) \propto \sin(\alpha_n x)\sin(\beta_m y)\sinh(\gamma_{nm} z)$, so write a general linear combination:

$$\boxed{\phi(x,y,z) = \sum_{n,m=1}^{\infty} A_{nm} \sin(\alpha_n x) \sin(\beta_m y) \sinh(\gamma_{nm} z)}$$

A_{nm} is determined by requiring $\phi(x,y,z=c) = V(x,y)$ for every x, y. This is a double Fourier series $\Rightarrow$ find A_{nm} by projecting coefficients:

$$V(x,y) = \sum_{n,m=1}^{\infty} \left[A_{nm} \sinh(\gamma_{nm} c)\right] \sin(\alpha_n x) \sin(\beta_m y)$$

Recall solving wave equation for a string with fixed ends using Fourier series

$\boxed{f(x) = \sum_{n=1}^{\infty} a_n \sqrt{\frac{2}{L}} \sin\left(\frac{n\pi x}{L}\right) \text{ with } f(0) = f(L) = 0}$ $\Rightarrow$ $\boxed{a_n = \sqrt{\frac{2}{L}} \int_0^L \sin \frac{n\pi x}{L} f(x) dx}$ using $\boxed{\int_0^L dx \sqrt{\frac{2}{L}} \sin \frac{n\pi x}{L} \sqrt{\frac{2}{L}} \sin \frac{n' \pi x}{L} = \delta_{nn'}}$

$$\Rightarrow A_{nm} \sinh(\gamma_{nm} c) = \left(\frac{2}{a}\right)\left(\frac{2}{b}\right) \int_0^a dx \int_0^b dy \, V(x,y) \sin(\alpha_n x) \sin(\beta_m y)$$

$$\Rightarrow \boxed{A_{nm} = \frac{4}{ab \sinh \gamma_{nm} c} \int_0^a dx \int_0^b dy \, V(x,y) \sin(\alpha_n x) \sin \beta_m y}$$

Problem solved!

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The sines here are one example of an orthonormal set of functions. We will discuss these more generally later as solutions to a Sturm–Liouville equation:

$$\hat{\mathcal{L}}_x u_n(x) + \lambda_n w(x) u_n(x) = 0 \qquad n = 1, 2, \dots \infty$$

weight function

eigenvalue

Sturm–Liouville operator $\hat{\mathcal{L}}_x = \frac{d}{dx} f(x) \frac{d}{dx} - g(x)$

normalization constant chosen to be 1 ⇒ orthonormal

Properties:

① orthogonal wrt $w(x)$: $\int_a^b w(x) u_m^*(x) u_n(x)\, dx = \delta_{mn}$

② completeness on $[a,b]$: $f(x) = \sum_{n=1}^{\infty} a_n u_n(x) \Rightarrow a_m = \int_a^b f(x) u_m^*(x) w(x)\, dx$

and $\delta(x-x') = \sum_n u_n^*(x) u_m(x') w(x)$

③ real eigenvalues

④ self-adjoint $\int_a^b u_n^* (\hat{\mathcal{L}}_x u_m)\, dx = \int_a^b (\hat{\mathcal{L}}_x u_n)^* u_m\, dx$ [from boundary conditions on the u_n's]

- For $\hat{\mathcal{L}}_x = \frac{d^2}{dx^2}$ [so $f(x) = 1$, $g(x) = 0$] and $w(x) = 1$ with Dirichlet BC's at $x = 0$ and L, we have the case just considered:

$$u_n(x) = \sqrt{\tfrac{2}{L}} \sin\frac{n\pi x}{L} \Rightarrow f(x) = \sum_{n=1}^{\infty} a_n \sqrt{\tfrac{2}{L}} \sin\frac{n\pi x}{L} \quad \text{with } a_n = \sqrt{\tfrac{2}{L}} \int_0^{\infty} \sin\frac{n\pi x}{L} f(x)\, dx$$

- As we already knew, the $\{\sin\frac{n\pi x}{L}\}$ are a complete set on $x \in [0, L]$ with these boundary conditions. (With Neumann or mixed BC's, we have to consider cosines as well.)

- From ②, we have $\delta(x-x') = \sum_{n=1}^{\infty} \sqrt{\tfrac{2}{L}} \sin\left(\frac{n\pi x}{L}\right) \sin\left(\frac{n\pi x'}{L}\right)$

We'll use this next time to find the Dirichlet Green function $G_D(\vec{x}, \vec{x}')$ for the rectangular box.