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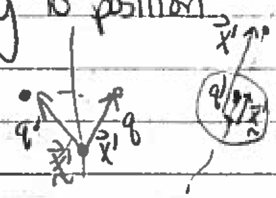
7701 Lecture 28

- Summary points for image charges (with outside conducting sphere and infinite plane as prototypes)

① Use boundary conditions (BCs) and symmetry to position the image charge (or charges)



- mirrored for infinite plane
- on line joining q and center of sphere
- more charges by superposition

② Determine q' and $\underset{\sim}{x}'$ (charge and location of image) by requiring BC to be satisfied

- grounded conductor (or G_D) $\Rightarrow$ surface has $\Phi = 0$
- other potential φ_0 of sphere from additional charge $q_0 = 4\pi\varepsilon_0\varphi_0$ at the center (superposition again!)

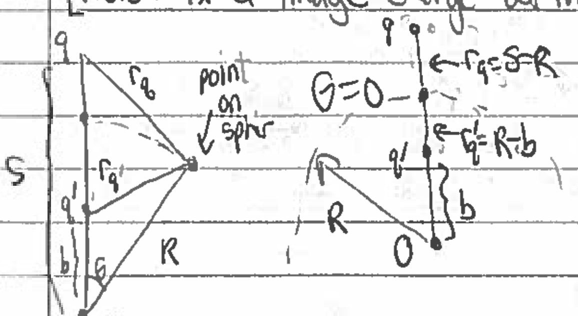
③ Force between "real" charge and conductor from Coulomb force on real charge from image charges

④ Surface charge density from Gauss's law

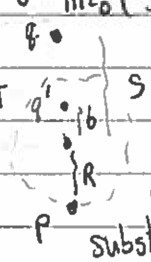
⑤ Dirichlet Green function $G_D(\vec{x}, \vec{x}')$ from potential at $\vec{x}'$ from unit charge $(q=1)$ at $\vec{x}'$. (or vice versa, since $G_D(\vec{x},\vec{x}') = G_D(\vec{x}',\vec{x})$)

spot the Error! had this $b-R$ last time!

[Note: Fix of image charge derivation.



$\Phi_p = 0 = \frac{1}{4\pi\varepsilon_0}\left(\frac{q}{S-R} + \frac{q'}{R-b}\right) \Rightarrow q(R-b) = q'(S-R)$



$\Phi_p = 0 \Rightarrow \frac{1}{4\pi\varepsilon_0}\left(\frac{q}{S+R} + \frac{q'}{b+R}\right) = 0$

$\Rightarrow q(R+b) = q'(S+R)$

add: $2qR = -2q'S \Rightarrow \boxed{q' = -\frac{R}{S}q}$

substitute $\frac{R}{S}(S+R) = b+R \Rightarrow \boxed{b = \frac{R^2}{S}}$]

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Dirichlet Green Function for Rectangular Box

Method I:

- Need to solve $\nabla^2 G_D(\vec{x},\vec{x}') = -4\pi\delta^3(\vec{x}-\vec{x}')$ with
  - $G_D(\vec{x},\vec{x}') = G_D(\vec{x}',\vec{x})$ symmetric (proved in bonus problem from master formula)
  - $G_D(\vec{x},\vec{x}') = 0$ for $\vec{x}$ (or $\vec{x}'$) on surface.
    - $\Rightarrow$ vanish on edges in $x, y, z \Rightarrow \sin\left(\frac{l\pi x}{a}\right)\sin\left(\frac{m\pi y}{b}\right)\sin\left(\frac{n\pi z}{c}\right)$ (separated in Cartesian coordinate)

Symmetric $\Rightarrow$ (depends on l,m,n but not x,x',y,y',z,z')

$$G_D(\vec{x},\vec{x}') = \sum_{l,m,n=1}^{\infty}\left(\frac{2}{a}\right)\left(\frac{2}{b}\right)\left(\frac{2}{c}\right) G_{lmn} \sin\left(\frac{l\pi x}{a}\right)\sin\left(\frac{l\pi x'}{a}\right)\sin\left(\frac{m\pi y}{b}\right)\sin\left(\frac{m\pi y'}{b}\right)\sin\left(\frac{n\pi z}{c}\right)\sin\left(\frac{n\pi z'}{c}\right)$$

$G_D(\vec{x}_s,\vec{x}')=0$ on surface:

- The sines mean that G_D vanishes at $x=x'=0$, $y=y'=0$, $z=z'=0$
- The arguments mean that G_D vanishes at $x=x'=a$, $y=y'=b$, $z=z'=c$

That is, any of these make it vanish

- We could absorb the $\frac{8}{abc}$ normalization into G_{lmn}
  - But it makes explicit the usual $\sqrt{\frac{2}{L}}$ normalization for each sine.

