Summary points for image charges (with outside conducting surface and infinite plane as prototypes)

1. Use boundary conditions (BCs) and symmetry to position the image charge for charges
   - mirrored for infinite plane
   - on line joining q and center of sphere
   - more charges by superposition

2. Determine q' and x' (charge and location of image) by requiring BC to be satisfied
   - grounded conductor (or \( G_0 \)) \( \Rightarrow \) surface has \( \phi = 0 \)
   - other potential of sphere from additional charge \( q_0 = 4 \pi \varepsilon_0 \delta \)
     at the center (superposition again!)

3. Force between "real" charge and conductor from Coulomb force on real charge from "image" charge

4. Surface charge density from Gauss's law

5. Dirichlet Green Function \( G_0(x_j, x_s) \) from potential of \( x_j \) from unit charge \( q_j \) at \( x_j \), (or vice versa, since \( G_0(x_j, x_s) = G_0(x_s, x_j) \))

\[ \text{Note: Fix } q \text{ and image charge derivation.} \]

\[ \begin{align*}
\Phi_0 &= 0 = 4 \pi \varepsilon_0 \left( \frac{q_R}{3R} + \frac{q_{R'}}{R'} \right) \Rightarrow q(R - b) = q'(5R - b) \\
\phi_p &= 0 = 4 \pi \varepsilon_0 \left( \frac{q}{3R} + \frac{q^2}{5R} \right) = 0 \\
\phi_p &= 0 = 4 \pi \varepsilon_0 \left( \frac{q}{3R} - \frac{q^2}{5R} \right) \Rightarrow \phi(R + b) = \phi'(5R + b) \\
\end{align*} \]

\[ \text{Spot the Error: had this last time!} \]

\[ q = - \frac{4q}{3} \text{ (odd)} \]

\[ q = \frac{8q}{5} \text{ (even)} \]

\[ \text{Substitute } \frac{q}{5} (5R) = 5q \Rightarrow b = \frac{3b}{5} \]
Dirichlet Green Function for Rectangular Box

Method 1:

- Need to solve $\nabla^2 G_D(x, y, z) = -4\pi \delta^3(x-x')$ with $|\nabla| = 0$.
- $G_D(x, y, z) = G_D(x', y', z')$ symmetric (proved in bonus problem).
- $G_D(x', y', z') = 0$ for $x', y', z' \neq x, y, z$ on surface. (Recall master formula)

$\Rightarrow$ vanishes on edges in $x, y, z \equiv \sin \left( \frac{lx}{a} \right) \sin \left( \frac{mx}{b} \right) \sin \left( \frac{nz}{c} \right)$

(separated in Cartesian coordinates)

(depending on $\frac{lx}{a}$, but not $x', y', z', z'$)

Symmetric $\Rightarrow \sum_{l, m, n = 1}^{\infty} \frac{\delta(x-x') \delta(y-y') \delta(z-z')}{l, m, n} G_{mn} \sin \left( \frac{lx}{a} \right) \sin \left( \frac{mx}{b} \right) \sin \left( \frac{nz}{c} \right) \frac{1}{l, m, n}$

$G_D(x, y, z) = 0$ on surface

$\Rightarrow$ In sines mean that $G_D$ vanishes at $x, x' = 0, y, y' = 0, z, z' = 0$.

$\Rightarrow$ In cosines mean that $G_D$ vanishes at $x, x' = 0, y, y' = b, z, z' = c$.

$\Rightarrow$ We could absorb the $x'$ normalization into $G_{mn}$.

$\Rightarrow$ But makes explicit the usual $x'$ normalization for each sine.

$\Rightarrow$ Now we need to determine $G_{mn}$ from $\nabla^2 G_D(x, y, z) = -4\pi \delta^3(x-x')$.

with $\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$ in Cartesian coordinates,

$G_D(x, y, z) = \frac{1}{\sqrt{x-x', y-y', z-z'}}$.

$\Rightarrow \sum_{l, m, n}^{\infty} \frac{\delta(x-x') \delta(y-y') \delta(z-z')}{l, m, n} G_{mn} \sin \left( \frac{lx}{a} \right) \sin \left( \frac{mx}{b} \right) \sin \left( \frac{nz}{c} \right) \frac{1}{l, m, n}$

$= -4\pi \left( \sum_{l=1}^{\infty} \frac{\sin \left( \frac{lx}{a} \right)}{l \cos \left( \frac{lx}{a} \right)} \right) \left( \sum_{m=1}^{\infty} \frac{\sin \left( \frac{mx}{b} \right)}{m \cos \left( \frac{mx}{b} \right)} \right) \left( \sum_{n=1}^{\infty} \frac{\sin \left( \frac{nz}{c} \right)}{n \cos \left( \frac{nz}{c} \right)} \right)$