- Now we need to determine G_{lmn} from $\nabla^2 G_D(\vec{x},\vec{x}') = -4\pi\delta^3(\vec{x}-\vec{x}')$.
  with $\nabla^2 = \frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}$ in Cartesian coordinates,
  and $\delta^3(\vec{x}-\vec{x}') = \delta(x-x')\,\delta(y-y')\,\delta(z-z')$.

$$\Rightarrow \nabla^2 G_D(\vec{x},\vec{x}') = \sum_{lmn}\underbrace{\left(\frac{-8}{abc}\right)}_{\text{from derivatives}} G_{lmn}\left(\frac{l^2\pi^2}{a^2}+\frac{m^2\pi^2}{b^2}+\frac{n^2\pi^2}{c^2}\right)\times \sin(\ )\sin(\ )\sin(\ )\sin(\ )\sin(\ )$$

$$= -4\pi \underbrace{\left(\sum_{l=1}^{\infty}\frac{2}{a}\sin\left(\frac{l\pi x}{a}\right)\sin\left(\frac{l\pi x'}{a}\right)\right)}_{\delta(x-x')}\underbrace{\left(\sum_{m=1}^{\infty}\frac{2}{b}\sin\left(\frac{m\pi y}{b}\right)\sin\left(\frac{m\pi y'}{b}\right)\right)}_{\delta(y-y')}\underbrace{\left(\sum_{n=1}^{\infty}\frac{2}{c}\sin\left(\frac{n\pi z}{c}\right)\sin\left(\frac{n\pi z'}{c}\right)\right)}_{\delta(z-z')}$$

from 193 $\rightarrow$

- Now we equate l,m,n terms (or we can project out coefficients)

$$\Rightarrow -\frac{8}{abc}G_{lmn}\pi^2\left(\frac{l^2}{a^2}+\frac{m^2}{b^2}+\frac{n^2}{c^2}\right) = -4\pi\frac{8}{abc} \Rightarrow G_{lmn} = \frac{4}{\pi}\,\frac{1}{\frac{l^2}{a^2}+\frac{m^2}{b^2}+\frac{n^2}{c^2}}$$

$$\Rightarrow \boxed{G_D(\vec{x},\vec{x}') = \frac{32}{\pi abc}\sum_{lmn=1}^{\infty}\frac{1}{l^2/a^2+m^2/b^2+n^2/c^2}\sin\left(\frac{l\pi x}{a}\right)\sin\left(\frac{l\pi x'}{a}\right)\sin\left(\frac{m\pi y}{b}\right)\sin\left(\frac{m\pi y'}{b}\right)\sin\left(\frac{n\pi z}{c}\right)\sin\left(\frac{n\pi z'}{c}\right)}$$

That's it! $\Rightarrow$ Use in Master Formula to find $\Phi(\vec{x})$

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Method II: Division of region or direct integration

- In this approach we use the sine expansion for two of the three dimensions (say x and y) but find the z, z' dependent part a different way.
  - This yields a different expansion that nonetheless gives the same result if you include all the terms
  - $\Rightarrow$ but for a truncated sum one of them may converge much faster.

- So the form we are looking is

$$G_D(\vec{x},\vec{x}') = \left(\sqrt{\frac{2}{a}}\sqrt{\frac{2}{b}}\right)^2 \sum_{l,m=1}^{\infty} g_{lm}(z,z') \sin\left(\frac{l\pi x}{a}\right)\sin\left(\frac{l\pi x'}{a}\right)\sin\left(\frac{m\pi y}{b}\right)\sin\left(\frac{m\pi y'}{b}\right)$$

where $g_{lm}(z,z')$ is a symmetric function [$g(z,z') = g(z',z)$, recall $G_D(\vec{x},\vec{x}') = G_D(\vec{x}',\vec{x})$] that is different for each l and m value. (cf. the constant G_{lm} in method I.)