$\Rightarrow G_{mn} = \frac{-1}{\alpha^2 + \beta^2 + \gamma^2} \sum_{l=1}^{\infty} \left( \frac{\sin \left( \frac{lx}{a} \right)}{l \cos \left( \frac{lx}{a} \right)} \right) \left( \sum_{m=1}^{\infty} \frac{\sin \left( \frac{mx}{b} \right)}{m \cos \left( \frac{mx}{b} \right)} \right) \left( \sum_{n=1}^{\infty} \frac{\sin \left( \frac{nz}{c} \right)}{n \cos \left( \frac{nz}{c} \right)} \right)$

That's it! $\Rightarrow$ Use in master formula to find $\phi(x')$.

$G_D(x, y, z) = \frac{1}{\alpha \beta \gamma} \sum_{l,m,n=1}^{\infty} \frac{\sin \left( \frac{lx}{a} \right) \sin \left( \frac{mx}{b} \right) \sin \left( \frac{nz}{c} \right)}{l \cos \left( \frac{lx}{a} \right) \cos \left( \frac{mx}{b} \right) \cos \left( \frac{nz}{c} \right) \sin \left( \frac{lx}{a} \right) \sin \left( \frac{mx}{b} \right) \sin \left( \frac{nz}{c} \right)}$.
Method II: Division of region or direct integration

In this approach we use the sine expansion for two of the three dimensions \((\sin x, y)\) but find the \(z, z'\) dependent part a different way.

This yields a different expansion that nonetheless gives the same result if you include all the terms but for a truncated sum one of the terms may converge much faster.

So the form we are looking is

\[
G_B(x, x') = \left( \frac{2\pi}{a^2 b} \right)^2 \sum_{\ell, m=1}^{\infty} g_{\ell m}(z, z') \sin \left( \frac{\ell \pi x}{a} \right) \sin \left( \frac{\ell \pi x'}{a} \right) \sin \left( \frac{m \pi y}{b} \right) \sin \left( \frac{m \pi y'}{b} \right)
\]

where \(g_{\ell m}(z, z')\) is a symmetric function \(g(z, z') = g(z', z)\), recall \(G_B(x, x') = G_B(x', x)\) that is different for each \(x, x'\) and \(z, z'\) value. (cf. As constant \(G_{\ell m}\) in method I.)

Boundary conditions: \(g_{\ell m}(x, z') = 0, g_{\ell m}(z, x') = 0\).

Determine the equation for \(g_{\ell m}(z, z')\) by requiring \(\nabla^2 G_B(x, x') = -4\pi^2 \delta(x-x') \delta(y-y') \delta(z-z')\)

\[
\nabla^2 G_B(x, x') = \frac{4\pi^2}{ab} \sum_{\ell, m=1}^{\infty} \sin \left( \frac{\ell \pi x}{a} \right) \sin \left( \frac{\ell \pi x'}{a} \right) \sin \left( \frac{m \pi y}{b} \right) \sin \left( \frac{m \pi y'}{b} \right)
\]

\[
\frac{d^2}{dx^2} \delta(x-x') \delta(y-y') \delta(z-z') = \frac{d^2}{dx^2} \left( \frac{2\pi}{a} \sin \left( \frac{\ell \pi x}{a} \right) \sin \left( \frac{\ell \pi x'}{a} \right) \sin \left( \frac{m \pi y}{b} \right) \sin \left( \frac{m \pi y'}{b} \right) \right)
\]

\[
\nabla \cdot \delta(x-x') \delta(y-y') \delta(z-z') = -4\pi^2 \delta(x-x') \delta(y-y') \delta(z-z')
\]

\[
eq \nabla \cdot \left[ \delta(x-x') \delta(y-y') \delta(z-z') \right] = \delta(x-x') \delta(y-y') \delta(z-z')
\]

\[
\nabla^2 \delta(x-x') \delta(y-y') \delta(z-z') = -4\pi^2 \delta(x-x') \delta(y-y') \delta(z-z')
\]

\[
\frac{\partial^2}{\partial x^2} g_{\ell m}(z, z') - \frac{\ell^2 \pi^2}{a^2} g_{\ell m}(z, z') = -4\pi^2 \delta(z-z')
\]