- Boundary conditions: $g_{lm}(0,z') = g_{lm}(z,0) = 0$.
- Determine the equation for $g_{lm}(z,z')$ by requiring $\nabla^2 G_D(\vec{x},\vec{x}') = -4\pi\delta^3(\vec{x}-\vec{x}')$

$$\Rightarrow \nabla^2 G_D(\vec{x},\vec{x}') = \frac{4}{ab}\sum_{l,m=1}^{\infty} \sin\left(\frac{l\pi x}{a}\right)\sin\left(\frac{l\pi x'}{a}\right)\sin\left(\frac{m\pi y}{b}\right)\sin\left(\frac{m\pi y'}{b}\right)$$

$$\times\left\{\left(-\frac{l^2\pi^2}{a^2} - \frac{m^2\pi^2}{b^2}\right)g_{lm}(z,z') + \frac{\partial^2}{\partial z^2}g_{lm}(z,z')\right\}$$

$\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}$

$$= -4\pi\delta^3(\vec{x}-\vec{x}') = -4\pi\delta(x-x')\,\delta(y-y')\,\delta(z-z')$$

$$= -4\pi \sum_{l=1}^{\infty}\frac{2}{a}\sin\left(\frac{l\pi x}{a}\right)\sin\left(\frac{l\pi x'}{a}\right)\cdot\sum_{m=1}^{\infty}\frac{2}{b}\sin\left(\frac{m\pi y}{b}\right)\sin\left(\frac{m\pi y'}{b}\right)\cdot\delta(z-z')$$

$\Rightarrow$ equate coefficients of each l, m term:

$$\boxed{\frac{\partial^2}{\partial z^2}g_{lm}(z,z') - \pi^2\left(\frac{l^2}{a^2}+\frac{m^2}{b^2}\right)g_{lm}(z,z') = -4\pi\delta(z-z')}$$

- We'll solve this by the "division-of-region" method.

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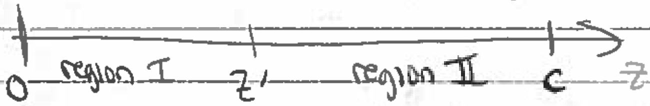
Let's first note that $\pi^2\left(\frac{l^2}{a^2}+\frac{m^2}{b^2}\right) > 0$, so define a new constant:

$$k_{lm} = \sqrt{\pi^2\left(\frac{l^2}{a^2}+\frac{m^2}{b^2}\right)}$$

$$\Rightarrow \boxed{\left(\frac{\partial^2}{\partial z^2} - k_{lm}^2\right) g_{lm}(z,z') = -4\pi\delta(z-z')} \quad \text{with} \quad \boxed{\begin{array}{l} g(0,z')=0 \\ g(z,0)=0 \end{array}}$$

- Some of you may recognize this as a problem from one-dimensional quantum mechanics; An attractive delta function (with $\hbar^2/2m = 1$)

- You may also remember the method to solve this equation: first solve the homogeneous problem in the two regions where the delta function is zero (ie $z<z'$ and $z>z'$) and then determine any remaining coefficients by a matching condition found by integrating across the delta function. (as well as continuity of G_D).



- The general solution to $\left(\frac{\partial^2}{\partial z^2} - k_{lm}^2\right) g_{lm}(z,z') = 0$ is
$g_{lm}(z,z') = Ae^{k_{lm}z} + Be^{-k_{lm}z} = A'\sinh k_{lm}z + B'\cosh k_{lm}z$

- Region I: $z<z'$ and $g_{lm}(0,z')=0 \Rightarrow B'=0$ and $g_{lm}^{(I)}(z,z') = A_I'\sinh k_{lm}z$ with $A_I' = A_I'(z')$.

- Region II: $z>z'$ and $g_{lm}(c,z')=0$.
  - We could use a combination of A and B or A' and B', but with some foresight we realize $g_{lm}^{(II)}(z,z') \propto \sinh k_{lm}(c-z)$.

- But we must have $g_{lm}(z',z) = g_{lm}(z,z')$ <u>everywhere</u>

$\Rightarrow z<z'$: $g_{lm}^{(I)}(z,z') \propto \sinh[k_{lm}(c-z')]\sinh k_{lm}z$

$z>z'$: $g_{lm}^{(II)}(z,z') \propto \sinh k_{lm}z' \cdot \sinh[k_{lm}(c-z)]$

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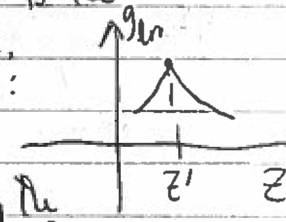
- In terms of Θ-functions this is

$$g_{lm}(z,z') = C\Big[\Theta(z'-z)(\sinh k_{lm}z)(\sinh k_{lm}(c-z')) + \Theta(z-z')(\sinh k_{lm}(c-z))(\sinh k_{lm}z')\Big]$$

but we usually use the compact notation $\left\{\begin{matrix} z_< \\ z_> \end{matrix}\right. = \left|\begin{matrix} \max\{z,z'\} \\ \min\{z,z'\} \end{matrix}\right.$

$$\Rightarrow \quad g_{lm}(z,z') = C \sinh(k_{lm}z_<)\sinh[k_{lm}(c-z_>)]$$

- Note that we have built in the continuity of $g_{lm}(z,z')$ at $z=z'$. This makes sense physically because the Green function is the potential from a point source, which should be continuous.
  - However, the slope does not need to be continuous:



- So how do we find C? It must be determined by the inhomogeneous part (ie, the delta function) of the equation for g_{lm}.
  - Standard procedure: integrate g_{lm} equation across delta function

$$\lim_{\varepsilon\to 0}\left|\ \underbrace{\int_{z'-\varepsilon}^{z'+\varepsilon} dz \frac{\partial^2}{\partial z^2} g_{lm}(z,z')}_{\frac{\partial}{\partial z} g_{lm}(z,z')\Big|_{z'_-}^{z'_+}} - \underbrace{k_{lm}^2 \int_{z'-\varepsilon}^{z'+\varepsilon} g_{lm}(z,z')}_{-k_{lm}^2\, 2\varepsilon\, g_{lm}(z',z') \to 0 \ \leftarrow \text{continuous}} = \underbrace{-4\pi \int_{-z'-\varepsilon}^{z'+\varepsilon} dz\, \delta(z-z')}_{-4\pi}\right.$$

constant

$z = z'-\varepsilon \equiv z'_-$ means region I: $\frac{\partial}{\partial z} C \sinh(k_{lm}z)\sinh k_{lm}(c-z')\Big|_{z=z'} = k_{lm} C \cosh(k_{lm}z')(\sinh k_{lm}(c-z'))$

$z = z'+\varepsilon = z'_+$ means region II: $\frac{\partial}{\partial z} C \sinh k_{lm}(c-z)\sinh(k_{lm}z')\Big|_{z=z'} = -k_{lm} C \cosh(k_{lm}(c-z'))\sinh(k_{lm}z')$

$$\Rightarrow -k_{lm} C\left[\sinh(k_{lm}z')(\cosh k_{l}(c-z')) + \cosh(k_{lm}z')\sinh(k_{lm}(c-z'))\right] = -4\pi$$

$$\Rightarrow k_{lm} C \sinh\left[k_l(c-z') + k_l z'\right] = -4\pi \quad \Rightarrow \quad C = \frac{4\pi}{k_{lm}\sinh(k_{lm}c)}$$

$$\Rightarrow \boxed{G_D(\vec{x},\vec{x}') = \frac{-16\pi}{ab}\sum_{l,m=1}^{\infty} \frac{1}{k_{lm}\sinh(k_{lm}c)} \times \sin\frac{l\pi x}{a}\sin\frac{l\pi x'}{a}\sin\frac{m\pi y}{b}\sin\frac{m\pi y'}{b}\sinh(k_{lm}z_<)\sinh(k_{lm}(c-z_>))}$$

with $\boxed{k_{lm} = \sqrt{\pi^2\left(\frac{l^2}{a^2}+\frac{m^2}{b^2}\right)}}$