We'll solve this by the "division-of-region" method.
Let's first note that if \( \frac{b^2}{a^2} + \frac{m^2}{b^2} > 0 \), so define a new constant:

\[
\chi_{\ell m} = \sqrt{1 - \left( \frac{b^2}{a^2} + \frac{m^2}{b^2} \right)}
\]

\[
\Rightarrow \left( \frac{\partial^2}{\partial z^2} - \chi_{\ell m}^2 \right) g_{\ell m}(z, z') = -4\pi S(z-z') \quad \text{with} \quad g(0, z') = 0, \quad g(z, 0) = 0
\]

So far you may recognize this as a problem from one-dimensional quantum mechanics: An attractive delta function (with \( \chi_{\ell m} = 1 \))

You may also remember the method to solve this equation:
First, solve the homogeneous problem in the two regions

where the delta function is zero (at \( z < z' \) and \( z > z' \)) and

then determine any remaining coefficients by a matching condition found by integrating across the delta function, as well

as continuity of \( G_R \).

The general solution to 

\[
\left( \frac{\partial^2}{\partial z^2} - \chi_{\ell m}^2 \right) g_{\ell m}(z, z') = 0,
\]

is

\[
g_{\ell m}(z, z') = A e^{\chi_{\ell m} z} + B e^{-\chi_{\ell m} z} = A \sinh \chi_{\ell m} z + B' \cosh \chi_{\ell m} z
\]

- Region I: \( z < z' \) and \( g_{\ell m}(0, z') = 0 \) \( \Rightarrow \) \( B = 0 \) and \( g_{\ell m}(z, z') = A' \sinh \chi_{\ell m} z \)

- Region II: \( z > z' \) and \( g_{\ell m}(c, z') = 0 \).

We could use a combination of \( A \) and \( B \) or \( A' \) and \( B' \), but with some foresight we realize:

\[
g_{\ell m}(z, z') \propto \sinh \chi_{\ell m}(c-z).
\]

But we must have \( g_{\ell m}(z, z') = g_{\ell m}(z', z) \) everywhere

\[
\Rightarrow \quad z < z': \quad g_{\ell m}(z, z') \propto \sinh(k_{\ell m}(c-z')) \sinh k_{\ell m} z
\]

\[
\quad z > z': \quad g_{\ell m}(z, z') \propto \sinh k_{\ell m} z' \cdot \sinh (k_{\ell m}(c-z))
\]
In terms of $\theta$-functions, this is
\[ g_{lm}(z, z') = C \left[ a_{lm}(z' - z) \frac{\sinh k_{lm} z}{\sinh k_{lm} z'} \right] \]
but we usually use the compact notation
\[ g_{lm}(z, z') = C \sinh(k_{lm} z') \sinh(k_{lm} z) \]
Note that we have built in the continuity of $g_{lm}(z, z')$ at $z = z'$.
This makes sense physically because the Green function is the potential from a point source, which should be continuous.

So how do we find $C$? It must be determined by the inhomogeneous part (i.e., the delta function) of the equation for $g_{lm}$.

Standard procedure: integrate $g_{lm}$ equation across delta function

\[ \lim_{\varepsilon \to 0} \int_{z - \varepsilon}^{z + \varepsilon} \frac{\partial g_{lm}(z, z')}{\partial z'} d z' = -4\pi \int d z' \delta(z - z') \]
\[ \frac{1}{z - z'} g_{lm}(z, z') \mid_{z - \varepsilon}^{z + \varepsilon} \] continuous

\[ z = z' \text{ means region I: } \frac{1}{z - z'} \frac{\partial g_{lm}(z, z')}{\partial z'} \mid_{z = z'} = x_{lm} C \cosh(k_{lm} z) \sinh(k_{lm} z') \]
\[ z' \text{ means region II: } \frac{1}{z - z'} \frac{\partial g_{lm}(z, z')}{\partial z'} \mid_{z = z'} = -x_{lm} C \cosh(k_{lm} z) \sinh(k_{lm} z') \]
\[ = -x_{lm} C \left[ \sinh(k_{lm} z') \cosh(k_{lm} z) + \sinh(k_{lm} z') \sinh(k_{lm} z) \right] = -4\pi \]
\[ \Rightarrow \frac{1}{z - z'} \frac{\partial g_{lm}(z, z')}{\partial z'} \mid_{z = z'} = -4\pi \Rightarrow C = \frac{4\pi}{x_{lm} \sinh(k_{lm} z)} \]
\[ C_{0}(\frac{z}{\pi}) = \frac{16\pi}{\alpha \beta} \sum_{l=1}^{\infty} x_{lm} \sinh(k_{lm} z) \sinh(k_{lm} z') \]
with
\[ x_{lm} = \sqrt{\frac{\alpha^2 + m^2}{\alpha^2 + \beta^2}} \]